PART (A) : PHYSICS

ANSWER KEY

1.	(D)	2.	(D)	3.	(B)	4.	(C)	5.	(D)
6.	(C)	7.	(C)	8.	(C)	9.	(B)	10.	(D)
11.	(D)	12.	(B)	13.	(D)	14.	(B)	15.	(B)
16.	(B)	17.	(C)	18.	(D)	19.	(D)	20.	(A)
21.	(5)	22.	(432)	23.	(3)	24.	(6)	25.	(4)
26.	(3)	27.	(17)	28.	(0)	29.	(3)	30.	(3)

SOLUTIONS

1. (D)

$\delta R / R = 2\delta u / u + 2\cot 2\theta\delta\theta$

2. (D)

In n-type semiconductor, intrinsic semiconductors are doped with donor atoms whose valency is 5.

3. (B)

The normal reaction vanishes when the person loses contact with the surface:

$mv^2 / r = mg\cos\theta;$

Conservation of energy gives:

$1/2\,mv^2 + mgr\cos\theta = 1/2\,mu^2 + mgr; \text{where } u^2 = 0.5\,gr$

4. (C)

The droplets fall a distance $\frac{1}{2}gt^2$ in time t, and the number of the droplets and hence their mass is proportional to dt. Computing the CM of the droplets using the definition, we get the result.

5. (D)

For an equilateral triangle of side a, moment of inertia is $\frac{ma^2}{12}$ about an axis passing through its CM and perpendicular to plane – the result of the integration is similar to that of a solid cone along its axis: a factor of 3/5.

$$I = \frac{3}{5}\left(\frac{ma^2}{12}\right)$$

6. (C)

Use Kepler's Law of Periods.

7. (C)
Taking torques about the hinge, we get:
In the 1st case, $mgL/2 = p\,AL/2$
In the second case, $mgL/4 = p'AL/2$

8. (C)
With reservoirs at 273 K, 173 K;
$\frac{W}{Q} = \frac{273-173}{273} = 0.37$
With reservoirs at 373 K, 173 K;
$\frac{W'}{Q} = \frac{373-173}{373} = 0.54$
$W' - W = (0.54 - 0.37)Q = 0.17\,Q$

9. (B)
$T = \frac{2w}{v}; f = \frac{1}{T}$

10. (D)
Using dimensional analysis or otherwise $t = \frac{V}{A}\sqrt{\frac{m}{RT}} \times$ some numerical factor
Substituting the values we get $\frac{t'}{t} = \frac{1}{2}\sqrt{\frac{1.7}{1.2}} \approx 0.6$

11. (D)
Force $= 25\left(0.4\hat{i} - 0.4\hat{k}\right) \times 2\hat{i} = -20\hat{j}$

12. (B)
The correct answer can be determined by dimensional analysis.

13. (D)
From geometry $\beta = \left(180^\circ - 2\theta\right)$
So, $\frac{d\beta}{dt} = -\frac{2d\theta}{dt}$

14. (B)
When the switch closes at $V_C = 2V/3$ the capacitor discharges through $R_2 = 3R$
T(discharge) $= 3RC\ell n2$ (since the voltage halves).
The charging occurs through $R_1 + R_2 = 9R$
$T(\text{charge}) = 9RC\ell n2$

$T = 12RC\ell n2 \approx 8.4RC$

15. (B)

When the left end is positive, upper diode conducts and lower diode is cut off.

$R_{eq} = R$

When right end is positive the lower diode conducts.

$R_{eq} = R + \frac{3R}{4} = \frac{7R}{4}$

$\therefore \text{Power} = \frac{(\Delta V)^2}{R}\left[1 + \frac{4}{7}\right] \times \frac{1}{2} \approx \frac{(\Delta V)^2}{R} \times 0.8$

16. (B)

The object must be located at the centre of the curvature of the mirror for this to happen.

For the inverted image formed directly by the lens we can write

$\frac{1}{v} - \frac{1}{u} = \frac{1}{10}, \frac{v}{u} = -\frac{3}{2}$

17. (C)

This occurs when the angle of incidence is equal to the Brewster angle: $\tan\theta_\beta = \mu$.

18. (D)

If the sphere has a uniform mass density (total mass m), then $\frac{\mu}{L} = \frac{q}{2m}$, where $L = \frac{2}{5}mR^2\omega$.

19. (D)

$h\nu = E_{n+1} - E_n = E_0\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \approx \frac{E_0 2n}{n^4}$, for large n.

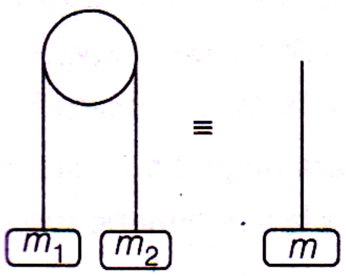
20. (A)

$m_n = \frac{m_1v_1 - m_2v_2}{v_2 - v_1}$

$= \frac{14 \times (4.7 \times 10^6) - 1 \times (3.3 \times 10^7)}{3.3 \times 10^7 - 4.7 \times 10^6} = 1.16$

21. (5)

A system of pulleys attached with two blocks of masses m_1 and m_2 is as shown below.



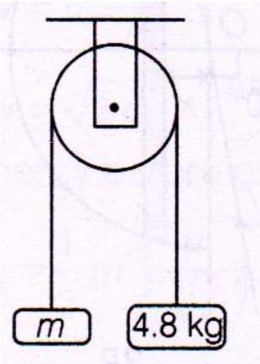
Equivalent mass of the above

Mentioned system, $m_{eq} = \frac{4m_1m_2}{m_1 + m_2}$

So, equivalent of 2 kg and 3 kg system given in the question

$m_{eq} = \frac{4\times2\times3}{2+3} = 4.8$

Given system can therefor be represented as



Therefore, for equilibrium m must equal to 4.8 kg.

22. (432)

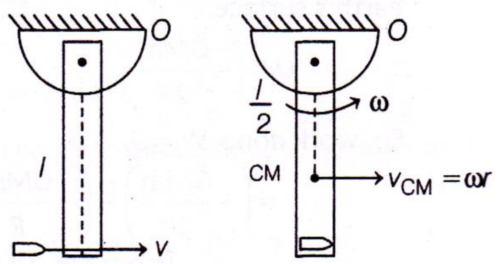
Work done by the fiction on block is = –216 J

Therefore the work done by the motor is

$= 2\times216\,\text{J} = 432\,\text{J}$

23. (3)

The position of the bullet and rod during its motion is as shown below



By conservation of angular momentum about hinge O,

$L_{final} = L_{initial}$

$\Rightarrow I\omega = mvl$

$$\Rightarrow \left(\frac{ml^2}{3} + ml^2\right)\omega = mvl$$

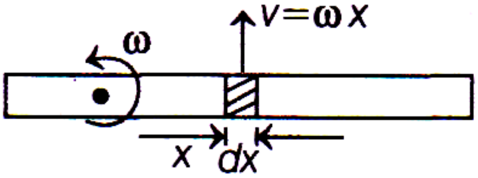
$$\Rightarrow \omega = \frac{3v}{4l}$$

$$\Rightarrow v_{CM} = \omega r = \left(\frac{3v}{4l}\right).\frac{l}{2} = \frac{3v}{8}$$

$$\Rightarrow x = \frac{8}{3} = 2.67$$

24. (6)

Consider an element at a distance x from axis having thickness dx as shown. Velocity of element,



$\Rightarrow$ Motion emf in element $= B.\omega x.dx$

$(\because e = Bvl)$

$\Rightarrow$Motional emf induced between ends of the rod

$$= \int_{-\frac{l}{3}}^{\frac{2l}{3}} B\omega x\, dx = B\omega\left[\frac{x^2}{2}\right]_{-\frac{l}{3}}^{\frac{2l}{3}}$$

$$= \frac{B\omega}{2}\left[\left(\frac{2l}{3}\right)^2 - \left(-\frac{l}{3}\right)^2\right] = \frac{B\omega l^2}{6}$$

$$\Rightarrow x = 6$$

25. (4)

For time scale to the uniform

$$v_{surface} = \text{constant(independent of y)}$$

$$\Rightarrow \frac{a\sqrt{2gy}}{\pi\left(\frac{y}{k}\right)^{\frac{2}{n}}} = \text{constant}$$

$$\Rightarrow n = 4$$

26. (3)

For equilibrium

$$\tau_{anticlockwise} = \tau_{clockwise}$$

$$\Rightarrow BlN\pi R^2 = v.\frac{dmd}{dt}.\frac{3L}{4}$$

$$\Rightarrow Bl\pi R^2 = v.\rho RLv.\frac{3L}{4}$$

$$\Rightarrow l = \frac{3\rho v^2L^2}{4N\pi RB} = 3A$$

27. (17)

As, angular frequency(coefficient time) does not depend on medium.

$$\Rightarrow \omega = \frac{\omega'}{2} \Rightarrow 0.5 \qquad \ldots\ldots(i)$$

Also, coefficient of x represents angular wave number. So, for electric field wave in liquid, angular wave number

$$\frac{2\pi}{\lambda'} = \frac{6\pi}{\lambda}$$

Where, λ' is wavelength in liquid.

$$\Rightarrow \lambda' = \frac{\lambda}{3}$$

$\therefore$ Refractive index of liquid $n = \frac{\lambda}{\lambda'} = 3$

Also, as refractive index in terms of relative permittivity ε_r is given by

$$n = \sqrt{\varepsilon_r, \mu_r} \Rightarrow \varepsilon_r \propto n^2$$

$$\Rightarrow \frac{(\varepsilon_r)_{liquid}}{(\varepsilon_r)_{air}} = \left(\frac{n_{liquid}}{n_{air}}\right)^2 = \left(\frac{3}{1}\right)^2 = 9$$

As $(\varepsilon_r)_{air} = 1$, therefore $(\varepsilon_r)_{liquid} = 9$. As, relative permittivity is same as dielectric constant

$\therefore$ Dielectric constant,

$$C = (\varepsilon_r)_{liquid} = 9 \qquad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$C - \frac{\omega}{\omega'} = 9 - 0.5 = 8.5$$

28. (0)

Velocity of CM of a system of two particles of masses m_1 and m_2 moving with velocities v_1 and v_2 respectively is given by

$$v_{CM} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

$$\Rightarrow m_1v_{CM} + m_2v_{CM} = m_1v_1 + m_2v_2$$

$$\Rightarrow m_1(v_1 - v_{CM}) + m_2(v_2 - v_{CM}) = 0$$

$\therefore$ Momentum of system is zero in CM frame irrespective if given information.

29. (3)

Fore on Dipole-2

= Force due to Dipole $-1+$ Force due to ring

$$=p\frac{dE_1}{dx}+p\frac{dE_2}{dx}$$

$$=\frac{pd\left[\frac{2Kp}{x^3}\right]}{dx}+0 \text{ (as field of ring is maximum at } x=\frac{R}{\sqrt{2}}\text{)}$$

$$=\frac{6Kp^2}{x^4}=\frac{6\left(\frac{1}{4\pi\varepsilon_0}\right)p^2}{\left(\sqrt{2}R\right)^4}$$

$$=\frac{3p^2}{8\pi\varepsilon_0 R^4}$$

$$\Rightarrow x=\frac{8}{3}=2.67$$

30. (3)

For the screw gauge shown,

$$LC=\frac{\text{Pitch}}{\text{No.of circular scale divisions}}$$

$$=\frac{0.5\text{mm}}{50}=0.01\text{mm}$$

For first wire shown in diagram,

Reading (Thickness) $=MSR+LC\times CSR$

$=7+0.01\times 47=7.47\text{mm}$

$\Rightarrow$Actual thickness of first wire,

$t_1=7.47-e\text{ mm}$ ....(i)

Here, $e=$ zero error

For second wire,

Reading (Thickness) $=MSR+LC\times CSR$

$=13+0.01\times 49$

$=13.49\text{mm}$

$\Rightarrow$ Actual thickness of second wire, t_2

$=13.49-e\text{ mm}$ .....(ii)

It is given that $t_2=2t_1$

$\Rightarrow 13.49-e=2(7.47-e)$ [using Eqs. (i) and (ii).]

$\Rightarrow e=1.45\times 2=2.9\simeq 3$

PART (B) : CHEMISTRY

ANSWER KEY

31.	(A)	32.	(C)	33.	(D)	34.	(B)	35.	(D)
36.	(A)	37.	(B)	38.	(C)	39.	(D)	40.	(D)
41.	(D)	42.	(D)	43.	(B)	44.	(C)	45.	(D)
46.	(B)	47.	(A)	48.	(D)	49.	(B)	50.	(B)
51.	(4)	52.	(3)	53.	(3)	54.	(1)	55.	(3)
56.	(9)	57.	(7)	58.	(2)	59.	(5)	60.	(2)

SOLUTIONS

31. (A)

H_2O_2 can act as 0.A as well as R.A. In H_2O_2, O.N> of O is -1. So, it can be oxidised to O.S. or reduced to O.S. -2

32. (C)

$$Cl_2 + H_2O \rightarrow HCl + HOCl$$

$$HCl + AgNO_3 \rightarrow AgCl \downarrow (\text{white}) + HNO_3$$

$$2HCl + Mg \rightarrow MgCl_2 + H_2(g) \uparrow$$

33. (D)

34. (B)

The order of chemical reactivity: $F_2 > Cl_2 > Br_2 > I_2$

35. (D)

The half reactions are

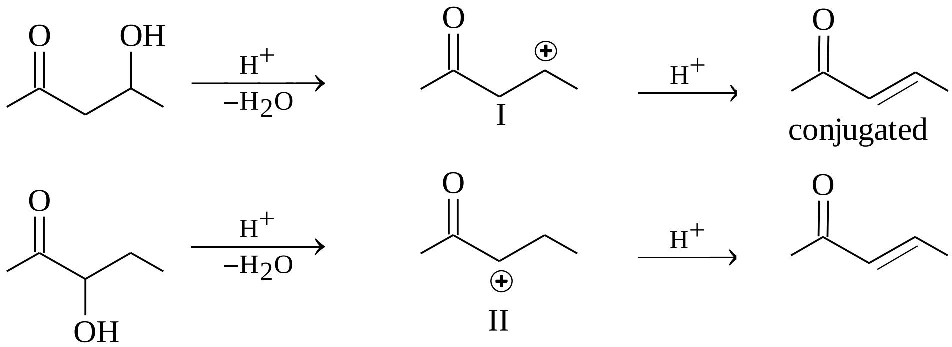
$$(Fe(s) \rightarrow Fe^{2+}(aq) + 2e^-) \times 2$$

$$O_2(g) + 4H^+ + 4e^- \rightarrow 2H_2O$$

$$2Fe(s) + O_2(g) + 4H^+ \rightarrow 2Fe^{2+}(aq) + 2H_2O(\ell)$$

$$E = E_0 - \frac{0.059}{4} \log \frac{(10^{-3})^2}{(10^{-3})^4(0.1)} = 1.57V$$

36. (A)



Although, both reactions are giving the same product, carbocation I is more stable than II.

37. (B)

$Ag_2CrO_4 \rightleftharpoons \underset{2x}{2Ag^+} + \underset{x}{CrO_4^{2-}}$

$AgNO_3 \rightarrow Ag^+ + NO_3^-$

$[Ag^+] = (2x + 0.1)M \simeq 0.1M (x << 0.1)$

$[CrO_4^{2-}] = xM$

Thus, $[Ag^+]^2[CrO_4^{2-}] = K_{Sp}$

$(0.1)^2(x) = 1.1 \times 10^{-12}$

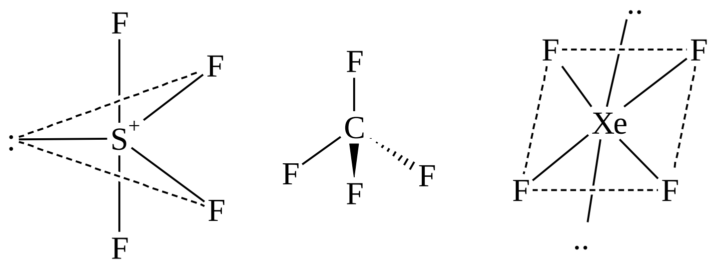
$x = 1.1 \times 10^{-10} M$

38. (C)

Rate of diffusion $\propto$ partial pressure

$\propto \frac{1}{\sqrt{M}}$

39. (D)



See-saw shape   Tetrahedral   square planar

40. (D)

Element	Relative mass	Molar mass	Relative mole	Simplest whole number rates
C	6	12	6/12 = 0.5	0.5/0.5=1=x
H	1	1	$\frac{1}{1}=1$	1/0.5=2=y

$$C_x + H_y + \left[x + \frac{y}{4}\right]O_2 \rightarrow xCO_2 + \frac{y}{2}H_2O$$

Number of oxygen atoms required = $2\times\left[x+\frac{y}{4}\right]=\left[2x+\frac{y}{2}\right]$

$$z = \frac{1}{2}\left[2x+\frac{y}{2}\right]=\left[x+\frac{y}{4}\right]$$

$$z = \left[x+\frac{y}{4}\right]=\left[1+\frac{2}{4}\right]=1.5$$

$$\therefore x = 1, y = 2, z = 1.5$$

$$C_xH_yO_z = C_1H_2O_{1.5}$$

So, empirical formula will be $((C_1H_2O_{1.5})\times 2 = C_2H_4O_3)$

41. (D)
Buna-N-Rubber is the

$$H_2C=CH-CH=CH_2\ \text{(buta-1, 3-diene)} + H_2C=\underset{\displaystyle CN}{\underset{|}{CH}}$$

(acrylonitrile). It is an example of both addition polymerisation as well as co-polymerisation.

42. (D)
Drinking water has BOD < 5 ppm. So, A and B both are polluted but B is more polluted than A.

43. (B)
Among these, only aniline can be converted into $(NH_4)_2SO_4$ during the process.

44. (C)

$$P_4^0 + 3NaOH + 3H_2O \rightarrow \overset{-3}{P}H_3 + NaH_2\overset{+1}{P}O_2$$

$$NaH_2\overset{+1}{P}O_2 \xrightarrow{\Delta} Na_2H\overset{+5}{P}O_4 + \overset{-3}{P}H_3$$

45. (D)

$$2NO(g) + O_2(g) \rightleftharpoons 2NO_2(g)$$

$$\Delta G_f^0 = 2\Delta G_f^0(NO_2) - [2\Delta G_f^0(NO) + \Delta G_f^0(O_2)]$$

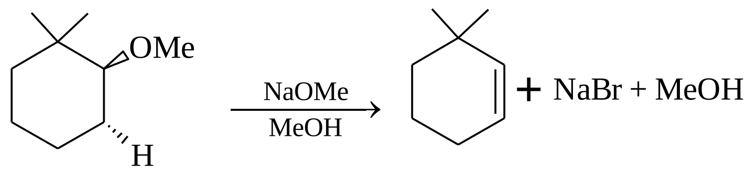
$$-RT\ell nK_p = 2\Delta G_f^0(NO_2) - 2\times 86600$$

$$\Delta G_f^0(NO_2)=\frac{1}{2}[2\times 86600-R\times 298\ln(1.6\times 10^{12})]$$

$$\Delta G_f^0(NO_2)=0.5[2\times 86600-Rx298\times \ln(1.6\times 10^{12})]$$

46. (B)

It is dihydrohalogenation, which is Belimination proceeding through E_2 mechanism.



47. (A)

$$\Delta G^0=-nF\varepsilon^0=-2\times 96000\times 2=-384000 Jmol^{-1}$$

$$\Delta S^0=nF\left(\frac{d\varepsilon^0}{dT}\right)=-96JK^{-1}mol^{-1}$$

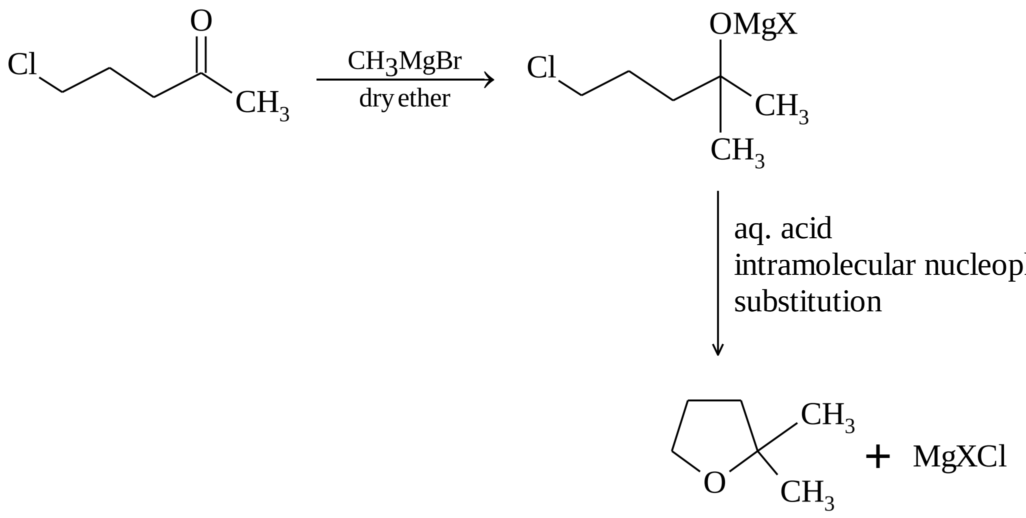
$$\Delta G^0=\Delta H^0-T\Delta S^0$$

$$\Rightarrow \Delta H^0=\Delta G^0+T\Delta S^0$$

$$=(-38400-28800)Jmol^{-1}=-412800Jmol^{-1}$$

$$=-412.800kJmol^{-1}$$

48. (D)



49. (B)

H.E. $\propto$ Charge

$\propto \frac{1}{\text{size}}$ Here, charge is considered first

50. (B)

Number of waves $= \dfrac{\text{circumference}}{\text{wavelength}}$

$\Rightarrow n = \dfrac{2\pi r}{\lambda}$

$\therefore\ 2\pi r = n\lambda$

Radius , $r = \dfrac{a_0 n^2}{z}$

$\Rightarrow 2\pi a_0 \dfrac{n^2}{z} = n\lambda$

Given $\lambda = 1.5\pi a_0$, so $2\pi a_0 \dfrac{n^2}{z} = n(1.5\pi a_0)$

$\Rightarrow \dfrac{n}{z} = \dfrac{1.5\pi a_0}{2\pi a_0} = \dfrac{1.5}{2} = 0.75$

51. (4)

$[Fe(SCN)_6]^{3-} = Fe^{3+}(3d^5)$ no pairing

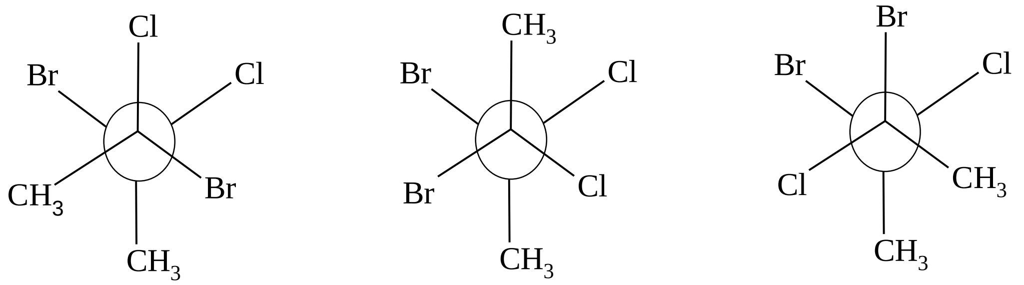
$Ms = \sqrt{5(5+2)}BM = \sqrt{35}BM$

$[Fe(CN)_6]^{3-} = Fe^{3+}(3d^5)$ pairing

$Ms = \sqrt{1(1+2)}BM = \sqrt{3}B.M$

Difference $= \sqrt{35} - \sqrt{3} \simeq 4BM$

52. (3)



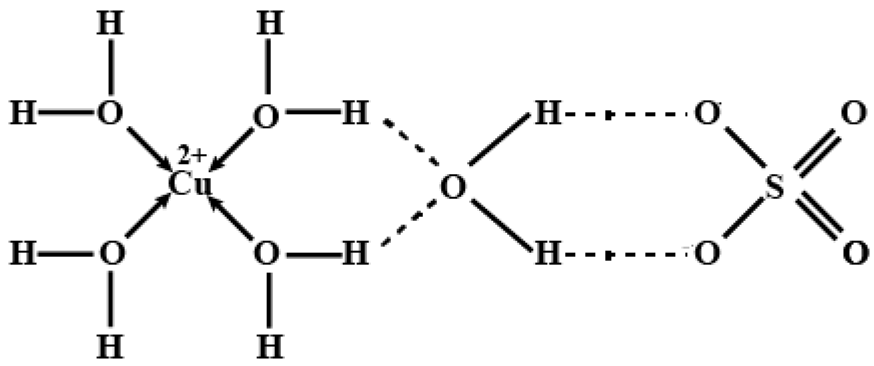
53. (3)

From charge balance

$2n + 6 + 24 - 36 = 0$

$\Rightarrow n = 3$

54. (1)



55. (3)

$$_{92}U^{235} \rightarrow {}_{54}Xe^{142} + {}_{38}Sr^{90} + 3\,{}_{0}n^{1}$$

56. (9)

$$kt = \ln\frac{[A]_0}{[A]}$$

$$kt = \ln\frac{[A]_0}{[A]}$$

$$\Rightarrow kt_{1/8} = \ln\frac{[A]_0}{[A]_0/8} = \ln 8$$

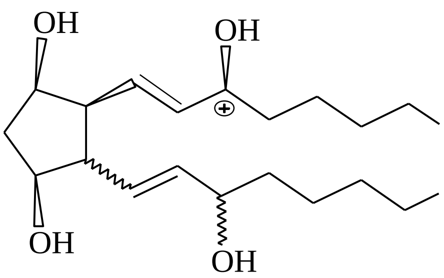
$$kt_{1/10} = \ln\frac{[A]_0}{[A]_0/10} = \ln 10$$

$$\therefore \frac{t_{1/8}}{t_{1/10}} = \frac{\ln_8}{\ln_{10}} = \log 8 = 3\log 2 = 3\times 0.3 = 0.9$$

$$\Rightarrow \frac{t_{1/8}}{t_{1/10}}\times 10 = 0.9\times 10 = 9$$

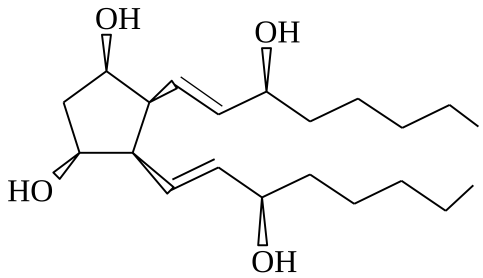
57. (7)

3 stereocenters are visible as follows



So, that number of stereoisomers = $2^3 = 8$

But



is inactive.

Hence, total number of optically active stereoisomers =7

58. (2)

Molality $=\left(\frac{20}{172}\right)\times\frac{1000}{50}=2.325\text{m}$

$\Delta T_f = 2 = i k_f m$

$\Rightarrow i = 0.5 = \frac{1}{2} = \frac{1}{x} \Rightarrow x = 2$

59. (5)

$d = \frac{4\times 6.023y}{6.023\times 10^{23}\times 8y\times 10^{-27}} = 5\times 10^{3}\text{g/m}^{3} = 5\text{kg/m}^{3}$

60. (2)

PART (C) : MATHEMATICS

ANSWER KEY

61.	(C)	62.	(B)	63.	(C)	64.	(D)	65.	(B)
66.	(C)	67.	(A)	68.	(C)	69.	(C)	70.	(A)
71.	(A)	72.	(A)	73.	(B)	74.	(D)	75.	(C)
76.	(D)	77.	(B)	78.	(A)	79.	(B)	80.	(C)
81.	(2)	82.	(4)	83.	(0)	84.	(14)	85.	(4)
86.	(28)	87.	(1)	88.	(2)	89.	(2)	90.	(4)

SOLUTIONS

61. (C)

Given, $f(x)+f\left(1-\frac{1}{x}\right)=1+x$ ...(i)

Replacing x by $1-\frac{1}{x}$ in Eq. (i), we get

$$f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-x}\right)=1+1-\frac{1}{x} \quad \text{...(ii)}$$

and replacing x by $\frac{1}{1-x}$ in Eq. (i), we get

$$f\left(\frac{1}{1-x}\right)+f(x)=1+\frac{1}{1-x} \quad \text{...(iii)}$$

Now, adding Eqs. (i) and (ii) and the subtracting Eq. (iii), we get

$$2f(x) = 1 + x + 1 + \frac{1}{1-x} - 2 + \frac{1}{x}$$

$$=x+\frac{1}{1-x}+\frac{1}{x}$$

$$2f(x)=\frac{x^3-x^2-1}{x(x-1)}$$

Now, given g(x) = 2f(x) – x + 1

$$=\frac{x^3-x^2-1}{x(x-1)}-x+1=\frac{-2+2x^2-2x}{2x(x-1)}$$

$$\Rightarrow \quad g(x)=\frac{x^2-x-1}{x(x-1)}$$

Now, for y = $\sqrt{g(x)}$, we must be

$$\frac{x^2-x-1}{x(x-1)}\geq 0$$

$$\Rightarrow \quad \frac{\left(x-\frac{1-\sqrt{5}}{2}\right)\left(x-\frac{1+\sqrt{5}}{2}\right)}{x(x-1)} \geq 0$$

$$\Rightarrow \quad x \in \left(-\infty \frac{1-\sqrt{5}}{2}\right) \cup (0,1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$$

62. (B)

Let $z = e^{ix}, x \in [0, 2\pi)$, then $\bar{z} = e^{-ix}$

Given, $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1 \qquad \Rightarrow |z^2 + \bar{z}^{-2}| = 1$

$\Rightarrow$ $|\cos 2x + i \sin 2x + \cos 2x - i \sin 2x| = 1$

$\Rightarrow$ $|\cos 2x| = \frac{1}{2}$

$\Rightarrow$ $\cos 2x = \pm\frac{1}{2}$

$\Rightarrow$ $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\left(\text{for } \cos x = \frac{1}{2}\right)$

$\Rightarrow$ $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\left(\text{for } \cos x = \frac{-1}{2}\right)$

63. (C)

t_{r+1} of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9 = {}^9C_r \left(\frac{3}{2}x^2\right)^r \left(\frac{-1}{3x}\right)^{9-r}$

$$= {}^9C_r \left(\frac{3}{2}\right)^r \left(\frac{-1}{3}\right)^{9-r} \cdot x^{3r-9}$$

t_{r+1} is independent of x, if $3r - 9 = 0$.

$\Rightarrow$ $r = 3$

For $r = 3$,

$${}^9C_r \left(\frac{3}{2}\right)^r \left(\frac{-1}{3}\right)^{9-r} = {}^9C_3 \left(\frac{3}{2}\right)^3 \left(\frac{-1}{3}\right)^{9-3}$$

$$= \frac{7}{18}$$

t_{r+1} contains $\frac{1}{x^3}$, if $3r - 9 = -3$

$\Rightarrow$ $r = 2$

For $r = 2$, ${}^9C_r \left(\frac{3}{2}\right)^r \left(\frac{-1}{3}\right)^{9-r}$

$$= {}^9C_2 \left(\frac{3}{2}\right)^2 \left(\frac{-1}{3}\right)^7 = \frac{-1}{27}$$

$\therefore$ Coefficient of the term independent of x in the given expression

$$=\frac{-2}{27}+\frac{7}{18}=\frac{-4+21}{54}=\frac{17}{54}$$

64. (D)

$$\text{RHL} = \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\tan^2\{x\}}{(x^2-[x]^2)}$$

$$=\lim_{x\to 0^+}\frac{\tan^2 x}{x^2}$$

[$\therefore$ $x \to 0^+$, then [x] = 0 and {x} = x]

$$=\lim_{x\to 0^+}\left(\frac{\tan^2 x}{x^2}\right)^2=1$$

$$\text{LHL} = \lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\sqrt{\{x\}\cot\{x\}}$$

[$\therefore$ $x \to 0^-$, then [x] = –1 and {x} = x – [x]

$\Rightarrow$ {x} = x + 1, when $x \to 0^-$]

$$\sqrt{1\cdot\cot 1}=\sqrt{\cot 1}$$

Also,

$$\cot^{-1}\left[\lim_{x\to 0^-} f(x)\right]^2=\cot^{-1}(\cot 1)=1$$

$$\text{and }\tan^{-1}\left[\lim_{x\to 0^+} f(x)\right]=\tan^{-1}1=\frac{\pi}{4}$$

$\therefore$ LHL $\neq$ RHL

$\therefore$ $\lim_{x\to 0} f(x)$ does not exist.

65. (B)

The straight lines I_1, I_2, I_3 are parallel and i.e. in the same plane

Total number of points = m + n + k total number of triangles formed with vertices = ${}^{m+n+k}C_3$.

By joining three given points on the same line we don't obtain a tringle.

Therefore, the maximum number of triangles

$$={}^{m+n+k}C_3-{}^{m}C_3-{}^{n}C_3-{}^{k}C_3$$

66. (C)

Method (1)

Given,	x + y = 2	…(i)
	3x – 4y = 6	…(ii)
	x – y = 0	…(iii)

Solving Eqs. (i), (ii) and (iii), we obtain vertices of triangle as (2, 0), (1, 1) and (–6, –6).

An equation of circle through (2, 0) and (1, 1) is

$(x – 2) (x – 1) + (y – 0) (y – 1) + \lambda$

$$\begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

As it passes through (–6, –6), we get (–6 –2) (–6 –1) + (–6 –0) (–6 –1) + λ

$$\begin{vmatrix} -6 & -6 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $56 + 42 + \lambda(14) = 0$

$\Rightarrow$ $\lambda = -7$

Thus, equation of required circle is

$(x^2 - 3x + 2) + (y^2 - y) - 7(-x - y + 2) = 0$

$\Rightarrow$ $x^2 + y^2 + 4x + 6y - 12 = 0$

Method (2)

Equation of any curve through intersection of three given lines is

$(x + y - 2)(3x - 4y - 6) + \alpha(3x - 4y - 6)(x - y) + \beta(x - y)(x + y - 2) = 0$

It will represent a circle, if coefficient of x^2 – coefficient of y^2 = 0 and coefficient of xy = 0.

i.e. $(3+3\alpha+\beta)-(-4+4\alpha-\beta)=0$

and $(-1)+\alpha(-7)+\beta(0)=0$

$\Rightarrow$ $7-\alpha+2\beta=0$ and $7\alpha=-1$

$\Rightarrow$ $\alpha=\frac{-1}{7}$ and $\beta=\frac{-25}{7}$

$\therefore$ equation of required circle is

$x^2(3+3\alpha+\beta)+y^2(-4+4\alpha-\beta)+x$

$(-6-6-6\alpha-2\beta)+y(8-6+6\alpha+2\beta)+12=0$

$\Rightarrow$ $-x^2 - y^2 - 4x - 6y + 12 = 0$

$\Rightarrow$ $x^2 + y^2 + 4x + 6y - 12 = 0$

67. (A)

Let the number of red and blue balls be r and b, respectively.

Then, the probability of drawing two red balls is

$$p_1 = \frac{{}^rC_2}{{}^{r+b}C_2} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2 = \frac{{}^bC_2}{{}^{r+b}C_2} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3 = \frac{{}^rC_1\,{}^bC_1}{{}^{r+b}C_2} = \frac{2br}{(r+b)(r+b-1)}$$

By hypothesis $p_1 = 5p_2$ and $p3 = 6p_2$

$\Rightarrow$ $r(r - 1) = 5b(b - 1)$ and $2br = 6b(b - 1)$

After solving these two equations, we get

$r = 6, b = 3$

68. (C)

The numbers are a, 2a, 3a…., 50a

The median of these numbers is

$$\frac{25a+26a}{2}=25.5a$$

∴ Mean deviation about median

$$=\frac{|a-25.5a|+|2a-25.5a|+\ldots+|-0.5\ a| +|0.5a|+\ldots+|50a-25.5a|}{50}$$

$$\frac{|a|}{25}(0.5+1.5+2.5+\ldots+24.5)$$

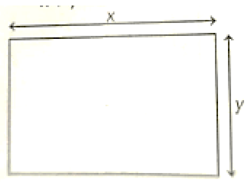
$$\frac{|a|}{25}\times\frac{25}{2}(0.5+24.5)=\frac{25|a|}{2}$$

$$\therefore\ 50=\frac{25|a|}{2}\Rightarrow |a|=4$$

69. (C)

$\therefore 2x + 2y = 4L$

$\Rightarrow\ x + y = 2L$ …(i)



∴ The area of the rectangle = xy

$x(2L-x)$ [from Eq. (i)]

But given, $x(2L-x)\le\frac{L^2}{4}$

$$\Rightarrow\ x^2-2Lx+\frac{L^2}{4}\ge 0$$

$$\Rightarrow\ (x-L)^2-L^2+\frac{L^2}{4}\ge 0$$

$$\Rightarrow\ (x-L)^2-\left(\frac{L\sqrt{3}}{2}\right)^2\ge 0$$

$$\Rightarrow\ \left(x-L+\frac{L\sqrt{3}}{2}\right)\left(x-L-\frac{L\sqrt{3}}{2}\right)\ge 0$$

$$\Rightarrow\ \left\{x-\left(\frac{2-\sqrt{3}}{2}\right)L\right\}\left\{x-\left(\frac{2+\sqrt{3}}{2}\right)L\right\}\ge 0$$

$\therefore \quad \therefore x \in \left[0, \left(\frac{2-\sqrt{3}}{2}\right)L\right] \cup \left[\left(\frac{2+\sqrt{3}}{2}\right)L, 2L\right]$

Now, x takes values in the interval [0, 2L].

$\therefore$ Required probability

$$= \frac{\int_0^{\left(\frac{2-\sqrt{3}}{2}\right)} dx + \int_{\left(\frac{2+\sqrt{3}}{2}\right)}^{2L} dx}{\int_0^{2L} dx}$$

$$= \frac{\left(\frac{2-\sqrt{3}}{2}\right)L + 2L - \left(\frac{2+\sqrt{3}}{2}\right)L}{2L-0}$$

$$= \frac{2L-\sqrt{3}L}{2L} = \frac{2-\sqrt{3}}{2}$$

70. (A)

$\equiv [(p \vee q) \wedge \sim p] \to q$	
$\equiv [(p \wedge \sim q) \vee (q \wedge \sim p)] \to q$	(Distributive law)
$\equiv [F \vee (q \wedge \sim p)] \to q$	(Complement law)
$\equiv (q \wedge \sim p) \to q$	(Identity law)
$\equiv \sim (q \wedge \sim p) \vee q$	(Conditional law)
$\equiv [(\sim q \vee \sim (\sim p)] \vee q$	(Negation of conjunction)
$\equiv (\sim q \vee p) \vee q$	(Commutative law)
$\equiv (p \vee \sim q) \vee q$	(Associative law)
$\equiv p \vee T$	(Complement law)
$\equiv T$	(Identity law)

Since, the truth value of the given statement pattern is T, therefore it is a tautology.

71. (A)

Continuous in (0, 2) and differentiable in $(0, 1) \cup (1, 2)$.

Given, $f(x) = x^3 - x^2 + x + 1$

$\therefore \quad f(t) = t^3 - t^2 + t + 1$

$\Rightarrow \quad f'(t) = 3t^2 - 2t + 1$

Discriminant of f'(t)

$= -2^{\,2} - 4 \times 3 \times 1 = -8 < 0$

And coefficient of $t^2 = 3 > 0$

Hence, $f'(t) > 0$ for all real t.

$\Rightarrow$ f(t) is strictly increasing. Thus, f(t) is maximum when t is

Maximum and $t_{max} = x \qquad (\because 0 \le t \le x)$

$\therefore$ max f(t) = f(x)

$$\Rightarrow \quad g(x)=\begin{cases} x^3-x^2+x+1 & , \quad 0\le x\le 1 \\ 3-x & , \quad 1<x\le 2 \end{cases}$$

Now, it can be easily seen that f(x) is continuous in (0, 2) and differentiable in (0, 2) except at x = 1 because at x = 1,
LHD > 0, while RHD = –1 < 0

72. (A)

Given, f is differentiable at x = 0

So $f'(0^-) = f'(0^+) = f'(0)$ …(i)

$$\therefore \lim_{h\to 0}\frac{f(h)-f(-2h)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)-[f(-2h)-f(0)]}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}+\lim_{h\to 0}\frac{2[f(-2h)-f(0)]}{h}$$

$= f'(0^+) + 2f'(0^-)$

$= f'(0^+) + 2f'(0^-) = 3 \cdot f'(0)$ [from Eq. (i)]

$= 3\times 1 = 3$

73. (B)

p, q, r are in AP

$\Rightarrow$ 2q = p + r …(i)

Given, condition : p + 1, q, r are in GP or p, q, r + 2 are in GP

Therefore, $q^2 = (p + 1)r$ …(ii)

or $q^2 = (r + 2)p$ …(iii)

from Eqs. (ii) and (iii),

(p + 1)r = p(r + 2)

$\Rightarrow$ pr + r = pr + 2p $\Rightarrow$ r = 2p

From Eq. (ii), $q^2 = (p + 1)2p$

Also, from Eq. (i),

2q = p + 2p = 3p

$$\Rightarrow \quad q=\frac{3}{2}p$$

$$\Rightarrow \quad \frac{9}{4}p^2=(p+1)2p$$

$$\Rightarrow \quad \frac{9}{8}p=p+1$$

$\Rightarrow$ p = 8, r = 16 and q = 12

$\Rightarrow$ q – p = 12 – 8 = 4

74. (D)

Given,

$$f(x)=\ln\left(\sqrt{\sqrt{x^2+1}+x}+\sqrt{\sqrt{x^2+1}-x}\right)$$

$$=\frac{1}{2}\ln\left[\sqrt{\sqrt{x^2+1}+x}+\sqrt{\sqrt{x^2+1}-x}\right]^2$$

$$=\frac{1}{2}\ln\left[\sqrt{x^2+1}+x+\sqrt{x^2+1}-x+2\sqrt{\sqrt{(x^2+1)^2-x^2}}\right]$$

$$\Rightarrow\quad f(x)=\frac{1}{2}\ln\left[2\sqrt{x^2+1}+2\right]$$

Also, $f(-x)=\frac{1}{2}\ln\left[2\sqrt{(-x)^2+1}+2\right]$

$$=\frac{1}{2}\ln\left[2\sqrt{x^2+1}+2\right]$$

Here, f(x) – f(–x) = 0

So, f(x) is an even function.

Hence, f(x) is neither one-one nor onto.

75. (C)

Since, $\sqrt{\frac{\sum x_i^2}{n}}\geq\frac{\sum x_i}{n}$

$$\therefore\quad \sqrt{\frac{400}{n}}\geq\frac{80}{n}\Rightarrow\frac{20}{\sqrt{n}}\geq\frac{80}{n}$$

$$\Rightarrow\quad \sqrt{n}\geq 4$$

[$\because$ n is positive, $\therefore$ $\sqrt{n}$ is positive]

$$\Rightarrow\quad n\geq 16$$

$\therefore$ n = 18 is possible value.

76. (D)

As, $T_n=\cot^{-1}\left[\frac{(n+1)(n+2)x}{2}+\frac{2}{x}\right]$

$$=\cot^{-1}\left[\frac{(n+1)(n+2)x^2+4}{2x}\right]$$

$$=\tan^{-1}\left[\frac{2x}{4+(n+1)(n+2)x^2}\right]$$

$$=\tan^{-1}\left[\frac{\left(\frac{n+2}{2}\right)x-\left(\frac{n+1}{2}\right)x}{1+\left(\frac{n+2}{2}\right)\left(\frac{n+1}{2}\right)x^2}\right]$$

$$= \tan^{-1}\left[\left(\frac{n+2}{2}\right)x\right] - \tan^{-1}\left[\left(\frac{n+1}{2}\right)x\right]$$

$$\therefore S_n = \sum T_n = \tan^{-1}\left[\left(\frac{n+2}{2}\right)x\right] - \tan^{-1}(x)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\tan^{-1}\left(\frac{n+2}{2}\right)x - \tan^{-1}x\right] = \frac{\pi}{2} - \tan^{-1}x$$

$\Rightarrow\ 1 = \cot^{-1}x$ $\qquad$ given $\lim_{n\to\infty} S_n = 1$

$\Rightarrow\ x = \cot 1$

77. (B)

Given,

$$\sin 2x\left(\frac{dy}{dx} - \sqrt{\tan x}\right) - y = 0$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{-1}{\sin 2x}\right)y = \sqrt{\tan x}$$

Which is a linear differential equation.

$$\therefore \quad IF = e^{-\int \operatorname{cosec} 2x dx}$$

$$= e^{\frac{-1}{2}\log_e \tan x}$$

$$= \sqrt{\cot x}$$

Therefore, general solution is

$$y\sqrt{\cot x} = \int \sqrt{\cot x}\sqrt{\tan x}\, dx$$

$$\Rightarrow y\sqrt{\cot x} = x + C$$

$$\Rightarrow y = \sqrt{\tan x}(x + C)$$

78. (A)

$$\text{Let } I = \int_0^{\pi} \frac{x}{1 - \cos\alpha \sin x} dx \qquad \ldots(i)$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1 - \cos\alpha \sin(\pi - x)} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1 - \cos\alpha \sin x} dx \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi}{1 - \cos\alpha \sin x} dx$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{1 - \cos\alpha \sin x)}$$

$$\Rightarrow \quad I = \pi\int_0^1 \frac{2dt}{(1+t^2)\left(1-\cos\alpha\dfrac{2t}{1+t^2}\right)}$$

$$\left[\begin{array}{l}\therefore \text{putting } \tan\dfrac{x}{2} = t \Rightarrow \dfrac{1}{2}\sec^2\dfrac{x}{2}dx = dt \\ \Rightarrow dx = \dfrac{2dt}{\sec^2\dfrac{x}{2}} = \dfrac{2dt}{1+\tan^2\dfrac{x}{2}} = \dfrac{2dt}{1+t^2}\end{array}\right]$$

$$= 2\pi\int_0^1 \frac{dt}{t^2+1-2t\cos\alpha}$$

$$= 2\pi\int_0^1 \frac{dt}{(t-\cos\alpha)^2+\sin^2\alpha}$$

$$= \frac{2\pi}{\sin\alpha}\tan^{-1}\left(\frac{t-\cos\alpha}{\sin\alpha}\right)\Bigg|_0^1$$

$$\left[\text{using} \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) - \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{2\sin^2\alpha/2}{2\sin\ \alpha/2\ \cos\ \alpha/2}\right) - \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\alpha\right)\right\}\right]$$

$$\Rightarrow \quad I = \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\tan\frac{\alpha}{2}\right) - \tan^{-1}\left(\tan\left(\frac{\pi}{2}-\alpha\right)\right)\right]$$

$$\Rightarrow \quad I = \frac{2\pi}{\sin\alpha} \qquad \left[\frac{\alpha}{2}+\left(\frac{\pi}{2}-\alpha\right)\right]$$

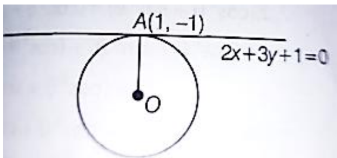
$$= 2\pi(\sin\alpha)^{-1}\times\frac{1}{2}[\pi-\alpha]$$

$$\Rightarrow \quad I = \pi(\pi-\alpha)(\sin\alpha)^{-1}$$

79. (B)

Centre of O lies on the line through the point A(1, – 1) and perpendicular to 2x + 3y + 1 = 0 i.e. on $y + 1 = \frac{3}{2}(x - 1)$

$$\Rightarrow \quad \frac{x-1}{2} = \frac{y+1}{3} = \lambda(\text{say})$$



Let centre of O be $(2\lambda+1, 3\lambda-1)$

Then, equation of C is

$$[x-(2\lambda+1)]^2+[y-(3\lambda-1)]^2=(2\lambda)^2+(3\lambda)^2$$

$$\Rightarrow x^2+y^2-2(2\lambda+1)x-2(3\lambda-1)\lambda-2\lambda+2=0 \quad \ldots(i)$$

Equation of circle C_1 is

$(x - 0)(x + 2) + (y + 1)(y - 3) = 0$

$\Rightarrow \quad x^2 + y^2 + 2x - 2y - 3 = 0 \quad \ldots(ii)$

As, Eqs. (i) and (ii) cut orthogonally.

So, $-2(2\lambda+1)(1)-2(3\lambda-1)(-1)=-2\lambda+2-3$

$$\Rightarrow \quad \lambda=\frac{3}{4}$$

Thus, required equation of circle is

$$x^2+y^2-5x-\left(\frac{5}{2}\right)y+\frac{1}{2}=0$$

80. (C)

Let $c=x\hat{i}+y\hat{j}+z\hat{k}$

$\therefore$ a, b and c are coplanar.

So, $\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}=0$

$\Rightarrow \quad 0\cdot x-2y-2z=0 \Rightarrow y+z=0$

Also, c is perpendicular to a

$\therefore \quad x + y - z = 0$

Therefore, $\dfrac{x}{-2}=\dfrac{y}{1}=\dfrac{z}{-11}$

And $x^2 + y^2 + z^2 = 1$

(c being a unit vector), we have

$$x=\frac{-2}{\sqrt{6}}, y=\frac{1}{\sqrt{6}} \text{ and } z=\frac{-1}{\sqrt{6}}$$

Thus, $c=\dfrac{1}{\sqrt{6}}(-2\hat{i}+\hat{j}+\hat{k})$

And $d=\dfrac{a\times c}{|a\times c|}=\dfrac{1}{\sqrt{6}}\cdot\dfrac{1}{|a\times c|}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -2 & 1 & -1 \end{vmatrix}$$

$$=\frac{1}{\sqrt{6}}\frac{1}{|a\times c|}(0\hat{i}+3\hat{j}+3\hat{k})$$

$$=\frac{1}{\sqrt{2}}(\hat{j}+\hat{k})\left[\because |a\times c|=\frac{1}{\sqrt{6}}\times 3\sqrt{2}\right]$$

81. (2)

Given, f(x) = f(3x – 4y) + f(4y – 2x) – (3x – 4y)(4y – 2x)

$\Rightarrow$ f ((3x – 4y) + (4y – 2x)) = f(3x – 4y) + f(4y – 2x) – (3x – 4y) + f(4y – 2x)

$\Rightarrow$ f(x + y) = f(x) + f(y) – xy …(i)

Now, $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0} \frac{f(2)+f(h)-2h-f(2)}{h}$ [using Eq. (i)]

$= \lim_{h\to 0} \frac{f(h)-2h}{h}$

$= \lim_{h\to 0} \frac{f(h)}{h} - \lim_{h\to 0} \frac{2h}{h}$

= 4 – 2 = 2

82. (4)

An equation of normal to the parabola $y^2 = 4x \text{ at } p(t_1^2, 2t_1)$ is

$y = -t_1 x + 2t_1 + t_1^3$

If meets the parabola again at $(t^2, 2t)$, then

$2t = t^2 t_1 + 2t_1 + t_1^3$

$\Rightarrow$ $2(t - t_1) = t_1(t_1^2 - t^2)$

$\Rightarrow$ $2 = -t_1(t + t_1)$

$\Rightarrow$ $t = -t_1 - \frac{2}{t_1}$

Similarly, if the normal at $Q(t_2^2, 2t_2)$ meets the parabola at $(t^2, 2t)$, then

$$t = -t_2 - \frac{2}{t_2}$$

If normal of P and Q meet at the same point on the parabola, then

$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$

$\Rightarrow$ $t_2 - t_2 = 2\left(\frac{1}{t_1} - \frac{1}{t_2}\right)$

$\Rightarrow$ $t_1 t_2 = 2$

As, $t_1^2 + t_2^2 = a + 4 - a = 4$

$(y_1 + y_2)^2 = 4(t_1 + t_2)^2$

$= 4(t_1^2 + t_2^2 + 2t_1 t_2) = 32$

$\Rightarrow$ $|y_1 + y_2| = 4\sqrt{2}$

$\therefore$ $\frac{1}{\sqrt{2}} |y_1 + y_2| = 4$

83. (0)

For idempotent matrix,

$A^2 = A$

$\Rightarrow A^{-1} A^2 = A^{-1} A$

($\therefore$ A is non-singular)

$\Rightarrow A = l$

Thus, non-singular idempotent matrix is always a unit matrix.

$\therefore \quad l^2 - 3 = 1 \Rightarrow l = \pm 2$

$\Rightarrow \quad m^2 - 8 = 1 \Rightarrow m = \pm 3$

$\Rightarrow \quad n^2 - 15 = 1 \Rightarrow n = \pm 4$

and $p = q = r = 0$

hence, the required sum = 0

84. (14)

$$\int_0^x f(t)\,dt + \int_0^x t \cdot f(x-t)dt = e^{-x} - 1$$

$$\int_0^x f(t)\,dt + \int_0^x (x-t)f(t)dt = e^{-x} - 1$$

$$\int_0^x f(t)\,dt + x\int_0^x f(t)dt - \int_0^x t \cdot f(t)dt = e^{-x} - 1$$

Differentiating w.r.t. x, we get

$$f(x) + x\,f(x) + \int_0^x f(t)dt - xf(x) = -e^{-x}$$

$$\Rightarrow \quad f(x) + \int_0^x f(t)dt = -e^{-x}$$

Again differentiating w.r.t. x, we get

$f'(x) + f(x) = e^{-x}$

$\Rightarrow e^x f'(x) + e^x f(x) = 1$

$\Rightarrow (e^x f(x))' = 1$

$\Rightarrow e^x f(x) = x + c$

[on integrating]

From Eq. (i), $f(0) = -1$

$\therefore \quad c = -1$

$\therefore \quad f(x) = e^x = x - 1$

Hence, $e^{15} f(15) = 14$

85. (4)

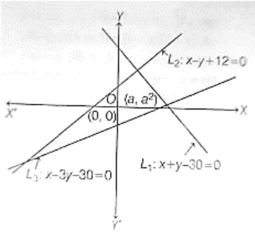
Method (1)

Given, $L_1: x + y - 30 = 0$ …(i)

$L_2: x - y + 12 = 0$ …(ii)

$L_3: x - 3y - 30 = 0$ …(iii)

From figure, O(0, 0) and (a, a^2) lying on same side of each of the lines L_1, L_2 and L_3.



Now, putting O(0, 0) in the LHS of L_1 must be negative i.e. $a + a^2 - 30 \le 0$

(here equality hold because (a, a^2) may lie on the side of triangle)

$\Rightarrow$ $a^2 + a - 30 \le 0$

$\Rightarrow$ $(a+6)(a-5) \le 0$

$\Rightarrow$ $a \in [-6, 5]$ ...(iv)

Similar concept for line L_2 and L_3 apply for $L_2 : x - y + 12 = 0$

At (a, a^2), $a - a^2 + 12 \ge 0$

$\Rightarrow$ $a^2 - a - 12 \le 0$

$\Rightarrow$ $(a + 3)(a - 4) \le 0$

$\Rightarrow$ $a \in [-3, 4]$ ...(v)

For $L_3 : x - 3y - 30 = 0$

At (a, a^2), $a - 3a^2 - 30 \le 0$

$3a^2 - a + 30 \ge 0$ $\Rightarrow$ $a \in R$ ...(vi)

(because coefficient of a^2 is positive and its discriminant is negative)

So, from Eqs. (iv), (v) and (vi), intersection value of a is $a \in [-3, 4]$

i.e. integral values of

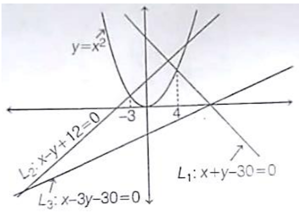
$a = \{-3, -2, -1, 0, 1, 2, 3, 4\}$

hence, its sum

$= -3 - 2 - 1 + 0 + 1 + 2 + 3 + 4 = 4$

Method (2)

Since, point (a, a^2) always lying on the parabola $y = x^2$.



For point lying inside or on the triangle, solving equations $y = x^2$ and $y = x + 12$, we get

$x^2 = x + 12$

$\Rightarrow$ $x^2 - x - 12 = 0$

$\Rightarrow$ $(x + 3)(x - 4) = 0 \Rightarrow x = -3, 4$

$\therefore$ $a \in [-3, 4]$

Hence, required sum of integers is 4.

86. (28)

The equation of the plane containing the given lines is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

[applying $R_3 \to R_3 - R_2$]

$$\Rightarrow \begin{vmatrix} x-1 & y-x-1 & z-x-2 \\ 2 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and

$C_3 \to C_3 - C_1$, we get

$\Rightarrow \quad 2y - 2x - 2 - z + x + 2 = 0$

$\Rightarrow \quad x - 2y + z = 0 \quad \ldots(i)$

This plane must be parallel to $Ax - 2y + z = d$ and so

$A = 1 \quad \ldots(ii)$

The distance between the lines (i) and (ii) is

$$\frac{|d|}{\sqrt{1+4+1}} = \sqrt{6} \Rightarrow |d| = 6$$

Hence, $10A + 3|d| = 10 \times 1 + 3 \times 6 = 28$

87. (1)

Let $I = \int \frac{1}{x + \sqrt{x^2 - x + 1}} dx \quad \ldots(i)$

let $x + \sqrt{x^2 - x + 1} = t$

$\Rightarrow \quad \sqrt{x^2 - x + 1} = t - x$

On squaring, we get

$x^2 - x + 1 = (t - x)^2 = t^2 + x^2 - 2tx$

$\Rightarrow \quad (2t - 1)x = t^2 - 1$

$$\Rightarrow \quad x = \frac{t^2 - 1}{2t - 1}$$

$$\Rightarrow \quad dx = \frac{(2t-1) - 2(t^2 - 1)}{(2t-1)^2} dt$$

$$\Rightarrow \quad dx = \frac{2t^2 - 2t + 2}{(2t-1)^2} dt$$

From Eq. (i)

$$I = \int \frac{1}{t} \cdot \frac{2t^2 - 2t + 2}{(2t-1)^2} dt = 2\int \frac{t^2 - t + 2}{t(2t-1)^2} dt$$

By partial fractions,

$$I = 2\int\left[\frac{1}{t} - \frac{3}{2(2t-1)} + \frac{3}{2(2t-1)^2}\right]dt$$

$\Rightarrow$ I = 2 loge |t| – $\frac{3}{2}$ loge |2t – 1| – $\frac{3}{2(2t-1)}$ + C,

Where t = x + $\sqrt{x^2 - x + 1}$

On comparing, m = 2, $n = \frac{-3}{2}$ and $p = \frac{-3}{2}$.

Hence, 2m + n + p

$$= 2\times 2 + \left(-\frac{3}{2}\right) + \left(\frac{-3}{2}\right) = 4 - 3 = 1$$

88. (2)

$$\lim_{x\to -2}\frac{\tan \pi x}{x+2} = \lim_{x\to -2}\frac{\tan \pi(x+2)}{x+2}$$

$= \lim_{t\to 0}\frac{\pi \tan t}{t}$ [put, t = π (x + 2)]

$= \pi$

And $\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{1}{x^2}\right)} = 1$

Hence, $\lambda = \pi + 1 \simeq 3.14 + 1 \simeq 4.14$

$\therefore \quad \frac{[\lambda]}{2} = \frac{4}{2} = 2$

89. (2)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$\Rightarrow$ $A^2 = A \cdot A = A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ $A^2 = 1$

$\therefore$ $A^4 = A^6 = A^8 = \ldots = 1$

Now, $A_x = I \Rightarrow x = 2, 4, 6, 8$ ………

$$\therefore \quad \sum(\cos^x\theta + \sin^x\theta) = (\cos^2\theta + \sin^2\theta) + (\cos^4\theta + \sin^4\theta) + (\cos^6\theta + \sin^6\theta) + (\cos^8\theta + \sin^8\theta) + \ldots.$$

$= (\cos^2\theta + \cos^4\theta + \cos^6\theta + \cos^8\theta + \ldots) +$

$(\sin^2\theta + \sin^4\theta + \sin^6\theta + \sin^8\theta + \ldots)$

$$= \frac{\cos^2\theta}{1-\cos^2\theta} + \frac{\sin^2\theta}{1-\sin^2\theta}$$

$\left[\because \text{sum of infinite}, S_\infty GP = \frac{a}{1-r}\right]$

$= \cot^2\theta + \tan^2\theta$ which has minimum value = 2

[using AM ≥ GM]

90. (4)

Equation of any tangent to ellipse

$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ is $\frac{x}{4}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0$

We have, $p_1 = \frac{\left|\frac{4e}{b}\sin\theta - 1\right|}{\sqrt{\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{b^2}}}$

$\Rightarrow \quad p_1^2\left(\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{b^2}\right) = \left(\frac{4e}{b}\sin\theta - 1\right)^2$

Similarly ...(i)

$\Rightarrow \quad p_2^2\left(\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{b^2}\right) = \left(\frac{4e}{b}\sin\theta + 1\right)^2$ ...(ii)

Adding Eqs. (i) and (ii), we get

$(p_1^2 + p_2^2)\left(\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{b^2}\right)$

$= 2\left(\frac{16e^2}{b^2}\sin^2\theta + 1\right)$

$= 2\left(\frac{16 - b^2}{b^2}\sin^2\theta + 1\right)\left[\begin{array}{l} \therefore b^2 = a^2(1-e^2) \\ \therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{b^2}{16} \end{array}\right]$

$= 2\left(\frac{16\sin^2\theta - b^2\sin^2\theta + b^2}{b^2}\right)$

$= 32\left(\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{16}\right)$

$\Rightarrow \quad \frac{p_1^2 + p_2^2}{8} = 4$