

Applications Of Derivatives

Exercise 1(A)

Q.1 (b)

$$\text{Given } x^2 + y^2 = 2c^2$$

$$\text{Differentiating w.r.t. } x, 2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{(c,c)} = -1$$

Q.2 (a)

$$\text{Given curve } x^2 = 3 - 2y$$

$$\text{diff. w.r.t. } x, 2x = -\frac{2dy}{dx}; \frac{dy}{dx} = -x$$

$$\text{Slope of the line} = -1$$

$$\frac{dy}{dx} = -x = -1; x = 1$$

$$\therefore y = 1 \text{ point } (1, 1)$$

Q.3 (b)

$$\text{Given } y = 2x^2 - x + 1$$

$$\text{Let the co-ordinate of } P \text{ is } (h, k) \text{ then } \left(\frac{dy}{dx} \right)_{(h,k)} = 4h - 1$$

$$\text{Clearly } 4h - 1 = 3.$$

$$h = 1 \Rightarrow k = 3. P \text{ is } (1, 2).$$

Q.4 (d)

$$x^2 = -4y \Rightarrow 2x = -4 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{2} \Rightarrow \left(\frac{dy}{dx} \right)_{(-4,-4)} = 2.$$

We know that equation of tangent is

$$(y - y_1) = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} (x - x_1)$$

$$\Rightarrow y + 4 = 2(x + 4)$$

$$\Rightarrow 2x - y + 4 = 0.$$

Q.5 (b)

$$y = \sin \frac{\pi x}{2} \Rightarrow \frac{dy}{dx} = \frac{\pi}{2} \cos \frac{\pi}{2} x$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{(1,1)} = 0$$

$$\therefore \text{Equation of normal is } y - 1 = \frac{1}{0}(x - 1)$$

$$\Rightarrow x = 1 .$$

Q.6

(d)

$$\text{Curve is } y = be^{-x/a}$$

Since the curve crosses y-axis (i.e., $x = 0$) $\therefore y = b$

$$\text{Now } \frac{dy}{dx} = \frac{-b}{a} e^{-x/a} .$$

$$\text{At point } (0, b), \left(\frac{dy}{dx} \right)_{(0,b)} = \frac{-b}{a}$$

$$\therefore \text{Equation of tangent is } y - b = \frac{-b}{a}(x - 0)$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1 .$$

Q.7

(d)

$$\text{Slope of the normal} = \frac{-1}{dy/dx}$$

$$\Rightarrow \tan \frac{3\pi}{4} = \frac{-1}{\left(\frac{dy}{dx} \right)_{(3,4)}}$$

$$\therefore \left(\frac{dy}{dx} \right)_{(3,4)} = 1 ; f'(3) = 1 .$$

Q.8

(d)

$$y^3 + 3x^2 = 12y \Rightarrow 3y^2 \cdot \frac{dy}{dx} + 6x = 12 \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(3y^2 - 12) + 6x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x}{12 - 3y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{12 - 3y^2}{6x}$$

Tangent is parallel to y-axis, $\frac{dx}{dy} = 0$

$$\Rightarrow 12 - 3y^2 = 0 \text{ or } y = \pm 2 .$$

Then $x = \pm \frac{4}{\sqrt{3}}$, for $y = 2$

$y = -2$ does not satisfy the equation of the curve,

$$\therefore \text{The point is } \left(\pm \frac{4}{\sqrt{3}}, 2 \right)$$

Q.9

(c)

Let the point be (x_1, y_1)

$$\therefore y_1 = be^{-x_1/a} \quad \dots\dots(i)$$

$$\text{Also, curve } y = be^{-x/a} \Rightarrow \frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-b}{a} e^{-x_1/a} = \frac{-y_1}{a} \quad (\text{by (i)})$$

Now, the equation of tangent of given curve at point (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{a}(x - x_1) \Rightarrow \frac{x}{a} + \frac{y}{y_1} = \frac{x_1}{a} + 1$$

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$, we get, $y_1 = b$ and $1 + \frac{x_1}{a} = 1 \Rightarrow x_1 = 0$

Hence, the point is $(0, b)$.

Q.10 (d)

$$y = x^3 - 3x^2 - 9x + 5 \Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 9.$$

We know that this equation gives the slope of the tangent to the curve.

The tangent is parallel to x -axis $\frac{dy}{dx} = 0$

$$\text{Therefore, } 3x^2 - 6x - 9 = 0$$

$$\Rightarrow x = -1, 3.$$

Q.11 (b)

Given curve $y^2 = x$ and $x^2 = y$

Differentiating w.r.t. x , $2y \frac{dy}{dx} = 1$ and $2x = \frac{dy}{dx}$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2} \text{ and } \left(\frac{dy}{dx}\right)_{(1,1)} = 2$$

Angle between the curve

$$\Rightarrow \tan \phi = \frac{2 - \frac{1}{2}}{1 + \frac{1}{2} \cdot 2}$$

$$\Rightarrow \tan \phi = \frac{3}{4} \Rightarrow \phi = \tan^{-1} \frac{3}{4}.$$

Q.12 (a)

Clearly the point of intersection of curves is $(0, 1)$

Now, slope of tangent of first curve, $m_1 = \frac{dy}{dx} = a^x \log a \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$

Slope of tangent of second curve, $m_2 = \frac{dy}{dx} = b^x \log b \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$

$$\therefore \tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\log a - \log b}{1 + \log a \log b}.$$

Q.13 (b)

The equation of two curves are $xy = 6$ and $x^2y = 12$ from (i) we obtain $y = \frac{6}{x}$

putting this value of y in equation (ii) to obtain $x^2 \left(\frac{6}{x}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2$

Putting $x = 2$ in (i) or (ii) we get, $y = 3$.

Thus, the two curves intersect at $P(2, 3)$

Differentiating (i) w.r.t. x , we get $x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = -\frac{3}{2} = m_1$

Differentiating (ii) w.r.t. x , we get $x^2 \frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = \frac{-2y}{x}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = -3 = m_2$$

$$\Rightarrow \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\left(\frac{-3}{2} + 3\right)}{\left(1 + \left(\frac{-3}{2}\right)(-3)\right)} = \frac{3}{11}$$

$$\Rightarrow \theta = \tan^{-1} \frac{3}{11}.$$

Q.14 (c)

Equation of the curve $x^2 y^2 = a^4$.

Differentiating the given equation,

$$x^2 2y \frac{dy}{dx} + y^2 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(-a,a)} = -\left(\frac{a}{-a}\right) = 1$$

Therefore, sub-tangent = $\frac{y}{\left(\frac{dy}{dx}\right)} = a$.

Q.15 (a)

$$y^n = a^{n-1} x \Rightarrow ny^{n-1} \frac{dy}{dx} = a^{n-1}$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{a^{n-1}}{ny^{n-1}}$$

$$\therefore \text{Length of the subnormal} = y \frac{dy}{dx} = \frac{ya^{n-1}}{ny^{n-1}} = \frac{a^{n-1}y^{2-n}}{n}$$

We also know that if the subnormal is constant,

then $\frac{a^{n-1}}{n} \cdot y^{2-n}$ should not contain y .

Therefore, $2-n=0$ or $n=2$.

Q.16 (a)

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\Rightarrow \therefore \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Hence tangent at (x, y) is $Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$

$$\text{or } X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}(\sqrt{x} + \sqrt{y}) = \sqrt{axy}$$

$$\text{or } \frac{X}{\sqrt{a}\sqrt{x}} + \frac{Y}{\sqrt{a}\sqrt{y}} = 1.$$

Clearly its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

$$\text{Sum of the intercepts} = \sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a} \cdot \sqrt{a} = a.$$

Q.17 (c)

Differentiating the given equation w.r.t. x , $2y \frac{dy}{dx} = 4$

$$\text{at point } (2, 4) \quad \frac{dy}{dx} = \frac{1}{2}$$

$$P = \frac{y_1 - x_1 \left(\frac{dy}{dx} \right)}{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}}$$

$$= \frac{4 - 2 \left(\frac{1}{2} \right)}{\sqrt{1 + \frac{1}{4}}} = \frac{6}{\sqrt{5}}.$$

Q.18 (b)

$$f(x) = 3x + \frac{2}{x} \Rightarrow f'(x) = 3 - \frac{2}{x^2}$$

Clearly $f'(x) > 0$ on the interval $(1, 3)$

$\therefore f(x)$ is strictly increasing.

Q.19 (c)

$$f(x) = (x-1)^2 - 1$$

Hence decreasing in $x < 1$

Alternative method:

$$f'(x) = 2x - 2 = 2(x-1)$$

To be decreasing, $2(x-1) < 0$

$$\Rightarrow (x-1) < 0 \Rightarrow x < 1.$$

Q.20 (a)

$$f'(x) = 6x^2 + 36x - 96 > 0, \text{ for increasing}$$

$$\Rightarrow f'(x) = 6(x+8)(x-2) \geq 0$$

$$\Rightarrow x \geq 2, x \leq -8.$$

Q.21 (a)

$$\text{Let } y = x^x \Rightarrow \frac{dy}{dx} = x^x (1 + \log x);$$

$$\text{For } \frac{dy}{dx} > 0, \quad x^x (1 + \log x) > 0$$

$$\Rightarrow 1 + \log x > 0$$

$$\Rightarrow \log_e x > \log_e \frac{1}{e}$$

For this to be positive, x should be greater than $\frac{1}{e}$.

Q.22 (b)

$f(x)$ will be monotonically decreasing, if $f'(x) < 0$.

$$\Rightarrow f'(x) = -\sin x - 2p < 0$$

$$\Rightarrow \frac{1}{2} \sin x + p > 0 \Rightarrow p > \frac{1}{2} \quad [\because -1 \leq \sin x \leq 1]$$

Q.23 (c)

$$f'(x) = 5x^4 - 60x^2 + 240$$

$$= 5(x^4 - 12x^2 + 48) = 5[(x^2 - 6)^2 + 12]$$

$$\Rightarrow f'(x) > 0 \forall x \in R$$

i.e., $f(x)$ is monotonically increasing everywhere.

Q.24 (d)

If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \leq 0$ for all $x \in R$

$$\Rightarrow 3(a+2)x^2 - 6ax + 9a \leq 0 \text{ for all } x \in R$$

$$\Rightarrow (a+2)x^2 - 2ax + 3a \leq 0 \text{ for all } x \in R$$

$$\Rightarrow a+2 < 0 \text{ and discriminant } \leq 0$$

$$\Rightarrow a < -2 \text{ and } -8a^2 - 24a \leq 0$$

$$\Rightarrow a < -2 \text{ and } a(a+3) \geq 0 \Rightarrow a < -2$$

$$\text{and } a \leq -3 \text{ or } a \geq 0 \Rightarrow a \leq -3$$

$$\Rightarrow -\infty < a \leq -3$$

Q.25 (d)

The function is monotonic increasing if, $f'(x) > 0$

$$\Rightarrow \frac{(2 \sin x + 3 \cos x)(\lambda \cos x - 6 \sin x)}{(2 \sin x + 3 \cos x)^2} - \frac{(\lambda \sin x + 6 \cos x)(2 \cos x - 3 \sin x)}{(2 \sin x + 3 \cos x)^2} > 0$$

$$\Rightarrow 3\lambda(\sin^2 x + \cos^2 x) - 12(\sin^2 x + \cos^2 x) > 0$$

$$\Rightarrow 3\lambda - 12 > 0 \Rightarrow \lambda > 4.$$

Q.26 (b)

$$\text{Let } f(x) = \frac{\ln(\pi+x)}{\ln(e+x)}$$

$$\therefore f'(x) = \frac{\ln(e+x) \times \frac{1}{\pi+x} - \ln(\pi+x) \frac{1}{e+x}}{\{\ln(e+x)\}^2}$$

$$= \frac{(e+x)\ln(e+x) - (\pi+x)\ln(\pi+x)}{\{\ln(e+x)\}^2 \times (e+x)(\pi+x)}$$

$$\Rightarrow f'(x) < 0 \text{ for all } x \geq 0 \quad \{ \because \pi > e \}.$$

Hence, $f(x)$ is decreasing in $[0, \infty)$.

Q.27 (c)

Obviously, here $\cos 3x$ is not decreasing in $\left(0, \frac{\pi}{2}\right)$ because $\frac{d}{dx} \cos 3x = -3 \sin 3x$.

But at $x = 75^\circ$, $-3 \sin 3x > 0$.

Hence the result.

Q.28 (b)

$$\text{We have } f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right) \Rightarrow f'(x) = 1 - e^x$$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

$$\Rightarrow 1 - e^x > 0 \Rightarrow e^x < 1$$

$$\Rightarrow x < 0 \Rightarrow x \in (-\infty, 0)$$

Q.29 (a)

$$f'(x) = e^{x(1-x)} + x \cdot e^{x(1-x)} \cdot (1-2x)$$

$$= e^{x(1-x)} \{1 + x(1-2x)\}$$

$$= e^{x(1-x)} \cdot (-2x^2 + x + 1)$$

Now by the sign-scheme for $-2x^2 + x + 1$

$$f'(x) \geq 0, \text{ if } x \in \left[-\frac{1}{2}, 1\right], \text{ because } e^{x(1-x)} \text{ is always positive.}$$

So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

Q.30 (b)

$$f(x) = x \sin x + \cos x + \cos^2 x$$

$$\therefore f'(x) = \sin x + x \cos x - \sin x - 2 \cos x \sin x$$

$$= \cos x(x - 2 \sin x)$$

Hence $x \rightarrow 0$ to π , then $f'(x) \leq 0$, i.e.,

$f(x)$ is decreasing function.

Q.31 (b)

$$\text{Let } f(x) = x^2 e^{-x}$$

$$\Rightarrow \frac{dy}{dx} = 2x e^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2)$$

Hence $f'(x) \geq 0$ for every $x \in [0, 2]$,

therefore it is non-decreasing in $[0, 2]$.

Q.32 (b)

$$f(x) = \sin^4 x + \cos^4 x$$

$$= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x$$

$$= 1 - \frac{4 \sin^2 x \cos^2 x}{2} = 1 - \frac{\sin^2 2x}{2} = 1 - \frac{1}{4}(2 \sin^2 2x)$$

$$= 1 - \left(\frac{1 - \cos 4x}{4}\right) = \frac{3}{4} + \frac{1}{4} \cos 4x$$

Hence function $f(x)$ is increasing when $f'(x) > 0$

$$f'(x) = -\sin 4x > 0 \Rightarrow \sin 4x < 0$$

$$\text{Hence } \pi < 4x < \frac{3\pi}{2} \text{ or } \frac{\pi}{4} < x < \frac{3\pi}{8}.$$

Q.33 (c)

$$\text{From mean value theorem } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$a = 0, f(a) = 0 \Rightarrow b = \frac{1}{2}, f(b) = \frac{3}{8}$$

$$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1),$$

$$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$$

$$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c, f'(c) = 3c^2 - 6c + 2$$

According to mean value theorem

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right) - 0}{\left(\frac{1}{2}\right) - 0} = \frac{3}{4} \Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36 - 15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}.$$

Q.34 (a)

$$f'(x_1) = \frac{-1}{x_1^2},$$

$$\therefore \frac{-1}{x_1^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab} \Rightarrow x_1 = \sqrt{ab}.$$

Q.35 (a)

Given that equation of curve $y = x^3 = f(x)$

So $f(2) = 8$ and $f(-2) = -8$

$$\text{Now } f'(x) = 3x^2 \Rightarrow f'(x) = \frac{f(2) - f(-2)}{2 - (-2)}$$

$$\Rightarrow \frac{8 - (-8)}{4} = 3x^2; \therefore x = \pm \frac{2}{\sqrt{3}}.$$

Q.36 (c)

To determine 'c' in Rolle's theorem, $f'(c) = 0$

$$\text{Here } f'(x) = (x^2 + 3x)e^{-(1/2)x} \cdot \left(-\frac{1}{2}\right) + (2x + 3)e^{-(1/2)x}$$

$$= e^{-(1/2)x} \left\{ -\frac{1}{2}(x^2 + 3x) + 2x + 3 \right\}$$

$$= -\frac{1}{2}e^{-(x/2)} \{x^2 - x - 6\}$$

$$\therefore f'(c) = 0 \Rightarrow c^2 - c - 6 = 0 \Rightarrow c = 3, -2.$$

But $c = 3 \notin [-3, 0]$, Hence $c = -2$.

Q.37 (a)

$$f(x) = x^3 - 6x^2 + ax + b \Rightarrow f'(x) = 3x^2 - 12x + a$$

$$\Rightarrow f'(c) = 0 \Rightarrow f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$$

$$\Rightarrow 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow 12 + 1 + 4\sqrt{3} - 24 - 4\sqrt{3} + a = 0$$

Q.38 (a)

$$y = x^5 - 5x^4 + 5x^3 - 10$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x^2 - 4x + 3)$$

$$= 5x^2(x - 3)(x - 1)$$

$$\frac{dy}{dx} = 0, \text{ gives } x = 0, 1, 3 \quad \dots(i)$$

$$\text{Now, } \frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$$

$$\text{and } \frac{d^3y}{dx^3} = 10(6x^2 - 12x + 3)$$

$$\text{For } x = 0: \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} \neq 0, \therefore \text{Neither minimum nor maximum}$$

$$\text{For } x = 1, \frac{d^2y}{dx^2} = -10 = \text{negative}, \therefore \text{Maximum value } y_{\max.} = -9$$

$$\text{For } x = 3, \frac{d^2y}{dx^2} = 90 = \text{positive}, \therefore \text{Minimum value } y_{\min.} = -37.$$

Q.39 (c)

$$y = \sin x(1 + \cos x) = \sin x + \frac{1}{2} \sin 2x$$

$$\therefore \frac{dy}{dx} = \cos x + \cos 2x \text{ and } \frac{d^2y}{dx^2} = -\sin x - 2 \sin 2x$$

$$\text{On putting } \frac{dy}{dx} = 0, \cos x + \cos 2x = 0$$

$$\Rightarrow \cos x = -\cos 2x = \cos(\pi - 2x)$$

$$\Rightarrow x = \pi - 2x$$

$$\therefore x = \frac{\pi}{3}, \therefore \left(\frac{d^2y}{dx^2}\right)_{x=\pi/3} = -\sin\left(\frac{1}{3}\pi\right) - 2\sin\left(\frac{2}{3}\pi\right)$$

$$= \frac{-\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2} \text{ which is negative.}$$

$$\therefore \text{at } x = \frac{\pi}{3} \text{ the function is maximum.}$$

Q.40 (d) $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$

$\Rightarrow a = -2b - 1$

and $\left(\frac{dy}{dx}\right)_{x=2} = \frac{a}{2} + 4b + 1 = 0 \Rightarrow \frac{-2b-1}{2} + 4b + 1 = 0$

$\Rightarrow -b + 4b + \frac{1}{2} = 0$

$\Rightarrow 3b = \frac{-1}{2} \Rightarrow b = \frac{-1}{6}$ and $a = \frac{1}{3} - 1 = \frac{-2}{3}$.

Q.41 (b)

$f(x) = \left(\frac{1}{x}\right)^x \Rightarrow f'(x) = \left(\frac{1}{x}\right)^x \left(\log \frac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \log \frac{1}{x} = 1 = \log e$

$\Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$.

Therefore, maximum value of function is $e^{1/e}$.

Q.42 (b)

$y = f(x) = -x^3 + 3x^2 + 9x - 27$

The slope of this curve $f'(x) = -3x^2 + 6x + 9$

Let $g(x) = f'(x) = -3x^2 + 6x + 9$

Differentiate with respect to x , $g'(x) = -6x + 6$

Put $g'(x) = 0 \Rightarrow x = 1$

Now, $g''(x) = -6 < 0$ and hence at $x = 1, g(x)$

(Slope) will have maximum value.

$\therefore [g(1)]_{\max.} = -3 \times 1 + 6 + 9 = 12$.

Q.43 (c)

$f(x) = \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt, \therefore f'(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$

For local minima, slope *i.e.*, $f'(x)$ should change sign from $-ve$ to $+ve$

$f'(x) = 0 \Rightarrow x = 0, 1, 2, 3$

If $x = 0 - h$, where h is a very small number, then $f'(x) = (-)(-)(-1)(-1)(-1) = -ve$

If $x = 0 + h$, $f'(x) = (+)(+)(-)(-1)(-1) = -ve$

Hence at $x = 0$ neither maxima nor minima.

If $x = 1 - h$, $f'(x) = (+)(+)(-)(-1)(-1) = -ve$

If $x = 1 + h$, $f'(x) = (+)(+)(+)(-1)(-1) = +ve$

Hence, at $x = 1$ there is a local minima.

If $x = 2 - h$, $f'(x) = (+)(+1)(+)(-)(-) = +ve$

If $x = 2 + h$, $f'(x) = (+)(+)(+)(+)(-1) = -ve$

Hence at $x = 2$ there is a local maxima.

If $x = 3 - h$, $f'(x) = (+)(+)(+)(+)(-) = -ve$

If $x = 3 + h$, $f'(x) = (+)(+)(+)(+)(+) = +ve$

Hence at $x = 3$ there is a local minima.

Q.44 (c)

$$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$$

$$f'(x) = 6x^2 - 18ax + 12a^2$$

$$f''(x) = 12x - 18a$$

$$\text{For maximum and minimum, } 6x^2 - 18ax + 12a^2 = 0$$

$$\Rightarrow x^2 - 3ax + 2a^2 = 0$$

$$x = a \text{ or } x = 2a \text{ at } x = a \text{ maximum and at } x = 2a \text{ minimum}$$

$$\therefore p^2 = q$$

$$a^2 = 2a \Rightarrow a = 2 \text{ or } a = 0 \text{ but } a > 0, \text{ therefore } a = 2.$$

Q.45 (c)

$$\phi(x) = \int_1^x e^{-t^2/2}(1-t^2)dt \Rightarrow \phi'(x) = e^{-x^2/2}(1-x^2)$$

$$\text{Now } \phi'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$$

Hence, $x = \pm 1$ are points of extrema of $\phi(x)$.

Q.46 (c)

$$\text{Let } y = x^3 - 18x^2 + 96x \Rightarrow \frac{dy}{dx} = 3x^2 - 36x + 96 = 0$$

$$\therefore x^2 - 12x + 32 = 0 \Rightarrow (x-4)(x-8) = 0, x = 4, 8$$

$$\text{Now, } \frac{d^2y}{dx^2} = 6x - 36 \text{ at } x = 4, \frac{d^2y}{dx^2} = 24 - 36 = -12 < 0$$

$$\therefore \text{ at } x = 4 \text{ function will be maximum and } [f(x)]_{\max.} = 64 - 288 + 384 = 160$$

$$\text{at } x = 8 \frac{d^2y}{dx^2} = 48 - 36 = 12 > 0$$

$$\therefore \text{ at } x = 8 \text{ function will be minimum and } [f(x)]_{\min.} = 128.$$

Q.47 (d)

$$y = 2 \cos 2x - \cos 4x$$

$$= 2 \cos 2x(1 - \cos 2x) + 1$$

$$= 4 \cos 2x \sin^2 x + 1$$

Obviously, $\sin^2 x \geq 0$

Therefore, to be least value of y , $\cos 2x$ should be least *i.e.*, -1 .

Hence least value of y is $-4 + 1 = -3$.

Q.48 (a)

$$f(x) = x^2 \log x \Rightarrow f'(x) = (2 \log x + 1)x$$

$$\text{Now } f'(x) = 0 \Rightarrow x = e^{-1/2}, 0$$

$$\therefore 0 < e^{-1/2} < 1, \therefore \text{ None of these critical points lies in the interval } [1, e]$$

\therefore So we only compare the value of $f(x)$ at the end points 1 and e .

$$\text{We have } f(1) = 0, f(e) = e^2$$

$$\therefore \text{ greatest value} = e^2$$

Q.49 (a)

$$xy = 1 \Rightarrow y = \frac{1}{x} \text{ and let } z = x + y$$

$$z = x + \frac{1}{x} \Rightarrow \frac{dz}{dx} = 1 - \frac{1}{x^2}$$

$$\Rightarrow \frac{dz}{dx} = 0 \Rightarrow 1 - \frac{1}{x^2} = 0$$

$$\Rightarrow x = -1, +1 \text{ and } \frac{d^2z}{dx^2} = \frac{2}{x^3}$$

$$\left(\frac{d^2z}{dx^2} \right)_{x=1} = \frac{2}{1} = 2 = +ve,$$

$\therefore x = 1$ is point of minima.

$$x = 1, y = 1,$$

$$\therefore \text{minimum value} = x + y = 2.$$

Q.50 (c)

$$\text{Let } x + y = 4 \text{ or } y = 4 - x$$

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} \text{ or } f(x) = \frac{4}{xy} = \frac{4}{x(4-x)}$$

$$f(x) = \frac{4}{4x-x^2}, f'(x) = \frac{-4}{(4x-x^2)^2} \cdot (4-2x)$$

$$\text{Put } f'(x) = 0 \Rightarrow 4 - 2x = 0 \Rightarrow x = 2 \text{ and } y = 2$$

$$\therefore \min. \left(\frac{1}{x} + \frac{1}{y} \right) = \frac{1}{2} + \frac{1}{2} = 1.$$

Q.51 (b)

$$\text{Let number} = x, \text{ then cube} = x^3$$

$$\text{Now } f(x) = x - x^3 \text{ (Maximum)} \Rightarrow f'(x) = 1 - 3x^2$$

$$\text{Put } f'(x) = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

$$\text{Because } f''(x) = -6x = -ve. \text{ when } x = +\frac{1}{\sqrt{3}}.$$

Q.52 (c)

$$2x + 2y = 100 \Rightarrow x + y = 50 \quad \dots(i)$$

$$\text{Let area of rectangle is } A, \text{ then } A = xy \Rightarrow y = \frac{A}{x}$$

$$\text{From (i), } x + \frac{A}{x} = 50 \Rightarrow A = 50x - x^2$$

$$\Rightarrow \frac{dA}{dx} = 50 - 2x$$

$$\text{for maximum area } \frac{dA}{dx} = 0$$

$$\therefore 50 - 2x = 0 \Rightarrow x = 25 \text{ and } y = 25$$

\therefore adjacent sides are 25 cm and 25 cm.

Q.53 (b)

If r be the radius and h the height, the from the figure,

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h^2 = 4(R^2 - r^2)$$

$$\text{Now, } V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

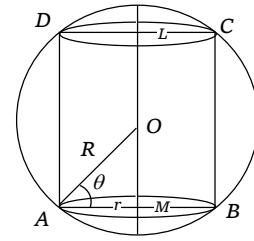
$$\therefore \frac{dV}{dr} = 4\pi r \sqrt{R^2 - r^2} + 2\pi r^2 \cdot \frac{1}{2} \frac{(-2r)}{\sqrt{R^2 - r^2}}$$

$$\text{For max. or min., } \frac{dV}{dr} = 0$$

$$\Rightarrow 4\pi r \sqrt{R^2 - r^2} = \frac{2\pi r^3}{\sqrt{R^2 - r^2}} \Rightarrow 2(R^2 - r^2) = r^2$$

$$\Rightarrow 2R^2 = 3r^2 \Rightarrow r = \sqrt{\frac{2}{3}}R \Rightarrow \frac{d^2V}{dr^2} = -ve.$$

$$\text{Hence } V \text{ is max. when } r = \sqrt{\frac{2}{3}}R.$$



Q.54 (a)

Let $OM = x$

Then height of cone *i.e.*, $h = x + a$ (where a is radius of sphere)

Radius of base of cone = $\sqrt{a^2 - x^2}$

$$\text{Therefore, volume } V = \frac{1}{3}\pi(a^2 - x^2)(x + a) \Rightarrow \frac{dV}{dx} = \frac{\pi}{3}(a + x)(a - 3x)$$

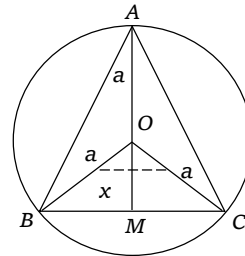
$$\text{Now, } \frac{dV}{dx} = 0 \Rightarrow x = -a, \frac{a}{3}$$

$$\text{But } x \neq -a, \text{ So, } x = \frac{a}{3}$$

$$\text{The volume is maximum at } x = \frac{a}{3}$$

$$\text{Height of a cone } h = a + \frac{a}{3} = \frac{4}{3}a$$

$$\text{Therefore ratio of height and diameter} = \frac{\frac{4}{3}a}{2a} = \frac{2}{3}.$$



Applications Of Derivatives

EXERCISE 1(B)

1. $y = 2x^3 + 13x^2 + 5x + 9$

$$\therefore \frac{dy}{dx} = 6x^2 + 26x + 5$$

Let the point be (h, k)

$$\therefore k = 2h^3 + 13h^2 + 5h + 9$$

$$\therefore \frac{y - (2h^3 + 13h^2 + 5h + 9)}{x - h} = 6h^2 + 26h + 5$$

substituting (0,0)

$$\therefore 2h^3 + 13h^2 + 5h + 9 = 6h^3 + 26h^2 + 5h$$

$$\therefore 4h^3 + 13h^2 - 9 = 0$$

$$\Rightarrow h = -1, k = 15.$$

2. $x = a(t + \sin t \cos t)$

$$y = a(1 + \sin t)^2$$

$$\therefore \frac{dx}{dt} = a(1 + \cos 2t)$$

$$\frac{dy}{dt} = 2a(1 + \sin t)\cos t$$

$$\therefore \frac{dy}{dx} = \frac{2\cos t + \sin 2t}{1 + \cos 2t}$$

$$= \frac{2\cos t(1 + \sin t)}{2\cos^2 t} = \frac{\left(\cos \frac{t}{2} + \sin \frac{t}{2}\right)^2}{\cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}}$$

$$= \frac{1 + \tan \frac{t}{2}}{1 - \tan \frac{t}{2}} = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

3. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$$\therefore \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}$$

At (a, b)

$$\frac{dy}{dx} = -\frac{b}{a}$$

$$\therefore b(x-a) + a(y-b) = 0$$

$$\therefore bx + ay = 2ab$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 2.$$

4. $y = be^{-x/a}$

$$x = 0 \Rightarrow y = b$$

$$y' = -\frac{b}{a}e^{-x/a}$$

$$\therefore \left. \frac{dy}{dx} \right|_{(0,b)} = -\frac{b}{a}$$

$$\therefore a(y-b) = -bx$$

$$\therefore bx + y = ab$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1.$$

5. $y = 3x^2 + bx + 2$

$$x = 0 \Rightarrow y = 2$$

$$\frac{dy}{dx} = 6x + b$$

$$\therefore \left. \frac{dy}{dx} \right|_{x=0} = b = 4 \quad \dots \text{Given}$$

$$\therefore b = 4$$

6. $y = \frac{8-x^2}{2}$

$$\therefore \frac{dy}{dx} = -x = -2 \quad [\text{Given}]$$

$$\therefore x = 2$$

$$\Rightarrow y = 2$$

$$\therefore y - 2 + 2(x - 2) = 0$$

$$\therefore 2x + y - 6 = 0.$$

7. Points are $(p, ap^2 + bp + c)$ & $(q, aq^2 + bq + c)$

$$\therefore \frac{y - (ap^2 + bp + c)}{x - p} = \frac{a(q^2 - p^2) + b(q - p)}{q - p}$$

$$\therefore \frac{y - (ap^2 + bp + c)}{x - p} = aq + ap + b$$

$$\therefore y = (aq + ap + b)x - apq + c$$

$$\therefore m = aq + ap + b$$

$$\& \quad m = 2ax + b = \frac{dy}{dx}$$

$$\therefore x = \frac{p+q}{2}$$

$$8. \quad \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\therefore \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0 \quad x - y = \sqrt{xy}$$

$$\therefore \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$\therefore \frac{Y-y}{X-x} = -\sqrt{\frac{y}{x}}$$

$$X=0 \Rightarrow Y = y + \sqrt{xy}$$

$$Y=0 \Rightarrow X = x + \sqrt{xy}$$

$$x + y + 2\sqrt{xy} = OA + OB = (\sqrt{x} + \sqrt{y})^2 = a$$

$$9. \quad x^{2/3} + y^{2/3} = a^{2/3}$$

$$\therefore \frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore \frac{Y-y}{X-x} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore Y = y + x^{2/3}y^{1/3} \quad \text{When } X = 0$$

$$X = x + x^{1/3}y^{2/3} \quad \text{When } Y = 0$$

$$Y^2 + X^2 = y^2 + x^{4/3}y^{2/3} + 2x^{2/3}y^{4/3} + x^2 + 2x^{4/3}y^{2/3} + x^{2/3}y^{4/3}$$

$$= x^2 + y^2 + 3x^{4/3}y^{2/3} + 3x^{2/3}y^{4/3}$$

$$= x^2 + x^{4/3}y^{2/3} + y^2 + x^{2/3}y^{4/3} + 2(x^{4/3}y^{2/3} + x^{2/3}y^{4/3})$$

$$= (x^{4/3} + y^{4/3})a^{2/3} + (2x^{2/3}y^{2/3})a^{2/3}$$

$$= a^{2/3}(x^{2/3} + y^{2/3})$$

$$= a^2$$

$$10. \quad xy^n = a^{n+1}$$

$$\therefore y^n = nxy^{n-1} \frac{dy}{dx} = 0, \quad n \neq -1$$

$$\therefore \frac{dy}{dx} = -\frac{y}{nx}$$

$$\therefore \frac{Y-y}{X-x} = -\frac{y}{nx}$$

$$X=0 \Rightarrow Y = y + \frac{y}{n}$$

$$Y=0 \Rightarrow X = x + nx$$

$$\therefore \Delta = \frac{1}{2}XY = \frac{1}{2}xy(1+n) \left(1 + \frac{1}{n}\right), \text{ Here } n \text{ is a constant.}$$

Δ is constant only when xy is constant but xy^n is constant

$$\therefore n = 1.$$

$$11. \quad f(x) = 2x^3 - 9x^2 + 12x - 3$$

$$\therefore f'(x) = 6x^2 - 18x + 12 > 0$$

$$\therefore x^2 - 3x + 2 > 0$$

$$\therefore x \in (-\infty, 1) \cup (2, \infty)$$

$$12. \quad f(x) = x^3 - ax^2 + 48x + 19$$

$$f'(x) = 3x^2 - 2ax + 48 \geq 0 \quad \forall x$$

$$\therefore (2a)^2 - 4(3)(48) \leq 0$$

$$\therefore a^2 - 144 \leq 0$$

$$\therefore a \in [-12, 12]$$

$$13. \quad f(x) = 2x^3 - 9x^2 - 60x + 81$$

$$\therefore f'(x) = 6x^2 - 18x - 60 < 0$$

$$\therefore x^2 - 3x - 10 < 0$$

$$\therefore x \in (-2, 5)$$

$$14. \quad f(x) = \frac{x^2}{x+2}$$

$$\therefore f'(x) = \frac{2x(x+2) - x^2}{x+2}$$

$$= \frac{x^2 + 4x}{x+2} < 0$$

$$\therefore \frac{x(x+4)}{(x+2)} < 0 \quad \therefore x \in (-\infty, -4) \cup (-2, 0)$$

15. $f(x) = x^2$

$$\therefore f'(x) = x^x (1 + \ln x) = 0$$

$$\therefore 1 + \ln x = 0$$

$$\therefore x = \frac{1}{e}$$

For $x < \frac{1}{e}$, $f'(x) < 0$

$$\therefore \text{Function decreases in } \left(0, \frac{1}{e}\right).$$

16. $f(x) = \frac{\log x}{x}$

$$\therefore f'(x) = \frac{1 - \log x}{x^2} < 0$$

$$\therefore \log x > 1 \quad \Rightarrow \quad x > e$$

$$\therefore x \in (e, \infty)$$

17. $f(x) = 2|x-2| + |x-3|$

For $x < 2$

$$f(x) = 2(2-x) + 3-x$$

$$= 7 - 3x \text{ is a decreasing function.}$$

For $2 < x < 3$

$$f(x) = 2(x-2) + 3-x$$

$$= x - 1 \text{ is increasing function.}$$

For $x > 3$

$$f(x) = 3x - 7 \text{ is an increasing function.}$$

$$\therefore x \in (2, \infty)$$

18. $f(x) = \cos x - \sin x$

$$\therefore f'(x) = -\sin x - \cos x < 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$g(x) = \cos x + \sin x$$

$$g'(x) = \cos x - \sin x > 0 \quad \forall x \in \left(0, \frac{\pi}{4}\right)$$

$$< 0 \quad \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$h(x) = \frac{\sin x}{x}$$

$$h'(x) = \frac{x \cos x - \sin x}{x^2} = 0 \quad \text{at} \quad x = 0,$$

$$h''(x) = \frac{x^2(-x \sin x) - 2x(x \cos x - \sin x)}{x^2} < 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore h'(x) < 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \frac{x}{\sin x} \text{ being reciprocal of } \frac{\sin x}{x} \text{ is an increasing function.}$$

$$19. \quad f(x) = \frac{a \sin x + b \cos x}{c \sin x + d \cos x}$$

$$f'(x) = \frac{(a \cos x - b \sin x)(c \sin x + d \cos x) - (a \sin x + b \cos x)(c \cos x - d \sin x)}{(c \sin x + d \cos x)^2}$$

$$= \frac{ad - bc}{(c \sin x + d \cos x)^2} > 0 \quad \forall x \quad \text{iff} \quad ad - bc > 0.$$

$$20. \quad \sin x - bx + c = f(x)$$

$$f'(x) = \cos x - b \leq 0 \quad \forall b \geq 1$$

$$21. \quad y = 2x^3 - 3x^2 - 36x + 10 = f(x)$$

$$f'(x) = 6x^2 - 6x - 36 = 0$$

$$\therefore x^2 - x - 6 = 0$$

$$\therefore x = 3 \text{ or } x = -2$$

$$f(3) = 2(27) - 3(9) - 36(3) + 10 \\ = -71$$

$$f(-2) = 2(-8) - 3(4) - 36(-2) + 10 \\ = -28 + 82 = 54$$

$$22. \quad f(x) = x^2 - 3x + 3$$

$$\therefore f'(x) = 2x - 3 = 0$$

$$\therefore x = \frac{3}{2} \quad f''(x) = 2 > 0$$

$$\therefore f(x) \text{ has minima at } x = \frac{3}{2}$$

$$f\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} + 3 = \frac{3}{4}$$

23. $f(x) = 2x^3 - 3x^2 - 12x + 8$

$$f'(x) = 6x^2 - 6x - 12 = 0$$

$$\therefore x = 2 \text{ or } x = -1$$

$$f''(x) = 12x - 6 < 0 \text{ for } x = -1$$

$$\text{and } > 0 \text{ for } x = 2$$

$$\therefore x = 2 \text{ is minima}$$

24. $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x = f(x)$

$$f'(x) = 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x = 0$$

$f(x)$ can have minima only as maxima is ∞ .

$$\therefore a^2 \sec^2 x \tan x = b^2 \operatorname{cosec}^2 x \cot x$$

$$\therefore a^2 \frac{\sin x}{\cos^3 x} = b^2 \frac{\cos x}{\sin^3 x}$$

$$\therefore \tan^4 x = \frac{b^2}{a^2}$$

$$\therefore \tan^2 x = \left| \frac{b}{a} \right|, \cot^2 x = \left| \frac{a}{b} \right|$$

For $a, b > 0$

$$\sec^2 x = \frac{a+b}{a}, \operatorname{cosec}^2 x = \frac{a+b}{b}$$

$$\begin{aligned} \therefore a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x \\ = a^2 + ab + ab + b^2 = (a+b)^2 \end{aligned}$$

25. $f(x) = \sin x + \sin x \cos x$

$$f'(x) = \cos x + \cos^2 x - \sin^2 x = 0 \qquad = \cos x + \cos 2x$$

$$\therefore \cos x + 2 \cos^2 x - 1 = 0$$

$$\therefore \cos x = -1 \Rightarrow x = \pi \text{ or } \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$$

$$f''(x) = -\sin x - 2 \sin 2x < 0 \text{ for } x = \frac{\pi}{3}$$

$$> 0 \text{ for } x = -\frac{\pi}{3}$$

$$\therefore x = \frac{\pi}{3}$$

26. $f(x) = x + \sin x$

$$f'(x) = 1 + \cos x \quad f''(x) = -\sin x$$

When $f'(x) = 0$, $f''(x) = 0$

$\therefore f(x)$ has neither minimum nor maximum.

27.
$$\Delta = \frac{1}{2} \begin{vmatrix} a & 0 & 1 \\ a \cos \theta & b \sin \theta & 1 \\ a \cos \theta & -b \sin \theta & 1 \end{vmatrix}$$

$$= \frac{1}{2} (2ab \sin \theta - 2ab \cos \theta \sin \theta)$$

$$= ab(\sin \theta - \sin \theta \cos \theta)$$

$$\therefore \frac{d\Delta}{d\theta} = ab(\cos \theta - \cos 2\theta) = 0$$

$$\therefore 2 \cos^2 \theta - \cos \theta - 1 = 0$$

$$\cos \theta = 1 \quad \text{or} \quad \cos \theta = -\frac{1}{2}$$

Now for a triangle, as it will be a st. line, $\theta \neq 0$, $\therefore \cos \theta \neq 1$

$$\therefore \cos \theta = -\frac{1}{2}$$

$$\therefore \theta = \frac{2\pi}{3} \quad \Rightarrow \quad \sin \theta = \frac{\sqrt{3}}{2}$$

$$\therefore \Delta = ab \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} \right) = \frac{3\sqrt{3}ab}{4}$$

28. $f'(x) = 3x^2 - 6x + 6 > 0$ for all x

\therefore Neither minimum nor maximum.

29. $f(x) = (x+6)^4 (8-x)^3$

Let $x+6 = t$

$$\therefore 8-x = 14-t$$

$$\therefore f(t) = t^4 (14-t)^3 \quad \& \quad f'(x) = f'(t)$$

$$\therefore f'(t) = 4t^3 (14-t)^3 - 3t^4 (14-t)^2 = 0$$

$$\therefore t^3 (14-t)^2 [4(14-t) - 3t] = 0$$

$$\therefore t^3 (4-t)^2 [56-7t] = 0$$

$$\therefore t = 0 \quad \text{or} \quad t = 14 \quad \text{or} \quad t = 8$$

Now $t \neq 0$ & $t \neq 14$ as product will become zero

$$\therefore t = 8 \quad \& \quad f(t) = 8^4 \cdot 6^3$$

30. $x^2 - (a - 2)x - (a + 1) = 0$

$\alpha + \beta = a - 2, \alpha\beta = -(a + 1)$

$\therefore \alpha^2 + \beta^2 = a^2 - 4a + 4 + 2a + 2$

$\therefore f(a) = a^2 - 2a + 6$

$\therefore f'(a) = 2a - 2 = 0 \qquad f''(a) > 0$

$\therefore a = 1$

$\therefore f(a) = 5 = \min(\alpha^2 + \beta^2)$

31. Function must be continuous and differentiable to apply Rolle's theorem.

32. Function must be continuous and differentiable to apply Rolle's theorem.

33. $f(x) = \log(\sin x) \qquad \text{in } \left[\frac{\pi}{6}, \frac{5\pi}{6} \right]$

$f\left(\frac{\pi}{6}\right) = f\left(\frac{5\pi}{6}\right)$

$f'(x) = \cot x = 0 \quad \text{at } x = \frac{\pi}{2}$

$\therefore c = \frac{\pi}{2}$

34. $f(x) = \log\left(\frac{x^2 + ab}{x(a+b)}\right) \text{ in } [a, b]$

$f'(x) = \frac{2x}{x^2 + ab} - \frac{(a+b)}{x(a+b)} = 0$

$\therefore 2x^2 = x^2 + ab$

$\therefore x = \text{GM of } a \text{ \& } b.$

35. $f(x) = x^3 + bx^2 + ax$ satisfies Rolle's theorem on $[1, 3]$

$c = 2 + \frac{1}{\sqrt{3}}$

$\therefore f(1) = f(3)$

$\therefore 1 + a + b = 27 + 9b + 3a \quad \text{and} \quad 3\left(1 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$

Solving, we get $(a, b) = (11, -6)$

36. $f(x) = \log x$ in $[1, e]$

$f'(x) = \frac{1}{x} \qquad f'(c) = \frac{1}{c}$

$\therefore \frac{1}{c} = \frac{\log e - \log 1}{e - 1} = \frac{1}{e - 1}$

$\therefore c = e - 1.$

37. Here, $f(0) = f(2) = 0$ in $[0, 2]$

$$\therefore f'(c) = 0$$

$$\therefore (c-2)^2 + 2c(c-2) = 0$$

$$\therefore (c-2)[3c-2] = 0$$

$$\therefore c = \frac{2}{3} \text{ as } c \in (0, 2)$$

38. $f(x) = \ell x^2 + mx + n$ in

$$\therefore f'(c) = 2\ell x + m = \frac{\ell(b^2 - a^2) + m(b-a)}{b-a}$$

$$\therefore 2\ell x + m = \ell(b+a) + m$$

$$\therefore x = \frac{a+b}{2}$$

39. Function should be differentiable in domain.

$$40. \sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x} + \sqrt{x+1}}$$

Now, $x > N^2$,

$$\therefore x+1 > N^2$$

$$\therefore \sqrt{x} > N, \sqrt{x+1} > N$$

$$\therefore \sqrt{x} + \sqrt{x+1} > 2N$$

$$\therefore \frac{1}{\sqrt{x} + \sqrt{x+1}} < \frac{1}{2N}$$

APPLICATIONS OF DERIVATIVES EXERCISE 1(C)

1. Let P be $\left(ct, \frac{c}{t}\right)$, then

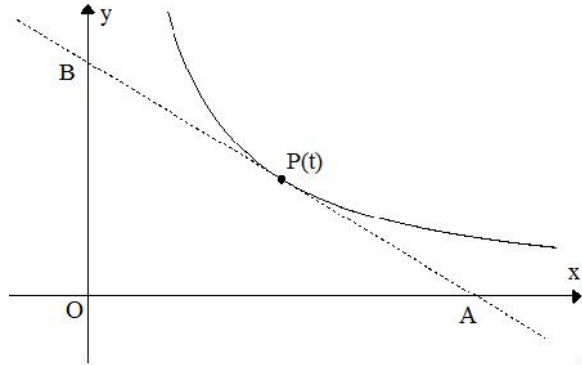
$$xy = c^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \text{ or } \frac{dy}{dx} = -\frac{1}{t^2}$$

tangent at P will be

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \text{ or } x + t^2y = 2ct$$

$$\text{Now } OA = |2ct|, OB = \left|\frac{2c}{t}\right|$$

$$\Delta = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 2c^2$$



2. $x^2 = 4y$

$$\therefore \frac{dy}{dx} = \frac{x}{2}$$

$$\therefore -\frac{dx}{dy} = -\frac{2}{x} = -2 \text{ at } (1, 2)$$

Find equation of line passing through (1,2) with slope -2 .

3. $\frac{dy}{dx} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$

$$\therefore -\frac{dx}{dy} = -\cot \theta$$

$$\therefore \frac{Y - a(\sin \theta - \theta \cos \theta)}{X - a(\cos \theta + \theta \sin \theta)} = -\cot \theta$$

$$\therefore Y - a \sin \theta + a\theta \cos \theta = -\cot \theta X + a \cos \theta \cot \theta + a\theta \cos \theta$$

$$\therefore Y + X \cot \theta - a(\sin \theta + \cos \theta \cot \theta) = 0$$

$$\therefore \text{Distance from origin} = \frac{|a(\sin \theta + \cos \theta \cot \theta)|}{\sqrt{1 + \cot^2 \theta}} = a$$

4. $\frac{dy}{dx} = \frac{2ae^\theta \cos \theta}{2ae^\theta \sin \theta} = \cot \theta$

$$-\frac{dy}{dx} = -\tan \theta$$

Equation of tangent is

$$\frac{y - ae^\theta (\sin \theta + \cos \theta)}{x - ae^\theta (\sin \theta - \cos \theta)} = \cot \theta$$

$$\therefore y - ae^\theta \sin \theta - ae^\theta \cos \theta = x \cot \theta - ae^\theta \cos \theta + ae^\theta \cos \theta \cot \theta$$

$$\therefore \frac{x \cos \theta}{\sin \theta} - y + ae^\theta \sin \theta + ae^\theta \frac{\cos^2 \theta}{\sin \theta} = 0$$

$$\therefore x \cos \theta - y \sin \theta + ae^\theta = 0$$

$$\therefore p = \frac{|ae^\theta|}{1} = ae^\theta$$

Equation of normal is

$$\frac{y - ae^\theta (\sin \theta + \cos \theta)}{x - ae^\theta (\sin \theta - \cos \theta)} = -\tan \theta$$

$$\therefore y - ae^\theta \sin \theta - ae^\theta \cos \theta = -x \tan \theta + ae^\theta \sin \theta \tan \theta - ae^\theta \sin \theta$$

$$\therefore y \cos \theta + x \sin \theta - ae^\theta = 0$$

$$\therefore q = \frac{|-ae^\theta|}{1} = ae^\theta$$

$$\therefore p = q$$

5. $x^{2/3} + y^{2/3} = a^{2/3}$

$$\therefore \frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore -\frac{dx}{dy} = \left(\frac{x}{y}\right)^{1/3}$$

Equation of tangent is

$$\frac{Y - y}{X - x} = -\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore x^{1/3} Y - x^{1/3} y = -y^{1/3} X + xy^{1/3}$$

$$\therefore y^{1/3} X + x^{1/3} Y - x^{1/3} y - xy^{1/3} = 0$$

$$p = \frac{|x^{1/3} y + xy^{1/3}|}{\sqrt{x^{2/3} + y^{2/3}}}$$

$$= |x^{1/3} y^{1/3} a^{1/3}|$$

Equation of normal is

$$\frac{Y-y}{X-x} = \left(\frac{x}{y}\right)^{1/3}$$

$$\therefore y^{1/3}Y - y^{4/3} = x^{1/3}X - x^{4/3}$$

$$\therefore x^{1/3}X - y^{1/3}Y - x^{4/3} + y^{4/3} = 0$$

$$\therefore q = \frac{|y^{4/3} - x^{4/3}|}{\sqrt{x^{2/3} + y^{2/3}}}$$

$$= |(x^{2/3} - y^{2/3})a^{1/3}|$$

$$\begin{aligned}\therefore 4p^2 + q^2 &= 4x^{2/3}y^{2/3} + a^{2/3}(x^{4/3} - 2x^{2/3}y^{2/3} + y^{4/3}) \\ &= a^{2/3}(x^{2/3} + y^{2/3}) \\ &= a^2.\end{aligned}$$

6. $y^2 = 2x, x^2 + y^2 = 8$

$$\therefore x^2 + 2x - 8 = 0 \quad \text{and} \quad x > 0 \quad \text{as} \quad x = \frac{y^2}{2}$$

$$\therefore x = 2$$

$$\Rightarrow y = 2$$

For $y^2 = 2x$

$$\therefore 2y \frac{dy}{dx} = 2$$

$$\therefore m_1 = \frac{1}{y} = \frac{1}{2}$$

For $x^2 + y^2 = 8$

$$\therefore 2x + 2y \frac{dy}{dx} = 0$$

$$\therefore m_2 = -\frac{x}{y} = -1$$

$$\therefore \tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = 3$$

7. $y = \frac{x+3}{x^2+1}$

$$\therefore \frac{dy}{dx} = \frac{(x^2+1) - (2x^2+6x)}{(x^2+1)^2} = \frac{-x^2-6x+1}{(x^2+1)^2}$$

$$\therefore m_1 = \frac{-4 - 12 + 1}{25} = \frac{-3}{5}$$

$$y = \frac{x^2 - 7x + 11}{x - 1}$$

$$\therefore \frac{dy}{dx} = \frac{(2x^2 - 9x + 7) - (x^2 - 7x + 11)}{(x - 1)^2} = \frac{x^2 - 2x - 4}{(x - 1)^2}$$

$$\therefore m_2 = \frac{4 - 4 - 4}{1} = -4$$

$$\therefore \tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = 4$$

8. $x^3 - 3xy^2 + 2 = 0$

$$\therefore 3x^2 - 3xy^2 - 6xy \frac{dy}{dx} = 0$$

$$\therefore x^2 - y^2 = 2xy \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1$$

$$3x^2y - y^3 + 2 = 0$$

$$\therefore 6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} (x^2 - y^2) = -2xy$$

$$\therefore \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = m_2$$

$$m_1 m_2 = -1$$

9. $x = y^2$

$$\therefore \frac{dy}{dx} = \frac{1}{2y} = m_1$$

$$xy = k$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x} = m_2$$

$$m_1 m_2 = -1$$

$$\therefore \frac{-1}{2x} = -1$$

$$\therefore x = \frac{1}{2}$$

$$\therefore y = \pm \frac{1}{\sqrt{2}}$$

$$\therefore k = \pm \frac{1}{2\sqrt{2}}$$

10. $ST = \frac{3}{8}, SN = 24$

$$y_0^2 = ST \cdot SN$$

$$= \frac{3}{8} \times 24 = 9$$

$$\therefore y_0 = \pm 3$$

11. $by^2 = (x+a)^3$

$$\therefore 2by \frac{dy}{dx} = 3(x+a)^2$$

$$\therefore \frac{dy}{dx} = \frac{3(x+a)^2}{2by} = \tan \theta$$

$$\cot \theta = \frac{2by}{3(x+a)^2}$$

$$ST = |y \cot \theta| = \left| \frac{2by^2}{3(x+a)^2} \right|$$

$$SN = |y \tan \theta| = \left| \frac{3(x+a)^2}{2b} \right|$$

$$\therefore \frac{3p(x+a)^2}{2b} = \frac{4qb^2y^4}{9(x+a)^4}$$

$$\therefore \frac{p}{q} = \frac{8b^3y^4}{27(x+a)^6} = \frac{8b}{27} \frac{(by^2)^2}{(x+a)^6} = \frac{8b}{27}$$

12. $xy^n = a^{n+1}$

$$\therefore \frac{dy}{dx} = \frac{-y}{nx} = \tan \theta$$

$$\therefore SN = |y \tan \theta|$$

$$= \left| \frac{-y^2}{nx} \right| = \text{constant}$$

But $xy^n = \text{constant}$

$$\Rightarrow n = -2$$

13. $x^2y^2 = a^5$

$$\therefore 2xy^2 + 2x^2y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-y}{x} = \tan \theta$$

$$\therefore \cot \theta = \frac{-x}{y}$$

$$\therefore ST = |y \cot \theta| = |-x|$$

14. $x^m y^n = a^{m+n}$

$$\therefore mx^{m-1}y^n + nx^m y^{n-1} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{my}{nx}$$

$$\therefore \cot \theta = -\frac{nx}{my}$$

$$\therefore ST = \left| -\frac{nx}{m} \right|$$

15. From information in Q.22,

$$(ST)^2 \propto (SN)$$

16. $-\frac{dx}{dy} = -\frac{a\theta \cos \theta}{a\theta \sin \theta} = -\cot \theta$

As shown in Q.14 of this exercise it is at a constant distance from origin.

17. $ax + by + c = 0$ normal to $xy = 1$

For $xy = 1$, $\frac{dy}{dx} = -\frac{y}{x}$

As xy is positive, $\frac{dy}{dx} < 0 \quad \forall x, y$

$$\therefore -\frac{dx}{dy} < 0 \quad \forall x, y$$

\therefore Slope of normal is positive.

$$\therefore a > 0, b < 0 \quad \text{or} \quad a < 0, b > 0$$

18. $(3-a)x + ay + a^2 - 1 = 0$

$$\therefore -\left(\frac{a}{3-a}\right) > 0$$

$$\therefore a \in (-\infty, 0) \cup (3, \infty)$$

19. $f(x) = 2x^2 - \log|x|$

$$\therefore f'(x) = 4x - \frac{1}{x} < 0$$

$$\therefore \frac{4x^2 - 1}{x} < 0$$

$$\therefore \frac{(2x+1)(2x-1)}{x} < 0$$

$$\therefore x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$$

20. $f(x) = \frac{x}{\sin x}, g(x) = \frac{x}{\tan x}$

$$f'(x) = \frac{\sin x - x \cos x}{x^2} > 0 \quad \forall x \in (0, 1)$$

$$g'(x) = \frac{\tan x - x \sec^2 x}{x^2} < 0 \quad \forall x \in (0, 1)$$

21. $f(x) = \tan^{-1}(\sin x + \cos x)$

Let $g(x) = \tan^{-1} x$

$$\therefore g'(x) = \frac{1}{1+x^2} > 0 \quad \forall x$$

$\therefore f(x)$ increases when $\sin x + \cos x$ increases

Let $h(x) = \sin x + \cos x$

$$\therefore h'(x) = \cos x - \sin x > 0$$

$$\therefore \cos x > \sin x \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

22. $f(x) = x^{100} + \sin x - 1$

$$f'(x) = 100x^{99} + \cos x < 0$$

23. $f(x) = |x| - |x-1|$

$x < 0$	\Rightarrow	$f(x) = -x + 1 - x = 1 - 2x$	MD
$0 < x < 1$	\Rightarrow	$f(x) = x + 1 - x = 1$	Constant
$x > 1$	\Rightarrow	$f(x) = 2x - 1$	MI

24. $f(x) = x(a^2 - 2a - 2) + \cos x$
 $f'(x) = a^2 - 2a - 2 - \sin x > 0 \quad \forall x$
 $\therefore a^2 - 2a - 2 > 1$
 $\therefore a \in (-\infty, -1) \cup (3, \infty)$

25. $\phi(x) = 3f\left(\frac{x^3}{3}\right) + f(3 - x^2) \quad \forall x \in (-3, 4)$
 $f''(x) > 0$
 $\therefore \phi'(x) = 3x^2 f'\left(\frac{x^3}{3}\right) - 2x f'(3 - x^2) > 0$

$$x^2 f'\left(\frac{x^3}{3}\right) > 2x f'(3 - x^2)$$

For $x > 0$

$$x f'\left(\frac{x^3}{3}\right) > 2 f'(3 - x^2)$$

$$\therefore \frac{x}{2} f'\left(\frac{x^3}{3}\right) > f'(3 - x^2)$$

26. $f'(x) \geq 0, g'(x) \leq 0$
 $\therefore h'(x) = f'(g(x)) g'(x) \leq 0$
 $\therefore h(2) = 1$ as $h(1) = 1$

27. $y = a \log|x| + bx^2 + x$
 $\therefore \frac{dy}{dx} = \frac{a}{x} + 2bx + 1 = 0$
 $\therefore \frac{a + 2bx^2 + x}{x} = 0 \quad \alpha = \frac{-4}{3}, \beta = 2$
 $\therefore \alpha + \beta = \frac{2}{3} = \frac{-1}{2b}$
 $\therefore b = \frac{-3}{4}$

$$\alpha\beta = \frac{-8}{3} = \frac{a}{2b}$$

$$\therefore a = \frac{-8}{3} \times 2 \times \frac{-3}{4} = 4$$

28. Point on $y^2 = 4x$ is $(t^2, 2t)$
Distance between point & $(2,1)$ is

$$d = \sqrt{(t^2 - 2)^2 + (2t - 1)^2}$$

$$d^2 = (t^2 - 2)^2 + (2t - 1)^2$$

$$= t^4 - 4t + 5 = f(t)$$

$$\therefore f'(t) = 4t^3 - 4 = 0$$

$\therefore t = 1$ we can show that $t = 1$ is minima

\therefore Point is $(1,2)$.

28. Point nearest to the required line will have common normal.

$$\therefore \frac{dy}{dx} = 3 = 2x + 7$$

$$\therefore x = -2, y = -8$$

point is $(-2, -8)$

30. $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$$x = a \cos \theta, y = 2 \sin \theta$$

$$\sqrt{a^2 \cos^2 \theta + 4(1 - \sin \theta)^2} = d$$

$$d^2 = f(\theta) = a^2 \cos^2 \theta + 4 - 8 \sin \theta + 4 \sin^2 \theta$$

$$= a^2 + 4 + (4 - a^2) \sin^2 \theta - 8 \sin \theta$$

$$\therefore f'(\theta) = 2(4 - a^2) \sin \theta \cos \theta - 8 \cos \theta = 0$$

$$\therefore \cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

\therefore point is $(0,2)$.

31. $r\theta + 2r = k$

$$\therefore r(\theta + 2) = k$$

$$\therefore \theta = \frac{k - 2r}{r}$$

$$\begin{aligned}\therefore A &= \frac{1}{2}r^2 \times \frac{(k-2r)}{r} \\ &= \frac{kr - r^2}{2}\end{aligned}$$

$$\therefore \frac{dA}{dr} = 0$$

$$\therefore r = \frac{k}{4}$$

$$\therefore \theta = \frac{k - \frac{k}{2}}{\frac{k}{4}} = 2^\circ$$

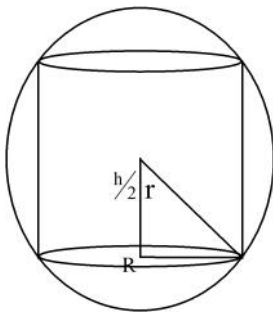
32. From above example, $\theta = 2^\circ$

$$\therefore 2r + 2r = 20$$

$$\therefore r = 5$$

$$\therefore A = \frac{1}{2} \times 25 \times 2 = 25 \text{ sq.cm.}$$

33.



$$R^2 + \frac{h^2}{4} = r^2$$

$$v = \pi R^2 h$$

$$= \pi h \left(r^2 - \frac{h^2}{4} \right)$$

$$= \pi r^2 h - \frac{\pi h^3}{4}$$

$$\therefore \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4} = 0$$

$$\therefore h^2 = \frac{4r^2}{3} \quad h = \frac{2r}{\sqrt{3}}$$

$$\begin{aligned}
 34. \quad s &= 2\pi r(r+h) \\
 &= 2\pi r\left(r + \frac{v}{\pi r^2}\right) \\
 &= 2\pi r^2 + \frac{2v}{r}
 \end{aligned}$$

$$\frac{ds}{dr} = 0 \quad \therefore \quad 4\pi r - \frac{2v}{r^2} = 0$$

$$\therefore \quad r^3 = \frac{v}{2\pi} \quad \therefore \quad r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$$

$$\pi r^2 = \left(\frac{\pi v^2}{4}\right)^{\frac{1}{3}}$$

$$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right)^{\frac{1}{3}} = h$$

$$h = 2r$$

$$35. \quad \frac{R}{h-H} = \tan \alpha$$

$$R = \tan \alpha (h - H)$$

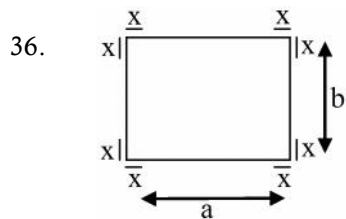
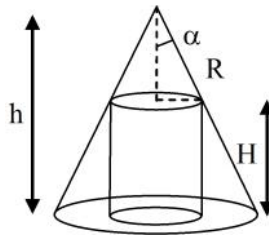
curved surface area

$$= S_c = 2\pi RH$$

$$= 2\pi \tan \alpha (hH - H^2)$$

$$\therefore \quad \frac{dS_c}{dH} = 2\pi \tan \alpha (h - 2H) = 0$$

$$\therefore \quad H = \frac{h}{2}$$



$$V = (a - 2x)(b - 2x)x$$

$$V = 4x^3 - 2(a+b)x^2 + abx$$

$$\therefore \quad \frac{dV}{dx} = 12x^2 - 4(a+b)x + ab = 0$$

$$\begin{aligned}\therefore x &= \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24} \\ &= \frac{(a+b) \pm \sqrt{a^2 + b^2 - ab}}{6}\end{aligned}$$

But $x < a, x < b$

$$\therefore x = \frac{(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}}{6} = \frac{1}{6} \left[(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}} \right]$$

37. $v = x(a - 2x)^2$

$$\therefore v = 4x^3 - 4ax^2 + a^2x$$

$$\therefore \frac{dv}{dx} = 12x^2 - 8ax + a^2 = 0 \quad \therefore x = \frac{a}{2} \quad \text{or} \quad x = \frac{a}{6}$$

But $x = \frac{a}{2}$ will make volume zero.

$$\therefore x = \frac{a}{6}$$

38. $a^2h = 32$

$a^2 + 4ah$ has to be minimised

$$\therefore h = \frac{32}{a^2}$$

$$\therefore f(a) = a^2 + \frac{128}{a}$$

$$\therefore f'(a) = 2a - \frac{128}{a^2} = 0$$

$$\therefore a = 4 \quad \& \quad h = 2$$

$$\therefore \text{Area} = 16 + 32 = 48$$

39. Line is $(y - 4) = m(x - 3)$

$$x = 0 \Rightarrow y = 4 - 3m$$

$$y = 0 \Rightarrow x = 3 - \frac{4}{m}$$

$$\Delta = \frac{1}{2} (4 - 3m) \left(3 - \frac{4}{m} \right)$$

$$= \frac{1}{2} \left(24 - 9m + \frac{16}{m} \right)$$

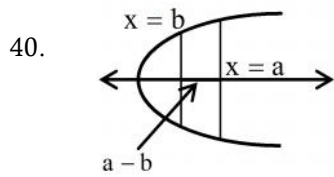
$$\therefore \frac{d\Delta}{dm} = \frac{-9}{2} + \frac{16}{2m^2} = 0$$

$$\therefore \frac{8}{m^2} = \frac{9}{2}$$

$$\therefore m^2 = \frac{16}{9}$$

$$\therefore m = \frac{-4}{3} \text{ as } m > 0 \Rightarrow \text{no } \Delta \text{ is formed}$$

$$\therefore \Delta = \frac{1}{2}(8)(6) = 24$$



$$x = b \Rightarrow y^2 = 4ab \quad y = \pm\sqrt{4ab} = \pm 2\sqrt{ab}$$

$$\therefore |2y| = \pm 4\sqrt{ab}$$

$$A = \frac{1}{2}(a - b)(4a + 4^2\sqrt{ab})$$

$$= 2a^2 + 2a^{\frac{3}{2}}b^{\frac{1}{2}} - 2ab - 2a^{\frac{1}{2}}b^{\frac{3}{2}}$$

$$\frac{dA}{db} = \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}} - 2a - 3a^{\frac{1}{2}}b^{\frac{1}{2}} = 0$$

$$\therefore -3a^{\frac{1}{2}}b - 2ab^{\frac{1}{2}} + a^{\frac{3}{2}} = 0$$

$$\Rightarrow b = \frac{a}{9}$$

41. $\therefore V = \frac{1}{3}\pi \ell^3 \sin^2 \alpha \cos \alpha$

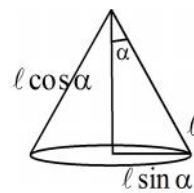
$$\frac{dV}{d\alpha} = \frac{1}{3}\pi \ell^2 [2 \sin \alpha \cos^2 \alpha - \sin^3 \alpha] = 0$$

$$\therefore \sin \alpha = 0 \text{ or } 2 \cos^2 \alpha = \sin^2 \alpha$$

Rejected

$$\therefore \tan \alpha = \sqrt{2} \text{ as } \alpha \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \alpha = \tan^{-1}(\sqrt{2})$$



42. $V = \frac{1}{3}\pi r^2 h = \text{constant}$

$S_c = \pi r \sqrt{r^2 + h^2}$ has to be maximized

$$\therefore h = \frac{3V}{\pi r^2}$$

$$\begin{aligned}\therefore S_c &= \pi r^2 \sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}} \\ &= \sqrt{\pi^2 r^4 + \frac{9V^2}{r^2}}\end{aligned}$$

$$\frac{dS_c}{dr} = \frac{dS_c^2}{dr} = 0$$

$$\therefore 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0$$

$$\therefore 4\pi^2 r^6 = 18V^2$$

$$\therefore r^6 = \frac{9V^2}{2\pi^2}$$

$$\therefore r = \frac{3^{\frac{1}{3}}}{2^{\frac{1}{6}}} \frac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$$

$$r^2 = \left(\frac{9V^2}{2\pi^2} \right)^{\frac{1}{3}}$$

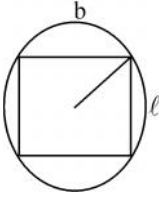
$$\begin{aligned}h &= \frac{3V}{\pi r^2} = \frac{3V}{\pi} \times \left(\frac{2\pi^2}{9V^2} \right)^{\frac{1}{3}} \\ &= \frac{3V}{\pi} \times \frac{2^{\frac{1}{3}} \pi^{\frac{2}{3}}}{9^{\frac{1}{3}} V^{\frac{2}{3}}} = \frac{(3^{1/3})(2^{1/3})V^{1/3}}{\pi^{1/3}}\end{aligned}$$

$$\frac{h}{r} = 2^{1/3} \times 2^{1/6} = \sqrt{2}$$

43. $\ell^2 = h^2 + r^2$
 $r = \sqrt{\ell^2 - h^2}$
 $V = \frac{1}{3} r^2 h = \frac{1}{3} (\ell^2 - h^2) h$
 $= \frac{\ell^2 h}{3} - \frac{h^3}{3}$
 $\therefore \frac{dV}{dh} = \frac{\ell^2}{3} - h^2 = 0$

$$\therefore h = \frac{\ell}{\sqrt{3}}$$

44.



$$b^2 + \ell^2 = 4r^2$$

$$b = \sqrt{4r^2 - \ell^2}$$

$$S = kb\ell^3$$

$$= k\ell^3 \sqrt{4r^2 - \ell^2}$$

$$\therefore \frac{dS}{d\ell} = 3k\ell^2 \sqrt{4r^2 - \ell^2} - \frac{k\ell^4}{\sqrt{4r^2 - \ell^2}} = 0$$

$$\therefore 3k\ell^2(4r^2 - \ell^2) = k\ell^4$$

$$\therefore \ell = 0 \quad \text{Rejected or}$$

$$3(4r^2 - \ell^2) = \ell^2$$

$$\therefore 12r^2 = 4\ell^2$$

$$\therefore \ell = \sqrt{3}r$$

$$\therefore b = r$$

45.

$$b^2 + d^2 = 4r^2$$

$$d^2 = 4r^2 - b^2$$

$$\therefore S = kb d^2$$

$$= kb(4r^2 - b^2) = 4kbr^2 - kb^3$$

$$\therefore \frac{dS}{dr} = 4kr^2 - 3kb^2 = 0$$

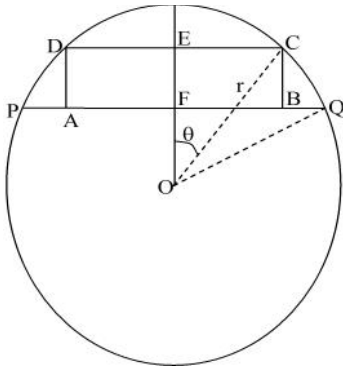
$$\therefore b = \frac{2r}{\sqrt{3}}$$

$$\therefore d^2 = 4r^2 - \frac{4r^2}{3} = \frac{8r^2}{3}$$

$$\Rightarrow d = \frac{2\sqrt{2}r}{3}$$

$$\therefore d = \sqrt{2}b = 2\sqrt{\frac{2}{3}}r$$

46. Let OABC be the sheet of paper as



The corner A of the rectangular sheet OABC is folded over along PQ so as to reach the opposite edge OC at R.

Let the crease PQ be of length x.

Let $\angle APQ = \theta$. Then $\angle PQR = \theta$ and $\angle OPR = \pi - 2\theta$.

In $\triangle APQ$, we have

$$\cos \theta = \frac{AP}{PQ}$$

$$\Rightarrow AP = x \cos \theta$$

In $\triangle OPR$, we have

$$\cos(\pi - 2\theta) = \frac{OP}{RP}$$

$$\Rightarrow -\cos 2\theta = \frac{OP}{AP} \quad [\because AP = RP]$$

$$\Rightarrow OP = -AP \cos 2\theta = -x \cos \theta \cos 2\theta$$

Now,

$$a = OA = OP + AP$$

$$\Rightarrow a = x \cos \theta - x \cos \theta \cos 2\theta$$

$$\Rightarrow x = \frac{a}{\cos \theta - \cos \theta \cos 2\theta} \quad \dots(i)$$

$$\Rightarrow \frac{a}{x} = \cos \theta - \cos \theta \cos 2\theta$$

Let $y = \frac{a}{x}$. Then y is maximum when x is minimum.

Now,

$$y = \cos \theta - \cos \theta \cos 2\theta$$

$$\Rightarrow \frac{dy}{d\theta} = -\sin \theta + \sin \theta \cos 2\theta + 2 \cos \theta \sin 2\theta$$

For maximum or minimum values of y we must have $\frac{dy}{d\theta} = 0$

$$\Rightarrow -\sin \theta + \sin \theta \cos 2\theta + 4 \sin \theta \cos^2 \theta =$$

$$\Rightarrow -\sin \theta(1 - \cos 2\theta) + 4 \sin \theta(1 - \sin^2 \theta) = 0$$

$$\Rightarrow -2 \sin^3 \theta + 4 \sin \theta - 4 \sin^3 \theta = 0$$

$$\Rightarrow 4 \sin \theta = 6 \sin^3 \theta$$

$$\Rightarrow \sin^2 \theta = \frac{2}{3} \text{ or } \sin \theta = 0$$

$$\Rightarrow \sin \theta = \sqrt{\frac{2}{3}} \text{ or } \theta = 0.$$

Now,

$$\frac{d^2y}{d\theta^2} = -\cos \theta + \cos \theta \cos 2\theta - 2 \sin \theta \sin 2\theta + 4 \cos \theta \cos 2\theta - 2 \sin \theta \sin 2\theta$$

$$\Rightarrow \frac{d^2y}{d\theta^2} = -\cos \theta + 5 \cos \theta \cos 2\theta - 4 \sin \theta \sin 2\theta$$

For $\sin \theta = \sqrt{\frac{2}{3}}$ and $\cos \theta = \sqrt{\frac{1}{3}}$, we have

$$\frac{d^2y}{d\theta^2} = -\frac{1}{\sqrt{3}} + 5 \times \sqrt{\frac{2}{3}} \times \left(\frac{2}{3} - 1\right) - 4 \times \sqrt{\frac{2}{3}} \times 2\sqrt{\frac{2}{3}} \times \sqrt{\frac{1}{3}} < 0.$$

So, y is maximum when $\sin \theta = \sqrt{\frac{2}{3}}$

Hence, x is minimum when $\sin \theta = \sqrt{\frac{2}{3}}$

Putting $\sin \theta = \sqrt{\frac{2}{3}}$ and $\cos \theta = \sqrt{\frac{1}{3}}$ in (i), we get

$$\text{Length of the crease} = x = \frac{a}{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \left(1 - 2 \times \frac{2}{3}\right)} = \frac{3\sqrt{3}a}{4}$$

47. Let speed of boat be v & walking speed be $v \sec \alpha$

$$\begin{aligned} \therefore t &= \frac{\sqrt{a^2 + (b-x)^2}}{v} + \frac{x \cos \alpha}{v} \\ &= \frac{\sqrt{a^2 + (b-x)^2} + x \cos \alpha}{v} \end{aligned}$$

$$\therefore v \frac{dt}{dx} = \cos \alpha + \frac{1}{2\sqrt{a^2 + (b-x)^2}} \times -2(b-x) = 0$$

$$\therefore \cos \alpha = \frac{b-x}{\sqrt{a^2 + (b-x)^2}}$$

$$\therefore (b-x)^2 = \cos^2 \alpha (a^2 + b^2 - 2bx + x^2)$$

$$\begin{aligned} \therefore (b-x)^2 &= a^2 \cot^2 \alpha \\ \therefore x &= b - a \cot \alpha = \frac{b \sin \alpha - a \cos \alpha}{\sin \alpha} \end{aligned}$$

$$\begin{aligned} 48. \quad \therefore T &= \frac{\sqrt{d^2 + x^2}}{u} + \frac{1-x}{v} \\ \therefore \frac{dT}{dx} &= \frac{x}{u\sqrt{d^2 + x^2}} - \frac{1}{v} = 0 \\ \therefore xv &= u\sqrt{d^2 + x^2} \\ \therefore x^2(v^2 - u^2) &= u^2d^2 \\ \therefore x &= \frac{ud}{\sqrt{v^2 - u^2}} \end{aligned}$$

For students. [Think for solution if $u > v$]

$$\begin{aligned} 49. \quad 2\ell + 2\pi r &= 440 \\ \therefore \ell + \pi r &= 220 \quad \& \quad \ell = 220 - \pi r \\ A &= 2(220r - \pi r^2) = 2\ell r \\ \frac{dA}{dr} &= 2(220 - 2\pi r) = 0 \\ \therefore r &= 35 \text{ ft} \\ \Rightarrow 2r &= 70 \text{ ft} \quad \& \quad \ell = 110 \text{ ft} \end{aligned}$$

$$\begin{aligned} 50. \quad \dots + a_1x^2 + a_2x^4 + \dots + a_nx^{2n} \\ 0 < a_1 < a_2 < \dots < a_n \\ \therefore P'(x) &= 2na_nx^{2n-1} + \dots + 4a_2x^3 + 2a_1x \\ &= 0 \quad \text{only at } x = 0 \quad \& \\ P''(x) &> 0 \quad \forall x \in \mathbb{R} \\ \therefore P(x) &\text{ has only one minimum.} \end{aligned}$$

$$\begin{aligned} 51. \quad x &= a \sec \theta, y = b \operatorname{cosec} \theta \\ \text{Minimum radius vector} &= ? \\ r &= \sqrt{x^2 + y^2} = \sqrt{a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta} \\ \text{From (Q.4),} \\ \text{Minimum value of } r &= \sqrt{(a+b)^2} = a+b \end{aligned}$$

$$\begin{aligned} 52. \quad \text{From (Q.18)} \\ s &= 2\pi r(r+h) \end{aligned}$$

$$= 2\pi r \left(r + \frac{v}{\pi r^2} \right)$$

$$= 2\pi r^2 + \frac{2v}{r}$$

$$\frac{ds}{dr} = 0 \quad \therefore \quad 4\pi r - \frac{2v}{r^2} = 0$$

$$\therefore \quad r^3 = \frac{v}{2\pi} \quad \therefore \quad r = \left(\frac{v}{2\pi} \right)^{\frac{1}{3}}$$

$$\pi r^2 = \left(\frac{\pi v^2}{4/3} \right)^{\frac{1}{3}}$$

$$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi} \right) = h$$

$h = 2r$, form this statement $h : r = 2 : 1$

53. $f(x) = (x-1)^p (x-2)^q$

$$\therefore \quad f'(x) = p(x-1)^{p-1} (x-2)^q + q(x-1)^p (x-2)^{q-1}$$

$$f''(x) = p(p-1)(x-1)^{p-2} (x-2)^q + 2pq(x-1)^{p-1} (x-2)^{q-1} + q(q-1)(x-1)^p (x-2)^{q-2}$$

If we go on taking derivatives, we find that the condition given in the question holds when (even)th derivative is non-zero for it, p & q should be even.

54. $f(x) = xe^x$

$$f'(x) = xe^x + e^x = 0$$

$$\therefore \quad x = -1$$

$$f''(x) = xe^x + 2e^x > 0 \quad \text{for } x = -1$$

$$\therefore \quad x = -1 \text{ is a minimum}$$

55. Time required = $T = \left(\frac{N}{x} \right) (\alpha + \beta x^2)$

$$\therefore \quad T = N \left(\frac{\alpha}{x} + \beta x \right)$$

$$\therefore \quad \frac{dT}{dx} = N \left(\beta - \frac{\alpha}{x^2} \right) = 0 \quad \therefore \quad x = \sqrt{\frac{\alpha}{\beta}}$$

56. $f(x) = \max \{x, x+1, 2-x\}$

By graph, $f(x) =$

$$f(x) = 2 - x, x \leq +\frac{1}{2}$$

$$x + 1, x > \frac{1}{2}$$

$\therefore x = \frac{1}{2}$ is point of minima and minimum value is $\frac{3}{2}$.

$$57. f(\alpha) = \left(1 + \frac{1}{\sin^n \alpha}\right) \left(1 + \frac{1}{\cos^n \alpha}\right)$$

$$= (1 + \sec^n \alpha)(1 + \operatorname{cosec}^n \alpha)$$

$$= 1 + \sec^n \alpha + \operatorname{cosec}^n \alpha + \sec^n \alpha \operatorname{cosec}^n \alpha$$

$$\therefore f'(\alpha) = n \sec^n \alpha \tan \alpha - n \operatorname{cosec}^n \alpha \cot \alpha$$

$$+ n \sec^n \alpha \operatorname{cosec}^n \alpha (\tan \alpha - \cot \alpha) = 0$$

$$\therefore \sec^n \alpha \tan \alpha (1 + \operatorname{cosec}^n \alpha) = \operatorname{cosec}^n \alpha \cot \alpha (1 + \sec^n \alpha)$$

$$\therefore \frac{(\sec^n \alpha)(\sec^2 \alpha - 1)}{1 + \sec^n \alpha} = \frac{\operatorname{cosec}^n \alpha}{1 + \operatorname{cosec}^n \alpha}$$

$$\therefore \frac{(\cos^n \alpha)(\sin^2 \alpha)}{(\cos^n \alpha + 1)(\cos^2 \alpha)} = \frac{\sin^n \alpha}{1 + \sin^n \alpha}$$

$$\therefore \frac{\sin^{n-2} \alpha}{1 + \sin^n \alpha} = \frac{\cos^{n-2} \alpha}{1 + \cos^n \alpha}$$

$$\Rightarrow \sin \alpha = \cos \alpha$$

For minima,

$$\sin \alpha = \cos \alpha = \frac{1}{\sqrt{2}}$$

$$\therefore \text{Minimum value} = (1 + 2^{n/2})^2$$

$$58. f(x) + f\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$\therefore f(x) + f\left(\frac{1}{x}\right) = x$$

$$\Rightarrow x = \frac{1}{x}$$

$\Rightarrow x = \pm 1$ Only

Here, $x = -1 \Rightarrow f(x) = -\frac{1}{2}$ and

$x = +1 \Rightarrow f(x) = +\frac{1}{2}$

$\therefore f(x)$ has maximum value $\frac{1}{2}$.

59. $f(x) = \cos 2\pi x + \{x\}$

At non-integral points,

$$f'(x) = -2\pi \sin 2\pi x + 1$$

It tends to achieving maximum values at points infinitesimally close to and less than integers but it has a discontinuity.

\therefore It has no maxima.

60. $f(x) = x - x^2$

x_1 & $x_2 \in y = x - x^2$ in $(0,1)$

maximum value of expression

$$= \max(x - x^2) = \frac{1}{4}$$

61. $f(x) = x^2, \quad x \in [-2, -1] \cup [1, 2]$

$$2 - x^2, \quad x \in (-1, 1)$$

\therefore Function has maximum at $x = 0$ & local as well as global minima at $x = \pm 1$

62. $x^3 - ax^2 + bx - 6 = 0$ has roots real and positive

$$\therefore \alpha\beta\gamma = 6, \alpha + \beta + \gamma = a, \alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{b}{6}$$

Now, sum is minimum when each of them is equal

$$\frac{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}{3} \geq \left(\frac{1}{\alpha\beta\gamma} \right)^{\frac{1}{3}} \quad [\text{AM-GM inequality}]$$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{3}{6^{1/3}} \quad \therefore b \geq \frac{3 \times 6}{6^{1/3}} = 3(36)^{1/3}$$

63. $f'(x) = \frac{2}{3(6-x)^{\frac{1}{3}}}$

Which is not diff. at $x = 6$

\therefore Theorems are not applicable.

64. By definition.

65. $f(0) = -6, f(4) = +6$

$$\therefore f'(x) = (x-2)(x-3) + (x-1)(x-2) + (x-1)(x-3)$$

$$f'(c) = \frac{6+6}{4-0} = 3$$

$$\therefore 3x^2 - 12x + 11 = 3 \quad \text{and} \quad x = c$$

$$\therefore 3c^2 - 12c + 8 = 0$$

$$c = 12 \pm \frac{\sqrt{144 - 96}}{6}$$

$$= 12 \pm \frac{\sqrt{48}}{6} = 6 \pm \frac{2\sqrt{3}}{3} = 2 \pm \frac{2}{\sqrt{3}}$$

66. $f(x) = x^\alpha \log x$

$$f'(x) = x^{\alpha-1} (1 + \alpha \log x) = 0$$

$$c = e^{-1/\alpha} \in (0, 1)$$

$$\therefore \alpha > 0$$

67. $a + b + c = 0$

$$f(x) = ax^3 + bx^2 + cx$$

Has roots 0 & 1

$$\therefore 3ax^2 + 2bx + c = 0$$

has at least one root in $(0, 1)$.

68. $f'(c) = \frac{13-5}{2} = 4$

69. Refer (Q.28) (above)

$$a + b + c = 0$$

$$f(x) = ax^3 + bx^2 + cx$$

Has roots 0 & 1

$$\therefore 3ax^2 + 2bx + c = 0$$

70. $x^3 - 3x + a = 0$ has two roots in $[0, 1]$

$$f'(x) = 3x^2 - 3 \neq 0 \quad \text{in} \quad (0, 1)$$

\therefore There is no value of a satisfying the conditions.

APPLICATIONS OF DERIVATIVES

EXERCISE 2(A)

1. $y = x^{1/3}(x - 1)$

$$\frac{dy}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3} \cdot \frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}} [4x - 1]$$

hence f is \uparrow for $x > \frac{1}{4}$

and $f \downarrow$ for $x < \frac{1}{4}$ } $x^{2/3}$ is always positive and at $x = 1/4$ the curves has a local minima

now $f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3} \cdot x^{-2/3}$ (non existent at $x = 0$, vertical tangent)

$$f''(x) = \frac{4}{9} \cdot \frac{1}{x^{2/3}} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{x^{5/3}}$$

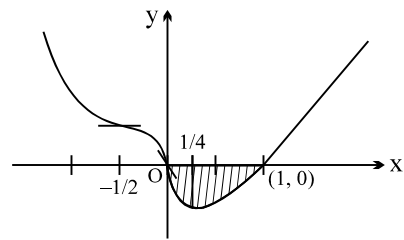
$$= \frac{2}{9x^{2/3}} \left[2 + \frac{1}{x} \right] = \frac{2}{9x^{2/3}} \left[\frac{2x+1}{x} \right]$$

$\therefore f''(x) = 0$ at $x = -\frac{1}{2}$ (inflection point)

graph of $f(x)$ is as

$$A = \int_0^1 (x^{4/3} - x^{1/3}) dx = \left[\frac{3}{7}x^{3/7} - \frac{3}{4}x^{4/3} \right]_0^1$$

$$= \left| \frac{3}{7} - \frac{3}{4} \right| = 3 \left| \frac{4-7}{28} \right| = \frac{9}{28} \Rightarrow \text{(D)}$$



2. $\frac{dy}{dx} =$ slope fo tangent

$$-\frac{1}{t^2} = -\frac{b}{a} \quad \therefore \frac{a}{b} = t^2 > 0 \quad \Rightarrow a \text{ and } b \text{ are of same sign.}$$

3. $f'(x) = \sqrt{1-x^4} > 0$ in $(-1, 1) \Rightarrow f$ is \uparrow

Now $f(x) + f(-x) = \int_0^x \sqrt{1-t^4} dt + \int_0^{-x} \sqrt{1-t^4} dt \Rightarrow \int_0^x \sqrt{1-t^4} dt + \left(-\int_0^y \sqrt{1-y^4} dy \right)$ ($t = -y$)

$$= 0 \Rightarrow f(x) \text{ is odd}$$

again $f''(x) = \frac{-4x^3}{2\sqrt{1-x^4}}$ which vanished at $x = 0$ and changes sign $\Rightarrow (0, 0)$ is inflection since f is

well defined in $[-1, 1] \Rightarrow A, B, C, D$

4. Since intercepts are equal in magnitude but opposite in sign $\Rightarrow \left. \frac{dy}{dx} \right|_p = 1$

now $\frac{dy}{dx} = x^2 - 5x + 7 = 1 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2 \text{ or } 3$]

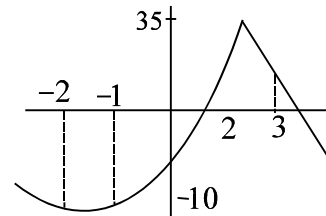
$$\begin{aligned}
 5. \quad h(x) &= \frac{\ln(f(x) \cdot g(x))}{\ln a} = \frac{\ln a^{\{a^{|x|} \cdot \operatorname{sgn} x\} + [a^{|x|} \cdot \operatorname{sgn} x]}}{\ln a} \\
 &= \{a^{|x|} \operatorname{sgn} x\} + [a^{|x|} \operatorname{sgn} x] = a^{|x|} \operatorname{sgn} x \quad (\because \{y\} + [y] = y) \\
 &= \begin{cases} a^x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -a^{-x} & \text{for } x < 0 \end{cases} \Rightarrow h(x) \text{ is an odd function]}
 \end{aligned}$$

$$6. \quad f'(x) = 100x^{99} + \cos x$$

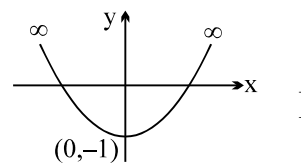
for $x \in (0, 1)$ and $\left(0, \frac{\pi}{2}\right)$, $\cos x$ and x are both +ve $\Rightarrow \uparrow$

for $x \in \left(\frac{\pi}{2}, \pi\right)$, $x > 1$ hence $100x^{99}$ obviously $> \cos x \Rightarrow \uparrow$]

7. Note that $f(x)$ is continuous at $x=2$ and f is decreasing for $(2, 3)$ and increasing for $[-1, 2]$. At $x=2$ f has a maxima hence (A) is not correct.]



8. Graph of $y = f(x) \Rightarrow$ (A) and (C)]



9. If f and g are inverse then $(f \circ g)(x) = x$

$$f'[g(x)] g'(x) = 1$$

if f is increasing $\Rightarrow f' > 0 \Rightarrow$ sign of g' is also +ve \Rightarrow (A) is correct

If f is decreasing $\Rightarrow f' < 0 \Rightarrow$ sign of g' is -ve \Rightarrow (B) is false

since f has an inverse $\Rightarrow f$ is bijective $\Rightarrow f$ is injective \Rightarrow (C) is correct

inverse of a bijective mapping is bijective

$\Rightarrow g$ is also bijective $\Rightarrow g$ is onto \Rightarrow (D) is correct]

10. $f(x) = \ln(1 - \ln x)$
domain $(0, e)$

$$f'(x) = -\frac{1}{(1 - \ln x)} \cdot \frac{1}{x} < 0 \Rightarrow \text{decreasing } \forall x \text{ in its domain } \Rightarrow \text{(A) \& (B) are incorrect}$$

$$f'(1) = -1 \Rightarrow \text{(C) is also incorrect}$$

$$\text{also } f(1) = 0; \quad \lim_{x \rightarrow e^-} f(x) \rightarrow -\infty; \quad \lim_{x \rightarrow 0^+} f(x) \rightarrow \infty$$

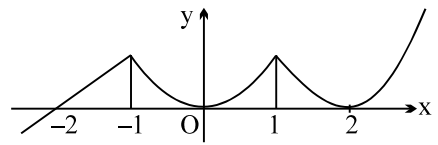
$$f''(x) = \frac{-\ln x}{x^2(1 - \ln x)^2}$$

$$f''(1) = 0 \text{ which is a point of inflection}$$

graph is as shown

y axis and $x = e$ are two asymptotes

11. f is obvious continuous $\forall x \in \mathbb{R}$ and not derivable at -1 and 1
 $f'(x)$ changes sign 4 times at $-1, 0, 1, 2$
 local maxima at 1 and -1
 local minima at $x = 0$ and 2]



12. Domain is $x \in \mathbb{R}$

Also $f(x) = [\cos(\tan^{-1}(\sin \theta))]^2$ where $\cot \theta = x$

$$= \left[\cos \left(\tan^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) \right) \right]^2 = (\cos \phi)^2 \text{ where } \tan \phi = \frac{1}{\sqrt{1+x^2}}$$

$$= \left(\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} \right)^2$$

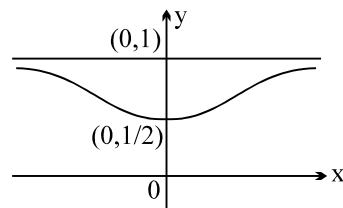
$$g(x) = \frac{1+x^2}{2+x^2} = 1 - \frac{1}{2+x^2}$$

range is $\left[\frac{1}{2}, 1 \right)$; $f'(x) = \frac{2x}{(2+x^2)^2}$

hence $f'(0) = 0$

also $\lim_{x \rightarrow \infty} f(x) = 1$

hence (B), (C), (D)]



13. Let the tangent line be $y = ax + b$

The equation for its intersection with the upper parabola is

$$x^2 + 1 = ax + b$$

$$x^2 - ax + (1 - b) = 0$$

This has a double root when $a^2 - 4(1 - b) = 0$ or $a^2 + 4b = 4$

For the lower parabola

$$ax + b = -x^2$$

$$x^2 + ax + b = 0$$

This has a double root when $a^2 - 4b = 0$

subtract these two equations to get $8b = 4$ or $b = 1/4$

add them to get $2a^2 = 4$ or $a = \pm \sqrt{2}$

The tangent lines are $y = \sqrt{2}x + \frac{1}{2}$ and $y = -\sqrt{2}x + \frac{1}{2}$

14. $f(x) = \int_0^{\pi} \cos t \cos(x-t) dt$ (1)

$$= \int_0^{\pi} -\cos t \cdot \cos(x-\pi+t) dt$$

$$f(x) = \int_0^{\pi} \cos t \cdot \cos(x+t) dt$$
(2)

(1) + (2) gives

$$2f(x) = \int_0^{\pi} \cos t (2 \cos x \cdot \cos t) dt$$

$$\therefore f(x) = \cos x \int_0^{\pi} \cos^2 t \, dt = 2 \cos x \int_0^{\pi/2} \cos^2 t \, dt$$

$$f(x) = \frac{\pi \cos x}{2} \text{ Now verify. Only (A) \& (B) are correct.}$$

15. (A) $f(x) = x - \tan^{-1}x$

$$f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \Rightarrow f \text{ is increasing in } (0, 1)$$

$$f(x) > f(0) \text{ but } f(0) = 0$$

$$f(x) > 0 \Rightarrow x > \tan^{-1}x \text{ in } (0, 1)$$

(B) $f(x) = \cos x - 1 + \frac{x^2}{2}$

$$f'(x) = -\sin x + x = x - \sin x > 0 \text{ in } (0, 1) \Rightarrow \text{(B) is not correct}$$

(C) $f(x) = 1 + x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$

$$f'(x) = x \left(\frac{1 + \frac{1}{2} \cdot \frac{2x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} \right) + \ln(x + \sqrt{1+x^2}) - \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}} + \ln(x + \sqrt{1+x^2}) - \frac{x}{\sqrt{1+x^2}} > 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \text{(C) is true}$$

(D) $f(x) = x - \frac{x^2}{2} - \ln(1+x)$

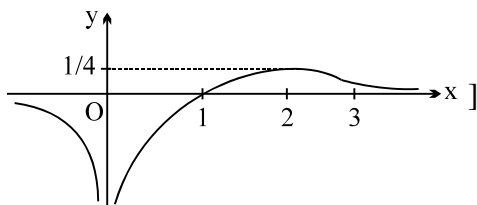
$$f'(x) = (1-x) - \frac{1}{1+x} = \frac{(1-x^2)-1}{1+x} = -\frac{x^2}{1+x} < 0 \Rightarrow \text{(D) is correct}$$

hence $f(x)$ is decreasing in $(0, 1)$

$$\therefore f(x) < f(0)$$

$$f(x) < 0 \Rightarrow x - \frac{x^2}{2} < \ln(1+x)]$$

16. $f'(x) = \frac{2-x}{x^3}$ and $f''(x) = \frac{x-3}{x^4}$. Now interpret

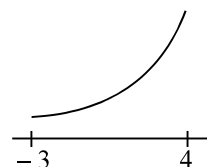


17. (A) $f(x)$ has no relative minimum on $(-3, 4)$

(B) $f(x)$ is continuous function on $[-3, 4]$

$$\Rightarrow f(x) \text{ has min. and max. on } [-3, 4] \text{ by IVT}$$

(C) $f''(x) > 0 \Rightarrow f(x)$ is concave upwards on $[-3, 4]$



- (D) $f(3) = f(4)$
 By Rolle's theorem
 $\exists c \in (3, 4)$, where $f'(c) = 0$
 $\Rightarrow \exists$ critical point on $[-3, 4]$

18. (A) False, e.g. $f(x) = \sin \sqrt{x}$
 (B) True, from IVT
 (C) True as $\lim_{x \rightarrow \infty} \sin^{-1}\left(1 + \frac{1}{x}\right) = \sin^{-1}(\text{a quantity greater than one}) \Rightarrow$ not defined
 (D) True, as the line passes through the centre of the circle.

19.

(A) Let $l = \lim_{x \rightarrow 0} \frac{\int_0^x e^{t^2} dt}{-(e^x - x - 1)} \left(\frac{0}{0}\right) = \lim_{x \rightarrow 0} \frac{\int_0^x e^{t^2} dt}{-x^2 \left(\frac{e^x - x - 1}{x^2}\right)} = -2 \lim_{x \rightarrow 0} \frac{\int_0^x e^{t^2} dt}{x} \left(\frac{0}{0}\right) = -2 \lim_{x \rightarrow 0} \frac{e^{x^2}}{1} = -2$

(B) $14x^2 - 7xy + y^2 = 2$

$$\frac{dy}{dx} = \frac{28x - 7y}{7x - 2y} \quad \dots(1)$$

if $x = 1$ then $14 - 7y + y^2 = 2 \Rightarrow y^2 - 7y + 12 = 0 \Rightarrow y = 3$ or 4
 hence $L(1, 3)$ and $M(1, 4)$

slope of tangent at $L = \frac{28 - 21}{7 - 6} = 7$; slope of tangent at $M = \frac{28 - 28}{7 - 8} = 0$

equation of tangent at L and M are

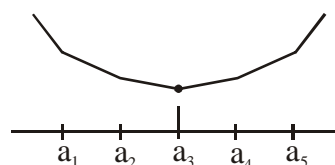
$$\begin{aligned} y - 3 &= 7(x - 1) &\Rightarrow & y = 7x - 4 \\ \text{and } y - 4 &= 0(x - 1) &\Rightarrow & y = 4 \end{aligned}$$

hence $N = \left(\frac{8}{7}, 4\right) \Rightarrow$ (C)

(C) If n is odd then graph of $f(x)$ is

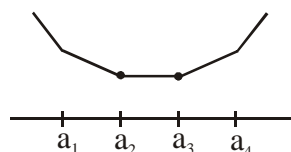
a_3 is the only point where

$f(x)$ has its minimum value



If n is even then graph of $f(x)$ is

From a_2 to a_3 at all values of x , $f(x)$ is minimum.



(D) $2lc + m = (lb^2 + mb) \frac{-(la^2 + my)}{b - a} = l(b^2 - a^2) + m(b - a) = l(b + a) + m$; $c = \frac{a + b}{2}$

20. We have $f'(x) = 5 \sin^4 x \cos x - 5 \cos^4 x \sin x = 5 \sin x \cos x (\sin x - \cos x)(1 + \sin x \cos x)$

$\therefore f'(x) = 0$ at $x = \frac{\pi}{4}$. Also $f'(0) = f'\left(\frac{\pi}{2}\right) = 0$

Hence \exists some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$ (By Rolle's Theorem) \Rightarrow (C) is correct.

Also in $\left(0, \frac{\pi}{4}\right)$ f is decreasing and in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ f is increasing \Rightarrow minimum at $x = \frac{\pi}{4}$

As $f(0) = f\left(\frac{\pi}{2}\right) = 0 \Rightarrow 2 \text{ roots} \Rightarrow \text{(D) is correct.}$

21. $f(x) = \tan^{-1}(x)$ is defined on \mathbb{R} and is strictly increasing but do not have its range \mathbb{R}

22. $f(0) = 1; f(2) = 2$
 $f(1^-) = f(1^+) = f(1) = 2$]

23. $f(x) = \ln(2+x) - \frac{2x+2}{x+3}$ is continuous in $(-2, \infty)$

$$f'(x) = \frac{1}{x+2} - \frac{4}{(x+3)^2} = \frac{(x+3)^2 - 4(x+2)}{(x+2)(x+3)^2}$$

$$= \frac{x^2 + 2x + 1}{(x+2)(x+3)^2} = \frac{(x+1)^2}{(x+2)(x+3)^2} > 0 \quad (f'(x) = 0 \text{ at } x = -1)$$

$\Rightarrow f$ is increasing in $(-2, \infty)$

also $\lim_{x \rightarrow -2^+} f(x) \rightarrow -\infty$ and $\lim_{x \rightarrow \infty} f(x) \rightarrow \infty \Rightarrow$ unique root]

24. Let $f(x) = 0$ has two roots say $x = r_1$ and $x = r_2$ where $r_1, r_2 \in [a, b]$

$\Rightarrow f(r_1) = f(r_2)$

hence \exists there must exist some $c \in (r_1, r_2)$ where $f'(c) = 0$

but $f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

for $|x| \geq 1$, $f'(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + 1 > 0$

for $|x| \leq 1$, $f'(x) = (1 - x) + (x^2 - x^3) + (x^4 - x^5) + x^6 > 0$

hence $f'(x) > 0$ for all x

\therefore Rolles theorem fails $\Rightarrow f(x) = 0$ can not have two or more roots.]

25. Consider the example of $f(x) = e^x$ and $f'(x) = e^x$ both increasing]

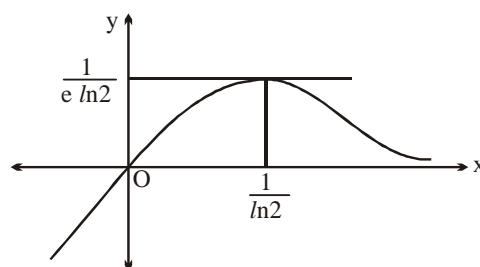
Paragraph for question nos. 26 to 27

(i) We have $f(x) = x 2^{-x}$

So, $f'(x) = 2^{-x} (1 - x \ln 2)$

and $f''(x) = 2^{-x} \ln 2 (x \ln 2 - 2)$

Clearly, $f(x)$ is increasing in $\left(-\infty, \frac{1}{\ln 2}\right)$ and decreasing in $\left(\frac{1}{\ln 2}, \infty\right)$.



Graph of $f(x) = x 2^{-x}$

(ii) $\in \left(0, \frac{1}{e \ln 2}\right)$.

(iii) Given $f(x) = x 2^{-x}$ and $g(x) = \max. \{f(t) : x \leq t \leq x + 1\}$

As $f(x)$ is increasing in $\left(-\infty, \frac{1}{\ln 2}\right)$, hence maximum value of $g(x)$ occurs at $t = x + 1$

$\therefore g(x) = f(x + 1) = (x + 1) 2^{-(x+1)}$

Let $I = \int_0^{\frac{1}{\ln 2} - 1} g(x) dx = \int_0^{\frac{1}{\ln 2} - 1} \underbrace{(x+1)}_I \underbrace{2^{-(x+1)}}_II dx$
 (I.B.P.)

$$= -\frac{(x+1) 2^{-(x+1)}}{\ln 2} \Big|_0^{\frac{1}{\ln 2} - 1} + \frac{1}{\ln 2} \int_0^{\frac{1}{\ln 2} - 1} 2^{-(x+1)} dx$$

$$= -\frac{(x+1) 2^{-(x+1)}}{\ln 2} \Big|_0^{\frac{1}{\ln 2} - 1} - \frac{1}{\ln^2 2} 2^{-(x+1)} \Big|_0^{\frac{1}{\ln 2} - 1}$$

$$= -\frac{1}{\ln 2} \left[\frac{1}{\ln 2} \frac{1}{e} - \frac{1}{2} \right] - \frac{1}{\ln^2 2} \left[\frac{1}{e} - \frac{1}{2} \right] = -\frac{1}{e \ln^2 2} + \frac{1}{2 \ln 2} - \frac{1}{e \ln^2 2} + \frac{1}{2 \ln^2 2}$$

$$= \frac{1}{2 \ln^2 2} + \frac{1}{2 \ln 2} - \frac{2}{e \ln^2 2} \text{ Ans.]}$$

Paragraph for question nos. 29 to 31

(1) $\lim_{x \rightarrow 0^+} x \ln \left(1 + \frac{1}{x}\right) = \lim_{x \rightarrow 0^+} \frac{\ln \left(\frac{x+1}{x}\right)}{\frac{1}{x}} \left(\frac{\infty}{\infty}\right)$

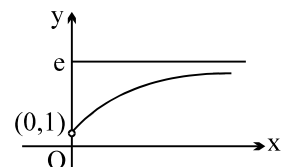
Using L'Hospital's Rule

$$l = \lim_{x \rightarrow 0} -\left(\frac{1}{x+1} - \frac{1}{x}\right) x^2 = \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{x+1}\right) \cdot x^2 = \lim_{x \rightarrow 0} \frac{1}{x(x+1)} \cdot x^2 = \lim_{x \rightarrow 0} \frac{x}{(x+1)} = 0 \text{ Ans.}$$

(2) $\lim_{x \rightarrow 0} f(x) = 1$ (can be verified)

$\lim_{x \rightarrow \infty} f(x) = e$

Also f is increasing for all $x > 0 \Rightarrow$ (D) (can be verified)



(3) $l = \left(\prod_{k=1}^n \left(1 + \frac{n}{k}\right)^{k/n} \right)^{1/n}$ { given $f(x) = (1 + 1/x)^x$ and $f(k/n) = \left(1 + \frac{n}{k}\right)^{k/n}$ }

taking log,

$$\ln l = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^n \ln \left(1 + \frac{n}{k}\right)^{k/n} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{k=1}^n \frac{k}{n} \ln \left(1 + \frac{1}{k/n}\right) dx$$

$$= \int_0^1 \underbrace{\ln \left(1 + \frac{1}{x}\right)}_I dx = \ln \left(1 + \frac{1}{x}\right) \cdot \frac{x^2}{2} \Big|_0^1 + \int_0^1 \left(\frac{1}{x} - \frac{1}{x+1}\right) \cdot \frac{x^2}{2} dx$$

$$= \left(\frac{1}{2} \ln 2 - 0 \right) + \frac{1}{2} \int_0^1 \frac{x+1-1}{x+1} dx = \frac{1}{2} \ln 2 + \frac{1}{2} [x - \ln(x+1)]_0^1$$

$$= \frac{1}{2} \ln 2 + \frac{1}{2} [(1 - \ln 2) - 0] = \frac{1}{2}$$

$$l = \sqrt{e}$$

Paragraph for question nos. 32 to 34

$$y = \frac{x^2}{x^2 - 1}; \text{ not defined at } x = \pm 1$$

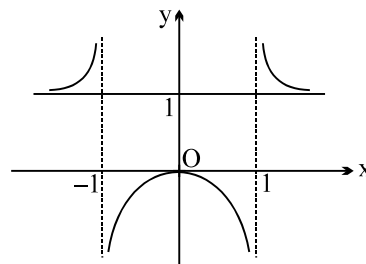
$$= 1 + \frac{1}{x^2 - 1}; \quad y' = -\frac{2x}{(x^2 - 1)^2}$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0 \text{ (point of maxima)}$$

as $x \rightarrow 1^+, y \rightarrow \infty$; $x \rightarrow 1^-, y \rightarrow -\infty$
 ||ly $x \rightarrow -1^+, y \rightarrow -\infty$; $x \rightarrow -1^-, y \rightarrow \infty$

The graph of $y = \frac{x^2}{x^2 - 1}$ is as shown

verify all alternatives from the graph.



Paragraph for question nos. 35 to 37

(i) $a = 1$
 $f(x) = 8x^3 + 4x^2 + 2bx + 1$
 $f'(x) = 24x^2 + 8x + 2b = 2(12x^2 + 4x + b)$
 for increasing function, $f'(x) \geq 0 \quad \forall x \in \mathbb{R}$
 $\therefore D \leq 0 \Rightarrow 16 - 48b \leq 0$

$$\Rightarrow b \geq \frac{1}{3} \Rightarrow \text{(C)}$$

(ii) if $b = 1$
 $f(x) = 8x^3 + 4ax^2 + 2x + a$
 $f'(x) = 24x^2 + 8ax + 2$ or $2(12x^2 + 4ax + 1)$
 for non monotonic $f'(x) = 0$ must have distinct roots
 hence $D > 0$ i.e. $16a^2 - 48 > 0 \Rightarrow a^2 > 3$;
 $\therefore a > \sqrt{3}$ or $a < -\sqrt{3}$
 $\therefore a \in 2, 3, 4, \dots$
 sum = 5050 - 1 = 5049 Ans.

(iii) If x_1, x_2 and x_3 are the roots then $\log_2 x_1 + \log_2 x_2 + \log_2 x_3 = 5$
 $\log_2(x_1 x_2 x_3) = 5$
 $x_1 x_2 x_3 = 32$
 $-\frac{a}{8} = 32$
 $\Rightarrow a = -256$ Ans.]

38. (A) R; (B) R, S, T; (C) Q; (D) Q

$$(A) \quad I = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{(x^2+1)^2 - (x^2-1)}{(x^2+1)^2} dx = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \left(1 - \frac{(x^2-1)}{(x^2+1)^2} \right) dx = 2 - \underbrace{\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{(x^2-1)}{(x^2+1)^2} dx}_{I_1}$$

$$I_1 = \int_{1/a}^a \frac{(x^2-1)}{(x^2+1)^2} dx \quad \text{where } (a = \sqrt{2}+1); \quad \text{put } x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$$

$$= \int_a^{1/a} \frac{\frac{1}{t^2}-1}{\left(\frac{1}{t^2}+1\right)^2} \cdot \left(-\frac{1}{t^2}\right) dt = - \int_a^{1/a} \frac{(1-t^2)t^4}{t^4(1+t^2)^2} dt = - \int_a^{1/a} \frac{(1-t^2)}{(1+t^2)^2} dt = \int_a^{1/a} \frac{t^2-1}{(t^2+1)^2} dt$$

$$= - \int_{1/a}^a \frac{t^2-1}{(t^2+1)^2} dt = -I_1$$

$$\Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0 \Rightarrow 2 \text{ is the answer.}]$$

(B) Domain of $f(x)$ is $(0, 1) \cup (1, \infty)$

$$\ln f(x) = 1 \Rightarrow f(x) = e = \text{constant}$$

$$f'(x) = 0, \text{ for all in } (0, \infty) - \{1\}$$

(C) Clearly $(1, 0)$ is the point of intersection of given curves.

$$\text{Now, } f'(x) = \frac{2^x}{x} + 2^x (\ln 2) (\ln x)$$

$$\therefore \text{ Slope of tangent to the curve } f(x) \text{ at } (1, 0) = m_1 = 2$$

$$\text{Similarly, } g'(x) = \frac{d}{dx} (e^{2x \ln x} - 1) = x^{2x} \left(2x \times \frac{1}{x} + 2 \ln x \right)$$

$$\therefore \text{ Slope of tangent to the curve } g(x) \text{ at } (1, 0) = m_2 = 2$$

since $m_1 = m_2 = 2$

$$\Rightarrow \text{ Two curves touch each other, so angle between them is } 0.$$

$$\text{Hence } \cos \theta = \cos 0 = 1$$

$$(D) \quad 3y^2y' - 3y - 3xy' = 0 \Rightarrow y' = \frac{y}{y^2 - x}$$

$$y' = 0 \Rightarrow y = 0, \text{ no real } x.$$

$$y' = \infty \Rightarrow y^2 = x \Rightarrow y^3 = 1, y = 1$$

The point is $(1, 1)$

39. (A) \rightarrow R, (B) \rightarrow Q, (C) \rightarrow P, (D) \rightarrow S

$$(A) \quad \frac{dy}{dx} = \frac{4t}{3}, \quad \text{Tangent is } y - at^4 = \frac{4t}{3}(x - at^3)$$

$$\text{x-intercept} = \frac{at^3}{4}$$

$$\text{y-intercept} = -\frac{at^4}{3}$$

the point of intersection of tangent with the axes are $\left(\frac{at^3}{4}, 0\right)$ and $\left(0, -\frac{at^4}{3}\right)$

$A\left(0, -\frac{at^4}{3}\right)$ $B\left(\frac{at^3}{4}, 0\right)$ $P(at^2, at^4)$

P divides AB externally in 4 : 3

$$\therefore \frac{m}{n} = \frac{4}{3} \quad \Rightarrow \quad m = 4 \text{ \& } n = 3$$

as m & n are coprime to each other

$$\therefore m + n = 7$$

(B) $\frac{dx}{dy} = e^{\sin y} \cos y$: slope of normal = -1

equation of normal is $x + y = 1$

$$\text{Area} = \frac{1}{2}$$

(C) $y = \frac{1}{x^2}$: $\frac{dy}{dx} = -\frac{1}{x^3}$: slope of tangent = -2

$$y = e^{2-2x} : \frac{dy}{dx} = e^{2-2x} \cdot (-2) : \text{slope of tangent} = -2$$

$$\therefore \tan_{\theta} = 0$$

(D) Length of subtangent = $\left| \frac{y}{y'} \right| = \left| \frac{be^{x/3}}{b \frac{1}{2} e^{x/3}} \right| = 3$

40 (A) \rightarrow (q), (B) \rightarrow (r), (C) \rightarrow (q), (D) \rightarrow (q)

$$y = f(x) + \frac{1}{x} \text{ so } y > 0$$

$$\text{Now } f(y) f\left(f(y) + \frac{1}{y}\right) = 1 \text{ also } f(x) f(y) = 1$$

$$\therefore f(x) = f\left(f(y) + \frac{1}{y}\right) = f\left(\frac{1}{f(x)} + \frac{1}{f(x) + 1/x}\right)$$

also f(x) is increasing

$$\therefore x = \frac{1}{f(x)} + \frac{1}{f(x) + 1/x}$$

$$\Rightarrow f(x) = \frac{1 \pm \sqrt{5}}{2x}$$

now $f(x) = \frac{1 + \sqrt{5}}{2x}$ is decreasing so discarding it $f(x) = \frac{1 - \sqrt{5}}{2x}$.

Applications Of Derivatives

Exercise 2(B)

- 1 Given $S = x^2 + 4xh = 1200$
and $V = x^2h$

$$V(x) = \frac{x^2(1200 - x^2)}{4x}; \quad V(x) = \frac{1}{4}(1200x - x^3)$$

Put $V'(x) = 0$ gives $x = 20$

If $x = 20, h = 10$

Hence $V_{\max.} = x^2h = (400)(10) = 4000$ cubic cm.

- 2 Note that C_1 is a semicircle and C_2 is a rectangular hyperbola.
PQ will be minimum if the normal at P on the semicircle is also a normal at Q on $xy = 9$

Let the normal at P be $y = mx$ (1) ($m > 0$)

solving it with $xy = 9$

$$mx^2 = 9 \Rightarrow x = \frac{3}{\sqrt{m}}; y = \frac{9\sqrt{m}}{3}$$

$$\therefore Q \equiv \left(\frac{3}{\sqrt{m}}, 3\sqrt{m} \right)$$

differentiating $xy = 9$

$$x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore \left. \frac{dy}{dx} \right|_Q = -\frac{3\sqrt{m} \cdot \sqrt{m}}{3} = -m$$

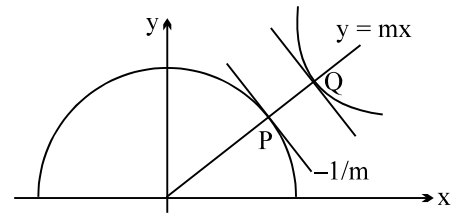
\therefore tangent at P and Q must be parallel

$$\therefore -m = -\frac{1}{m} \Rightarrow m^2 = 1 \Rightarrow m = 1$$

\therefore normal at P and Q is $y = x$

solving P(1, 1) and Q(3, 3)

$$\therefore (PQ)^2 = d^2 = 4 + 4 = 8 \text{ Ans.]}$$



- 3 The given expression resembles with $(x_1 - x_2)^2 + (y_1 - y_2)^2$, where $y_1 = \frac{x_1^2}{20}$ and

$$y_2 = \sqrt{(17 - x_2)(x_2 - 13)}$$

Thus, we can think about two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the curves $x^2 = 20y$ and $(x - 15)^2 + y^2 = 4$ respectively.

Let D be the distance between P_1 and P_2 then the given expression simply represents D^2 .

Now, as per the requirements, we have to locate the point on these curves (in the first quadrant) such that the distance between them is minimum.

Since the shortest distance between two curves always occurs along the common normal, it implies that we have to locate a point $P(x_1, y_1)$ on the parabola $x^2 = 20y$ such that normal drawn to parabola at this point passes through (15, 0).

Now, equation of the normal to the parabola at (x_1, y_1) is $\left(y - \frac{x_1^2}{20} \right) = \frac{-10}{x_1}(x - x_1)$. It should pass through (15, 0).

$$\Rightarrow x_1^3 + 200x_1 - 3000 = 0 \Rightarrow x_1 = 10 \Rightarrow y_1 = 5$$

$$\Rightarrow D = \sqrt{(10-15)^2 + 5^2} - 2 = (5\sqrt{2} - 2)$$

The minimum value of the given expression is $(5\sqrt{2} - 2)^2 = (a\sqrt{2} - b)^2$

$$\therefore a = 5 \text{ \& } b = 2$$

4 $x = t^2 ; y = t^3$

$$\frac{dx}{dt} = 2t ; \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{3t}{2}$$

$$y - t^3 = \frac{3t}{2} (x - t^2) \quad \dots(1)$$

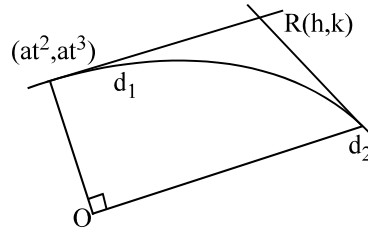
$$2k - 2t^3 = 3th - 3t^3$$

$$t^3 - 3th + 2k = 0$$

$$t_1 t_2 t_3 = -2k \quad (\text{put } t_1 t_2 = -1); \quad \text{hence } t_3 = 2k$$

now t_3 must satisfy the equation (1) which gives $4y^2 = 3x - 1$.

Comparing with $ay^2 = bx - 1$, we have $a = 4$ and $b = 3$.



5 We have
$$F(x) = \begin{cases} -2x + \log_{1/2}(k^2 - 6k + 8), & -2 \leq x < -1 \\ x^3 + 3x^2 + 4x + 1, & -1 \leq x \leq 3 \end{cases}$$

Also $F(x)$ is increasing on $[-1, 3]$ because $F'(x) > 0 \forall x \in [-1, 3]$.

And $F'(x) = -2 \forall x \in [-2, -1)$, so $F(x)$ is decreasing on $[-2, -1)$.

\therefore If $F(x)$ has smallest value at $x = -1$, then we must have

$$\lim_{h \rightarrow 0} F(-1-h) \geq F(-1)$$

$$\Rightarrow 2 + \log_{1/2}(k^2 - 6k + 8) \geq -1 \Rightarrow \log_{1/2}(k^2 - 6k + 8) \geq -3 \Rightarrow k^2 - 6k + 8 \leq 8$$

$$\Rightarrow k^2 - 6k \leq 0 \Rightarrow k \in [0, 6] \quad \dots(1)$$

But in order to define $\log_{1/2}(k^2 - 6k + 8)$,

We must have $k^2 - 6k + 8 > 0$

$$\Rightarrow (k-2)(k-4) > 0 \Rightarrow k < 2 \text{ or } k > 4 \quad \dots(2)$$

\therefore From (1) and (2), we get $k \in [0, 2) \cup (4, 6]$

\Rightarrow Possible integer(s) in the range of k are 0, 1, 5, 6

Hence the sum of all possible positive integer(s) in the range of $k = 1 + 5 + 6 = 12$ **Ans.]**

6 We have
$$F(x) = \frac{x^3}{3} + (a-3)x^2 + x - 13$$

\therefore For $F(x)$ to have negative point of local minimum, the equation $F'(x) = 0$ must have two distinct negative roots.

$$\text{Now, } F'(x) = x^2 + 2(a-3)x + 1$$

\therefore Following condition(s) must be satisfied simultaneously.

(i) Discriminant > 0 ; (ii) Sum of roots < 0 ; (iii) Product of roots > 0

Now, $D > 0$

$$\Rightarrow 4(a-3)^2 > 4 \Rightarrow (a-3)^2 - 1 > 0 \Rightarrow (a-2)(a-4) > 0$$

$$\therefore a \in (-\infty, 2) \cup (4, \infty) \quad \dots(i)$$

$$\text{Also } -2(a-3) < 0 \Rightarrow a-3 > 0 \Rightarrow a > 3 \quad \dots(ii)$$

And product of root(s) = $1 > 0 \forall a \in \mathbb{R}$

$$\therefore (i) \cap (ii) \cap (iii) \Rightarrow a \in (4, \infty) \quad \dots(iii)$$

Hence sum of value(s) of $a = 5 + 6 + 7 + \dots + 100 = 5040$ **Ans.]**

7. Consider $y = x + \frac{1}{x} - 3$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2} = 0$$

$$\therefore \frac{dy}{dx} = 0 \Rightarrow x = 1 \text{ or } -1$$

As $x \rightarrow 0^+$, $y \rightarrow \infty$ and $x \rightarrow 0^-$, $y \rightarrow -\infty$

Also roots of $x + \frac{1}{x} - 3 = 0 \Rightarrow x^2 - 3x + 1 = 0$

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

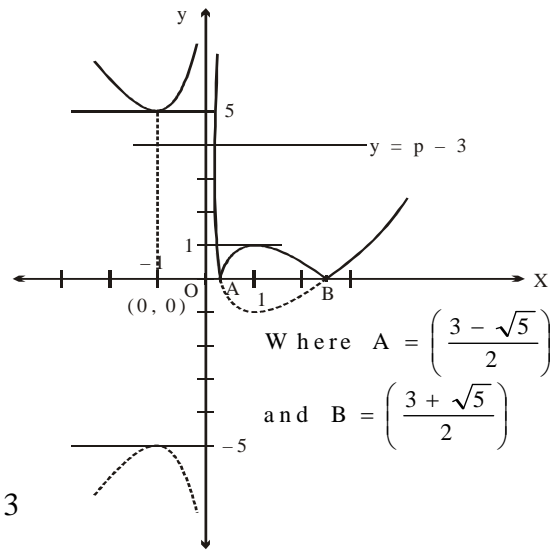
For two distinct solutions either $p - 3 = 0 \Rightarrow p = 3$

or $1 < p - 3 < 5$

$4 < p < 8$

Hence $p \in \{3\} \cup (4, 8)$

$p = \{3, 5, 6, 7\} \Rightarrow \text{Sum} = 21 \text{ Ans.]}$



8. Let $f''(x) = 6a(x - 1)$ ($a > 0$) then $f'(x) = 6a\left(\frac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b$.

Now $f'(-1) = 0 \Rightarrow 9a + b = 0 \Rightarrow b = -9a$.

$\therefore f'(x) = 3a(x^2 - 2x - 3) = 0 \Rightarrow x = -1$ and 3 .

So $y = f(-1)$ and $y = f(3)$ are two horizontal tangents.

Hence distance between its two horizontal tangents = $|f(3) - f(-1)| = |-22 - 10| = 0032$. **Ans.]**

9. Volume $(V) = \frac{1}{3} A_1 h_1 \Rightarrow h_1 = \frac{3V}{A_1}$

||ly $h_2 = \frac{3V}{A_2}$, $h_3 = \frac{3V}{A_3}$ and $h_4 = \frac{3V}{A_4}$

So $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4) = (A_1 + A_2 + A_3 + A_4)\left(\frac{3V}{A_1} + \frac{3V}{A_2} + \frac{3V}{A_3} + \frac{3V}{A_4}\right)$

$$= 3V(A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)$$

Now using A.M.-H.M inequality in A_1, A_2, A_3, A_4 , we get

$$\frac{A_1 + A_2 + A_3 + A_4}{4} \geq \frac{4}{\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)}$$

$$\Rightarrow (A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right) \geq 16$$

Hence the minimum value of $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4) = 3V(16) = 48V = 48 \times 5 = 240$ **Ans.]**

$$10 \quad y = x^2 \text{ and } y = -\frac{8}{x}; \quad q = p^2 \text{ and } s = -\frac{8}{r} \quad \dots(1)$$

Equating $\frac{dy}{dx}$ at A and B, we get

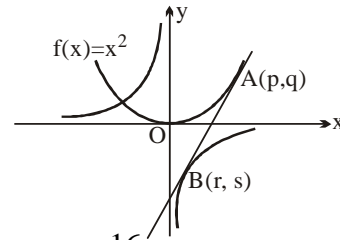
$$2p = \frac{8}{r^2} \quad \dots(1) \Rightarrow pr^2 = 4$$

$$\text{Now } m_{AB} = \frac{q-s}{p-r} \Rightarrow 2p = \frac{p^2 + \frac{8}{r}}{p-r} \Rightarrow p^2 = 2pr + \frac{8}{r} \Rightarrow p^2 = \frac{16}{r}$$

$$\Rightarrow \frac{16}{r^4} = \frac{16}{r} \Rightarrow r = 1 \quad (r \neq 0) \Rightarrow p = 4$$

$$\therefore r = 1, p = 1$$

$$\text{Hence } p + r = 5$$



$$11 \quad x = 0 \text{ and } x = 1]$$

$$12 \quad y = x^2$$

$$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = 10 \text{ m/sec.}$$

$$\tan \theta = \frac{x^2}{x} = x$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = 10 \times \cos^2 \theta = 10 \times \frac{1}{10} = 1 \quad \{ \text{at } x = 3 \text{ m} \}$$

$$13 \quad 3x^2 - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\text{slope of tangent at } (4a^2, 8a^3) = \frac{3(16a^4)}{2(8a^3)} = 3a$$

let this tangent at this point also cuts the curve at $(4b^2, 8b^3)$ and normal at this point slope of

$$\text{normal at } (4b^2, 8b^3) = -\frac{1}{3b}.$$

$$\Rightarrow 3a = -\frac{1}{3b} \Rightarrow ab = -\frac{1}{9} \quad \dots(i)$$

$$\text{slope of line} = \frac{8a^3 - 8b^3}{4a^2 - 4b^2} = \frac{2(a^3 - b^3)}{(a^2 - b^2)} = \frac{2(a^2 + b^2 + ab)}{a + b}$$

$$= 3a \text{ [it is equal to slope of target]}$$

$$\Rightarrow 2a^2 + 2b^2 + 2ab = 3a^2 + 3ab$$

$$\Rightarrow 2b^2 = a^2 + ab \Rightarrow \frac{2}{81a^2} = \frac{a^2 - 1}{9}$$

$$2 = 81a^4 - 9a^2$$

$$\Rightarrow 81a^4 - 9a^2 - 2 = 0$$

$$81a^4 - 18a^2 + 9a^2 - 2 = 0$$

$$9a^2(9a^2 - 2) + (9a^2 - 2) = 0$$

$$\Rightarrow (9a^2 - 2)(9a^2 + 1) = 0$$

$$9a^2 = 2$$

14. Let $x = r \cos \theta$ and $y = r \sin \theta$

$$\Rightarrow r^2 = x^2 + y^2; \quad \tan \theta = \frac{y}{x} \quad \theta \in (0, \pi/2)$$

$$N = \frac{r^2}{r^2[\cos^2 \theta + \sin \theta \cos \theta + 4 \sin^2 \theta]} = \frac{r^2}{(1 + \cos 2\theta) + \sin 2\theta + 4(1 - \cos 2\theta)} = \frac{2}{5 + \sin 2\theta + 3 \cos 2\theta}$$

$$N_{\max} = \frac{2}{5 - \sqrt{10}} = \frac{2}{15}(5 + \sqrt{10}) = M$$

$$N_{\min} = \frac{2}{5 + \sqrt{10}} = \frac{2}{15}(5 - \sqrt{10}) = m$$

$$A = \frac{M+m}{2} = \frac{2 \cdot 10}{15 \cdot 2} = \frac{2}{3} \Rightarrow 2007 \times \frac{2}{3} = 1338 \text{ Ans.]}$$

15. $\frac{f(3)}{f(6)} = \frac{2^{3k} + 9}{2^{6k} + 9} = \frac{1}{3}; \quad f(9) - f(3) = (2^{9k} + 9) - (2^{3k} + 9) = 2^{9k} - 2^{3k} \quad \dots(1)$

$$\Rightarrow 3(2^{3k} + 9) = 2^{6k} + 9$$

$$2^{6k} - 3(2^{3k}) - 18 = 0$$

$$2^{3k} = y$$

$$y^2 - 3y - 18 = 0$$

$$(y - 6)(y + 3) = 0$$

$$y = 6; \quad y = -3 \text{ (rejected)}$$

$$2^{3k} = 6$$

now $f(9) - f(3) = 2^{9k} - 2^{3k} \quad \{ \text{from (1)} \}$

$$= (2^{3k})^3 - 2^{3k}$$

$$= 6^3 - 6 = 210$$

hence $N = 210 = 2 \cdot 3 \cdot 5 \cdot 7$

Total number of divisor = $2 \cdot 2 \cdot 2 \cdot 2 = 16$

number of divisors which are composite = $16 - (1, 2, 3, 5, 7) = 11 \text{ Ans.]}$

16. $f(-3) = f(3) = 2 \quad [f(x) \text{ is an even function, } \therefore f(-x) = f(x)]$

again $f(-1) = f(1) = -3$

$$\therefore 2|f(-1)| = 2|f(1)| = 2|-3| = 6$$

from the graph, $-3 < f\left(\frac{7}{8}\right) < -2$

$$\therefore \left[f\left(\frac{7}{8}\right) \right] = -3$$

$$f(0) = 0 \quad (\text{obviously from the graph})$$

$$\begin{aligned}\cos^{-1}(f(-2)) &= \cos^{-1}(f(2)) = \cos^{-1}(1) = 0 \\ f(-7) &= f(-7+8) = f(1) = -3 \quad [f(x) \text{ has period } 8] \\ f(20) &= f(4+16) = f(4) = 3 \quad [f(nT+x) = f(x)] \\ \text{sum} &= 2 + 6 - 3 + 0 + 0 - 3 + 3 \\ \therefore \text{sum} &= 5\end{aligned}$$

17 We have $f(x) = (b^2 - 3b + 2)(\cos^2 x - \sin^2 x) + (b-1)x + \sin 2x$
 $\therefore f'(x) = (b-1)(b-2)(-2\sin 2x) + (b-1)$
 Now, $f'(x) \neq 0$ for every $x \in \mathbb{R}$,
 so $(b-1)(1-2(b-2)\sin 2x) \neq 0 \forall x \in \mathbb{R}$
 $\therefore b \neq 1$

Also, $\left| \frac{1}{2(b-2)} \right| > 1 \Rightarrow b \in \left(\frac{3}{2}, 2 \right) \cup \left(2, \frac{5}{2} \right)$

Now, when $b = 2$, $f(x) = x + \sin 2x \Rightarrow f'(x) = 1 (\neq 0)$.

Hence, $b \in \left(\frac{3}{2}, \frac{5}{2} \right) \Rightarrow b_1 = \frac{3}{2}$ and $b_2 = \frac{5}{2}$

$\Rightarrow (b_1 + b_2) = \frac{3}{2} + \frac{5}{2} = \frac{8}{2} = 4$

18. Let $x = r \cos \theta$ and $y = r \sin \theta$
 $E = (x+5)(y+5) = (r \cos \theta + 5)(r \sin \theta + 5) = r^2 \sin \theta \cos \theta + 5r(\cos \theta + \sin \theta) + 25$
 Now put $x = r \cos \theta$ and $y = r \sin \theta$ in $x^2 + xy + y^2 = 3$
 $\Rightarrow r^2 \cos^2 \theta + r^2 \sin \theta \cos \theta + r^2 \sin^2 \theta = 3$

$\Rightarrow r^2(1 + \sin \theta \cos \theta) = 3 \Rightarrow r^2 = \frac{3}{1 + \sin \theta \cos \theta} = \frac{6}{2 + \sin 2\theta}$

hence $r^2]_{\min.} = 2 + \sin 2\theta]_{\max.} = 3$ occurs at $\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\frac{5\pi}{2}$ i.e. $\frac{\pi}{4}$ or $\frac{5\pi}{4}$

Hence $E = \frac{r^2}{2}(\sin 2\theta) + 5r(\cos \theta + \sin \theta) + 25$

put $r^2 = 2$ and $\theta = \frac{\pi}{4} \Rightarrow E = 1 + 5\sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + 25 = 36$

put $r^2 = 2$ and $\theta = \frac{5\pi}{4} \Rightarrow E = 1 + 5\sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) + 25 = 16$

hence minimum value of E is 16

19. Using LMVT for f in $[1, 2]$

$\forall c \in (1, 2) \quad \frac{f(2) - f(1)}{2-1} = f'(c) \leq 2$

$f(2) - f(1) \leq 2 \Rightarrow f(2) \leq 4 \dots(1)$

again using LMVT in $[2, 4]$

$\forall d \in (2, 4) \quad \frac{f(4) - f(2)}{4-2} = f'(d) \leq 2$

$\therefore f(4) - f(2) \leq 4$

$$\begin{aligned} 8 - f(2) &\leq 4 \\ 4 &\leq f(2) \quad \Rightarrow \quad f(2) \geq 4 \quad \dots(2) \\ \text{from (1) and (2)} \quad f(2) &= 4 \end{aligned}$$

20. Let x tree be added then

$$P(x) = (x + 50)(800 - 10x)$$

$$\text{now } P'(x) = 0 \quad \Rightarrow \quad x = 15$$

APPLICATIONS OF DERIVATIVES

EXERCISE 2(C)

1. $y = ax^3 + bx^2 + cx + 5$

It has repeated root (-2)

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$y(-2) = 0$$

$$\therefore -8a + 4b - 2c + 5 = 0$$

$$y'(-2) = 0$$

$$\therefore 12a - 4b + c = 0$$

$$y'(0) = 3$$

$$\therefore c = 3$$

$$\therefore 12a - 4b + 3 = 0 \quad \text{and} \quad -8a + 4b - 1 = 0$$

$$\therefore a = -\frac{1}{2} \text{ and } b = -\frac{3}{4}, c = 3$$

2. $xy = 4, x^2 + y^2 = 8$

$$m_1 = -\frac{y}{x}, m_2 = -\frac{x}{y}$$

Clearly curves intersect at $(\pm 2\sqrt{2}, \pm 2\sqrt{2})$

Here, $m_1 = m_2$

\therefore Curves touch each other.

3. $y = \cos(x + y)$

$$\therefore \frac{dy}{dx} = -\sin(x + y) \left[1 + \frac{dy}{dx} \right]$$

$$\therefore \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2} \quad \dots\dots[\text{Given}]$$

$$\therefore 2 \sin(x + y) = 1 + 1 \sin(x + y)$$

$$\therefore \sin(x + y) = 1$$

$$\therefore x + y = (4n + 1) \frac{\pi}{2} \quad \text{and} \quad y = \cos\left(\frac{(4n + 1)\pi}{2}\right) = 0$$

$$\therefore x = \frac{(4n + 1)\pi}{2}$$

But $-2\pi \leq x \leq 2\pi$

$$\therefore x = \frac{-3\pi}{2} \quad \text{or} \quad \frac{\pi}{2}$$

$$\therefore \text{Equation of tangents are } x + 2y = \frac{\pi}{2} \text{ and } x + 2y = \frac{-3\pi}{2}$$

4. $x^2 = 4y$

$$\frac{dy}{dx} = \frac{x}{2}$$

$$\therefore -\frac{dx}{dy} = -\frac{2}{x} = m_N$$

$$\therefore m_N = -2$$

$$y - 2 = -2(x - 1)$$

$$\therefore 2x + y = 4$$

5. Given curve is $y = \sin x$...(1)

$$\therefore \frac{dy}{dx} = \cos x$$

Equation of tangent at (x, y) is $Y - y = \frac{dy}{dx}(X - x)$

or $Y - y = \cos x(X - x)$...(2)

Since tangent are drawn from origin then origin $(0, 0)$ lies on (2)

$$\therefore 0 - y = \cos(0 - x) \quad \text{or} \quad \cos x = \frac{y}{x} \quad \dots(2)$$

From (1) and (2),

$$\sin^2 x + \cos^2 x = y^2 + \frac{y^2}{x^2}$$

$$\Rightarrow 1 = \frac{x^2 y^2 + y^2}{x^2} \quad \Rightarrow \quad x^2 y^2 = x^2 - y^2.$$

6. $\tan \theta = n a^{1-n} x^{n-1}$

$$\begin{aligned} \therefore SN &= y \tan \theta \\ &= n a^{2-2n} x^{2n-1} \\ &= \text{Constant} \end{aligned}$$

$$\therefore n = \frac{1}{2}$$

7. Given $x^{m+n} = a^{m-n} \cdot y^{2n}$...(1)

Taking logarithm of both sides, we get $(m + n) \ln x = (m - n) \ln a + 2n \ln y$

Differentiating of both sides w.r.t. x , we get $\frac{(m+n)}{x} = 0 + \frac{2n}{y} \frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{(m+n)}{2n} \cdot \frac{y}{x}$$

$$\begin{aligned} \text{Now } \frac{(\text{Sub-tangent})^m}{(\text{Sub-normal})^n} &= \frac{\left(y \frac{dx}{dy}\right)^m}{\left(y \frac{dy}{dx}\right)^n} = \frac{y^{m-n}}{\left(\frac{dy}{dx}\right)^{m+n}} \\ &= \frac{y^{m-n}}{\left\{\frac{(m+n)}{2n} \cdot \frac{y}{x}\right\}^{m+n}} = \frac{x^{m+n}}{\left(\frac{m+n}{2n}\right)^{m+n} \cdot y^{2n}} = \frac{a^{m-n}}{\left(\frac{m+n}{2n}\right)^{m+n}} \quad \{\text{from (1)}\} \end{aligned}$$

$$(\text{Sub-tangent})^m \propto (\text{Sub-normal})^n$$

8. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

$$f'(x) = 4x^3 - 24x^2 + 44x - 24 > 0$$

$$\therefore x^3 - 6x^2 + 11x - 6 > 0$$

$$\therefore (x-1)(x-2)(x-3) > 0 \quad \therefore x \in (1,2) \cup (3,\infty)$$

9. $f(x) = x - 2 \sin x$

$$f'(x) = 1 - 2 \cos x > 0$$

$$\therefore \cos x < \frac{1}{2} \text{ in } 0 \leq x \leq 2\pi$$

$$\therefore x \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right).$$

10. $f(x) = 2x^2 - \ln|x|$

$$\therefore f'(x) = 4x - \frac{1}{x}$$

$$= \frac{(2x+1)(2x-1)}{x}$$

$$\text{M.I. in } \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right) \text{ \& M.D. in } \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$$

11. $f(x) = \sin x + \cos x \quad x \in [0, 2\pi]$

$$\therefore f'(x) = \cos x - \sin x$$

$\therefore f(x)$ is M I in $\left[\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)\right]$ & MD in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

12. $g(x) = f(x) + f(1-x) \quad x \in [0, 1]$

$$g'(x) = f'(x) - f'(1-x)$$

$$f''(x) < 0$$

$$\therefore f'(x) > f'(1-x)$$

$$\Rightarrow x < 1-x$$

$$\Rightarrow \text{in } x \in \left[0, \frac{1}{2}\right) \quad g'(x) > 0 \quad \& \quad g'(x) < 0 \text{ in } x \in \left(\frac{1}{2}, 1\right]$$

13. TPT $\ln(1+x) < 0 \quad ; x > 0$

i.e. $x > \ln(1+x)$

Let $f(x) = x - \ln(1+x)$

$$\begin{aligned} \therefore f'(x) &= 1 - \frac{1}{1+x} \\ &= \frac{x}{1+x} > 0 \quad \forall x > 0. \end{aligned}$$

Now, $f(0) = 0$

$$\therefore f(x) > 0 \quad \forall x > 0$$

$$\therefore x > \ln(1+x)$$

14. $ax^2 + \frac{b}{x} \geq c$

Let $f(x) = ax^2 + \frac{b}{x}$

$$\begin{aligned} \therefore f'(x) &= 2ax - \frac{b}{x^2} \\ &= \frac{2ax^3 - b}{x^2} = 0 \text{ at point of minima.} \end{aligned}$$

as $2ax^3 - b$ is an increasing function $\forall a > 0$.

$$\therefore x = \left(\frac{b}{2a}\right)^{\frac{1}{3}}$$

$$\therefore ax^2 + \frac{b}{x} = a \left(\frac{b^2}{4a^2}\right)^{\frac{1}{3}} + b \times \left(\frac{2a}{b}\right)^{\frac{1}{3}}$$

$$= \frac{a^{\frac{1}{3}} b^{\frac{2}{3}}}{4^{\frac{1}{3}}} + 2^{\frac{1}{3}} b^{\frac{2}{3}} a^{\frac{1}{3}}$$

$$= \frac{3 a^{\frac{1}{3}} b^{\frac{2}{3}}}{4^{\frac{1}{3}}} \geq c$$

$\therefore 27ab^2 \geq 4c^3$ [Cubing both sides]
as a,b,c are positive.

15. $0 < x_1 < x_2 < \frac{f}{2}$

T.P. $\frac{\tan x_2}{\tan x_1} > \frac{x_1}{x_2}$

i.e. T.P. $x_2 \tan x_2 > x_1 \tan x_1$

Let $f(x) = x \tan x$

$\therefore f'(x) = \tan x + x \sec^2 x > 0 \quad \forall x \in \left(0, \frac{\pi}{2}\right)$

as $\tan x$ & $x \sec^2 x$ are positive

16. P.T. $x > \sin x > x - \frac{x^3}{6}$ for $0 < x \leq \frac{\pi}{2}$

Let $f(x) = x - \sin x$

$\therefore f'(x) = 1 - \cos x > 0$ for $0 < x \leq \frac{\pi}{2}$

$\therefore x > \sin x$

Let $g(x) = \sin x - x + \frac{x^3}{6}$

$\therefore g'(x) = \cos x - 1 + \frac{x^2}{2}$

$\therefore g''(x) = x - \sin x > 0$ for $0 < x \leq \frac{\pi}{2}$ as shown

$\therefore g'(x) = 0$ for $x = 0$ & increases as x increases

$\therefore g'(x) > 0$ for $0 < x \leq \frac{\pi}{2}$

$g(x) = 0$ for $x = 0$

$\therefore g(x) > 0$ for $0 < x \leq \frac{\pi}{2}$

$$\therefore \sin x > x - \frac{x^3}{6}$$

$$17. f(x) = \frac{1}{8} \ln x - bx + x^2$$

$$\therefore f'(x) = 2x - b + \frac{1}{8x} = 0$$

$$\therefore 2x^2 - bx + 1 = 0$$

$$\therefore x = \frac{b \pm \sqrt{b^2 - 4}}{4} \quad \text{If } b > 2$$

$$f''(x) = 2 - \frac{1}{8x} > 0 \text{ if } x = \frac{b + \sqrt{b^2 - 4}}{4} \text{ and}$$

$$< 0 \text{ if } x = \frac{b - \sqrt{b^2 - 4}}{4}$$

$$\therefore \text{It has maxima at } x = \frac{b - \sqrt{b^2 - 4}}{4} \text{ and}$$

$$\text{minima at } x = \frac{b + \sqrt{b^2 - 4}}{4}.$$

$$18. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{Equation of tangent is } \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

$$x = 0 \Rightarrow y = b \operatorname{cosec} \theta \quad \text{and}$$

$$y = 0 \Rightarrow x = a \sec \theta$$

$$\text{Now, length of intercept} = \sqrt{(a \sec \theta - 0)^2 + (b \operatorname{cosec} \theta - 0)^2} = \sqrt{a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta}$$

Which has minimum value $a + b$ is

$$a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x = f(x)$$

$$f'(x) = 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x = 0$$

$f(x)$ can have minima only as maxima is ∞ .

$$\therefore a^2 \sec^2 x \tan x = b^2 \operatorname{cosec}^2 x \cot x$$

$$\therefore a^2 \frac{\sin x}{\cos^3 x} = b^2 \frac{\cos x}{\sin^3 x}$$

$$\therefore \tan^4 x = \frac{b^2}{a^2}$$

$$\therefore \tan^2 x = \left| \frac{b}{a} \right|, \quad \cot^2 x = \left| \frac{a}{b} \right|$$

For $a, b > 0$

$$\sec^2 x = \frac{a+b}{a}, \operatorname{cosec}^2 x = \frac{a+b}{b}$$

$$\begin{aligned} \therefore a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x \\ = a^2 + ab + ab + b^2 = (a+b)^2 \end{aligned}$$

$$\begin{aligned} 19. S &= \pi r \sqrt{r^2 + b^2} + \pi r^2 \\ &= \pi r^2 \operatorname{cosec} \alpha + \pi r^2 = \text{constant} \end{aligned}$$

$$V = \frac{1}{3} \pi r^3 \cot \alpha \quad \therefore r = \sqrt{\frac{S}{\pi(\operatorname{cosec} \alpha + 1)}}$$

$$= \frac{1}{3} \frac{\pi S^{\frac{3}{2}} \cot \alpha}{\pi^2 (\operatorname{cosec} \alpha + 1)^{\frac{3}{2}}}$$

$$\begin{aligned} V &= \sqrt{\frac{S^3}{9\pi}} \times \frac{(\cos \alpha) \sqrt{\sin \alpha}}{(1 + \sin \alpha)^{\frac{3}{2}}} \\ &= \frac{(\cos \alpha) \sqrt{\sin \alpha} (1 - \sin \alpha)^{\frac{3}{2}}}{\cos^3 \alpha} \times \sqrt{\frac{S^3}{9\pi}} \\ &= \sqrt{\sin \alpha} (1 - \sin \alpha)^{\frac{3}{2}} \sec^2 \alpha \times \sqrt{\frac{S^3}{9\pi}} \end{aligned}$$

$$\begin{aligned} \therefore \frac{dV}{d\alpha} &= \sqrt{\frac{S^3}{9\pi}} \left[\frac{\cos \alpha (1 - \sin \alpha)^{\frac{3}{2}} \sec^2 \alpha + 2\sqrt{\sin \alpha} (1 - \sin \alpha)^{\frac{3}{2}} \sec^2 \alpha \tan \alpha}{2\sqrt{\sin \alpha}} \right. \\ &\quad \left. - \frac{3\sqrt{\sin \alpha} \sec^2 \alpha (1 - \sin \alpha)^{\frac{1}{2}} \times \cos \alpha}{2} \right] = 0 \end{aligned}$$

$$\therefore 1 - \sin \alpha = 0$$

$$\Rightarrow \sin \alpha = 1 \text{ rejected as } \alpha \in \left(0, \frac{\pi}{2}\right)$$

OR

$$\therefore \frac{\cos \alpha (1 - \sin \alpha) \sec^2 \alpha + 2\sqrt{\sin \alpha} (1 - \sin \alpha) \sec^2 \alpha \tan \alpha}{2\sqrt{\sin \alpha}}$$

$$= 3 \frac{\sqrt{\sin \alpha} \sec^2 \alpha \cos \alpha}{2}$$

$$\therefore \frac{\cos \alpha (1 - \sin \alpha) + 2 \sin \alpha (1 - \sin \alpha) \tan \alpha}{2} = \frac{3 \sin \alpha \cos \alpha}{2}$$

$$\therefore \cos \alpha + 4 \sin \alpha \tan \alpha - 4 \sin^2 \alpha \tan \alpha = 4 \sin \alpha \cos \alpha$$

$$\therefore \cos^2 \alpha + 4 \sin^2 \alpha - 4 \sin^3 \alpha = 4 \sin \alpha (1 - \sin^2 \alpha)$$

$$\therefore 3 + 3 \sin^2 \alpha - 4 \sin^3 \alpha = 4 \sin \alpha - 4 \sin^3 \alpha$$

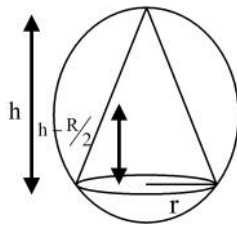
$$\therefore 3 \sin^2 \alpha - 4 \sin \alpha + 3 = 0$$

$$\Rightarrow \sin \alpha = \frac{1}{3} \quad \text{hence proved.}$$

20. Refer (Q.21) Assifnment

$$\text{Ans. } \frac{1}{3} \text{ m}$$

21.



$$r^2 + (h - R)^2 = R^2$$

$$\therefore r^2 = 2Rh - h^2$$

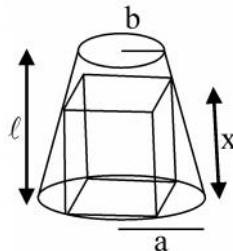
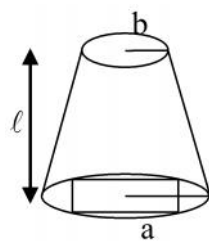
$$\therefore V = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi (2Rh^2 - h^3)$$

$$\therefore \frac{dV}{dh} = \frac{1}{3} (4Rh - 3h^2) = 0$$

$$\therefore h = 0 \text{ Rejected or } h = \frac{4R}{3}$$

22.



Where h is the height of cut off conical portion

$$\frac{b}{h} = \frac{a}{\ell + h} = \frac{s}{\sqrt{2}(h + \ell - x)}$$

$$\therefore s = \frac{\sqrt{2} b (h + \ell - x)}{h}$$

$$V = \frac{2b^2 (h + \ell - x)^2}{h^2} \times x$$

$$= \frac{2b^2}{h^2} (h^2x + \ell^2x + x^3 + 2h\ell x - 2hx^2 - 2\ell x^2)$$

$$\therefore \frac{dV}{dx} = \frac{2b^2}{h^2} (h^2 + \ell^2 + 2h\ell + 3x^2 - 4hx - 4\ell x) = 0$$

$$\therefore 3x^2 - 4(h - \ell)x + (h + \ell)^2 = 0$$

$$\therefore x = h + \ell \text{ rejected or}$$

$$x = \frac{(h + \ell)}{3}$$

$$\text{Now, } bl + bh = ah$$

$$\therefore h = \frac{b\ell}{a - b}$$

$$\therefore x = \frac{1}{3} \left(\frac{b\ell}{a - b} + \ell \right) = \frac{a\ell}{3(a - b)}$$

23. Profit = SP - CP

$$= \frac{n}{2}(100 - n) - \frac{n^2}{4} - 35n - 25$$

$$P = -\frac{3n^2}{4} + 15n - 25$$

$$\therefore \frac{dP}{dn} = -\frac{3n}{2} + 15 = 0$$

$$\therefore n = 10$$

24. $R = (500 - x)(300 + x)$ $x = \text{Surcharge}$

$$\therefore \frac{dR}{dx} = -300 - x + 500 - x = 0$$

$$\therefore x = 100$$

25. $f(x) = |2x - 1| - 2|x - 1| - 3$

$$\therefore f(x) = 1 - 2x + 2x - 2 - 3 = -4, \quad x \leq \frac{1}{2}$$

$$= 2x - 1 + 2x - 2 - 3 = 4x - 6, \quad \frac{1}{2} < x \leq 1$$

$$= 2x - 1 - 2x + 2 - 3 = -2, \quad x > 1$$

$$\therefore \text{Range of } f(x) \text{ is } [-4, -2]$$

26. $x + y = 20$ $y = 20 - x$

$$x^3 y^2 = \max = P$$

$$\therefore P = x^3 (20 - x)^2$$

$$= x^5 - 40x^4 + 400x^3$$

$$\therefore \frac{dP}{dx} = 5x^4 - 160x^3 + 1200x^2 = 0$$

$$\therefore x = 0 \Rightarrow \text{Rejected or}$$

$$5x^2 - 160x + 1200 = 0$$

$$\Rightarrow x^2 - 32x + 240 = 0$$

$$\therefore x = \frac{32 \pm \sqrt{64}}{2} = 20 \text{ or } 12$$

$$x = 20 \text{ rejected}$$

$$\therefore x = 12 \text{ \& } y = 8$$

$$27. f(a)g(b) - f(b)g(a) = (b-a)[f(a)g'(c) - g(a)f'(c)]$$

$$29. f(x) = x(x+3)e^{-x/2} \quad [-3, 0]$$

$$f'(x) = (2x+3)e^{-x/2} - \frac{1}{2}x(x+3)e^{-x/2} = 0$$

$$\text{at } x = c$$

$$\therefore 2(2x+3) = x^2 + 3x$$

$$\therefore x^2 - x - 6 = 0$$

$$\therefore x = -2 \text{ i.e. as } c < 0$$

$$\text{i.e. } c = -2$$

$$30. f(x) = x(x-1)(x-2)$$

$$\therefore f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2} = \frac{3}{8}$$

$$\therefore f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$$

$$f'(c) = \frac{\frac{3}{8}}{\frac{1}{2}} = \frac{3}{4}$$

$$\therefore \frac{3}{4} = 3c^2 - 6c + 2$$

$$\therefore 12c^2 - 24c + 5 = 0$$

$$\therefore c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24} = \frac{6 - \sqrt{21}}{6} \text{ as } c < \frac{1}{2}$$

APPLICATIONS OF DERIVATIVES

EXERCISE 3

1. Since the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at P(-2,0) then x-axis is the tangent

at (-2,0). The curve meets y-axis in (0,5). We have $\frac{dy}{dx} = 3ax^2 + 2bx + c$

$$\therefore \left. \frac{dy}{dx} \right|_{(0,5)} = 0 + 0 + c = 3 \quad (\text{given})$$

$$\therefore c = 3 \quad \dots(1)$$

$$\text{and } \left. \frac{dy}{dx} \right|_{(-2,0)} = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$$

$$\Rightarrow 12a - 4b + 3 = 0 \quad \{\text{From (1)}\} \quad \dots(2)$$

and (-2,0) lies on the curve then

$$0 = -8a + 4b - 2c + 5 \quad \Rightarrow \quad 0 = -8a + 4b - 1 \quad (\because c = 3)$$

$$\Rightarrow 8a - 4b + 1 = 0 \quad \dots(3)$$

From (2) and (3) we get $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$.

2. $x^3 - 3xy^2 + 2 = 0$

$$\therefore 3x^2 - 3y^2 - 6xy \frac{dy}{dx} = 0$$

$$\therefore x^2 - y^2 = 2xy \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = m_1$$

$$3x^2y - y^3 + 2 = 0$$

$$\therefore 6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} (x^2 - y^2) = -2xy$$

$$\therefore \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2} = m_2$$

$$m_1 m_2 = -1$$

3. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$$\therefore \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}$$

At (a, b)

$$\frac{dy}{dx} = -\frac{b}{a}$$

$$\therefore b(x-a) + a(y-b) = 0$$

$$\therefore bx + ay = 2ab$$

$$\therefore \frac{x}{a} + \frac{y}{a} = 2.$$

4. $ay^2 = (x+b)^3$

$$2ay \frac{dy}{dx} = 3(x+b)^2$$

$$\frac{dy}{dx} = \frac{3(x+b)^2}{2ay}$$

$$ST = \frac{2ay^2}{3(x+b)^2}, SN = \frac{3(x+b)^2}{2a}$$

$$\frac{(ST)^2}{SN} = \frac{4a^2 y^4}{3(x+b)^4} \times \frac{2a}{3(x+b)^2}$$

$$= \frac{8a(ay^2)^2}{27(ay^2)^2} = \frac{8a}{27}.$$

5. $y = (1+x)^y + \sin^{-1}(\sin^2 x)$

if $x = 0$; $y = 1$

now $y = e^{y \ln(1+x)} + \sin^{-1}(\sin^2 x)$

differentiating $\frac{dy}{dx} = (1+x)^y \left[\frac{y}{1+x} + \ln(1+x) \frac{dy}{dx} \right] + \frac{\sin 2x}{\sqrt{1-\sin^4 x}}$

put $x = 0$ and $y = 1$

$$\frac{dy}{dx} = 1$$

slope of normal = -1

$$y - 1 = -1(x - 0)$$

$$x + y - 1 = 0$$

6. $2y = -5x^5 + 10x^3 - x - 6$

$$\therefore \frac{dy}{dx} = -\frac{25}{2}x^4 + 15x^2 - \frac{1}{2}$$

$$\text{At } x = 0, m_N = -\frac{dx}{dy} = 2$$

$$\therefore y + 3 = 2x$$

$$\therefore y = 2x - 3$$

$$\therefore 4x - 6 = -5x^5 + 10x^3 - x - 6$$

$$\therefore 5x^5 - 10x^3 + 5x = 0$$

$$\therefore x(x^4 - 2x + 1) = 0$$

$$\therefore x(x+1)^2(x-1)^2 = 0$$

$$\therefore x+1 = 0 \quad \text{or} \quad x-1 = 0$$

$$\therefore x = \pm 1$$

$$x = 1 \quad \Rightarrow \quad y = -1$$

$$x = -1 \quad \Rightarrow \quad y = -5$$

$$m_{x=1} = 2 = m_{x=-1}$$

\therefore Equation of tangents are

$$y + 5 = 2(x + 1) \quad \text{and} \quad y + 1 = 2(x - 1)$$

i.e. $y = 2x - 3$ i.e. the normal becomes the tangent.

7. $x^3 + y^3 = a^3$

$$\therefore 3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$\therefore \frac{y - y_1}{x - x_1} = -\frac{x_1^2}{y_1^2}$$

$$\therefore y_1^2 y - y_1^3 = -x_1^2 x + x_1^3$$

$$\therefore y_1^2 y + x_1^2 x = a^3 \quad x_1^3 + y_1^3 = a^3$$

Now, $y_1^2 y_2 + x_1^2 x_2 = y_1^3 + x_1^3$ [Given]

$$\therefore y_1^2 (y_2 - y_1) = x_1^2 (x_1 - x_2)$$

$$x_2^3 + y_2^3 = a^3 \quad \text{By solving}$$

We have,

$$x_1^3 + y_1^3 = a^3$$

$$3x_1^2 x_2 + 3y_1^2 y_2 = 3a^3$$

$$3x_1 x_2^2 + 3y_1 y_2^2 = 3a^3$$

$$x_2^3 + y_2^3 = a^3$$

$$\therefore (x_1 + x_2)^3 + (y_1 + y_2)^3 = (2a)^3$$

8. Given $\frac{x^2}{a^2 + k_1} + \frac{y^2}{b^2 + k_1} = 1$... (1)

and $\frac{x^2}{a^2 + k_2} + \frac{y^2}{b^2 + k_2} = 1$... (2)

Subtracting (2) from (1), we get

$$x^2 \left(\frac{1}{a^2 + k_1} - \frac{1}{a^2 + k_2} \right) + y^2 \left(\frac{1}{b^2 + k_1} - \frac{1}{b^2 + k_2} \right) = 0$$

$$\Rightarrow x^2 \left(\frac{k_2 - k_1}{(a^2 + k_1)(a^2 + k_2)} \right) + y^2 \left(\frac{k_2 - k_1}{(b^2 + k_1)(b^2 + k_2)} \right) = 0$$

$$\therefore \frac{x^2}{y^2} = - \frac{(a^2 + k_1)(a^2 + k_2)}{(b^2 + k_1)(b^2 + k_2)} \quad \dots (3)$$

Now from (1), $\frac{2x}{(a^2 + k_1)} + \frac{2y}{(b^2 + k_1)} \frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = - \frac{x(b^2 + k_1)}{y(a^2 + k_1)} = m_1 \text{ (say)}$$

Similarly from (2), $\frac{dy}{dx} = - \frac{x(b^2 + k_2)}{y(a^2 + k_2)} = m_2 \text{ (say)}$

$$\begin{aligned} \therefore m_1 m_2 &= \frac{x^2 (b^2 + k_1)(b^2 + k_2)}{y^2 (a^2 + k_1)(a^2 + k_2)} && \{\text{From (3)}\} \\ &= -1 \end{aligned}$$

Hence given curves intersect orthogonally.

9. Let $P_1(t_1, t_1^3)$ is a point on the curve $y = x^3$

$$\therefore \left. \frac{dy}{dx} \right|_{(t_1, t_1^3)} = 3t_1^2$$

Tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$... (1)

The intersection of (1) and $y = x^3$

$$\Rightarrow x^3 - t_1^3 = 3t_1^2(x - t_1) \quad \Rightarrow (x - t_1)(x^2 + xt_1 + t_1^2) - 3t_1^2(x - t_1) = 0$$

$$\Rightarrow (x - t_1)^2(x + 2t_1) = 0$$

If $P_2(t_2, t_2^3)$, then $(t_2 - t_1)^2(t_2 + 2t_1) = 0$

$$\therefore t_2 = -2t_1 \quad (t_2 \neq t_1)$$

Similarly, the tangent at P_2 will meet the curve at the point $P_3(t_3, t_3^3)$ when $t_3 = -2t_2 = 4t_1$ and so on.

The abscissae of P_1, P_2, \dots, P_n are $t_1, -2t_1, 4t_1, \dots, (-2)^{n-1} t_1$ in G.P.

$$\therefore \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \dots = -2 \text{ (r say)}$$

$$\therefore t_2 = t_1 r, t_3 = t_2 r \text{ and } t_4 = t_3 r$$

$$\therefore \text{Area of } \Delta P_1 P_2 P_3 = \frac{1}{2} \begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}$$

$$\text{and Area of } \Delta P_2 P_3 P_4 = \frac{1}{2} \begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} r t_1 & r^3 t_1^3 & 1 \\ r t_2 & r^3 t_2^3 & 1 \\ r t_3 & r^3 t_3^3 & 1 \end{vmatrix} = r^4 \text{ (Area of } \Delta P_1 P_2 P_3)$$

$$\therefore \frac{\text{Area of } (\Delta P_1 P_2 P_3)}{\text{Area of } (\Delta P_2 P_3 P_4)} = \frac{1}{r^4} = \frac{1}{(-2)^4} = \frac{1}{16}.$$

10. Given curve is $x^3 + y^3 = c^3$

$$\therefore \frac{dy}{dx} = -\frac{x^2}{y^2} \quad \Rightarrow \quad \left. \frac{dy}{dx} \right|_{(a,b)} = -\frac{a^2}{b^2} \quad \dots(1)$$

and tangent at (a,b) cuts the curve again at (a_1, b_1)

$$\therefore \text{Slope of tangent} = \frac{b_1 - b}{a_1 - a} = -\frac{a^2}{b^2} \quad \{\text{From (1)}\} \quad \dots(1)$$

$$\text{Also } a^3 + b^3 = c^3 \quad \dots(2)$$

$$\text{and } a_1^3 + b_1^3 = c^3 \quad \dots(3)$$

subtracting (3) from (2), we get $(a^3 - a_1^3) + (b^3 - b_1^3) = 0$

$$\Rightarrow (a - a_1)(a^2 + a a_1 + a_1^2) + (b - b_1)(b^2 + b b_1 + b_1^2) = 0$$

$$\Rightarrow \frac{b_1 - b}{a_1 - a} = -\frac{a^2 + a a_1 + a_1^2}{b^2 + b b_1 + b_1^2} \quad \dots(4)$$

$$\text{From (1) and (4),} \quad -\frac{a^2}{b^2} = -\frac{a^2 + a a_1 + a_1^2}{b^2 + b b_1 + b_1^2}$$

$$\Rightarrow a^2 b^2 + a^2 b b_1 + a^2 b_1^2 = a^2 b^2 + a b^2 a_1 + a_1^2 b^2$$

$$\Rightarrow ab(ab_1 - ba_1) + a^2 b_1^2 + a_1^2 b^2 = 0$$

$$\Rightarrow ab(ab_1 - ba_1) + (ab_1 + a_1 b)(ab_1 - a_1 b) = 0$$

$$\Rightarrow (ab_1 - a_1 b) + (ab + ab_1 + a_1 b) = 0$$

If $ab_1 - a_1b = 0$

then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{\sqrt[3]{a^3 + b^3}}{\sqrt[3]{a_1^3 + b_1^3}} = \frac{c}{c} = 1$ (Law of proportion)

$\therefore a = a_1$ and $b = b_1$ which is impossible.

Hence $ab + ab_1 + a_1b = 0$ or $\frac{a_1}{a} + \frac{b_1}{b} = -1$

11. $|\sin x| + |\cos x| \in [1, \sqrt{2}]$

$\therefore [|\sin x| + |\cos x|] = 1$

For $x^2 + y^2 = 5, \frac{dy}{dx} = -\frac{x}{y}$

For $y = 1, x = \pm 2$

$\therefore m_1 = 0, m_2 = \mp 2$

$\therefore \tan \alpha = \left| \frac{\pm 2 - 0}{1 + 0} \right| = 2$

$\therefore \alpha = \tan^{-1}(2)$

12. $x^m y^n = k^{m+n}$

$\therefore \frac{dy}{dx} = -\frac{my}{nx}$

$\therefore \frac{Y-y}{X-x} = -\frac{my}{nx}$

$\therefore nxY - nxy = -myX + mxy$

$\therefore myX + nxY = xy(m+n)$

$X=0 \Rightarrow Y = \frac{y(m+n)}{n}$

$Y=0 \Rightarrow X = \frac{x(m+n)}{m}$

Now, $x = \frac{Rx(m+n)}{k+1} + 0$, Where R is ratio in which point of contact divides the segments

$\therefore mR + m = Rm + Rn$

$\therefore R = \frac{m}{n}$

$\therefore \text{Ratio} = m : n$

$$13. \quad \frac{x^2}{a} + \frac{y^2}{b} = 1 \quad ; \quad \frac{x^2}{a_1} + \frac{y^2}{b_1} = 1$$

$$m_1 = -\frac{bx}{ay}, \quad m_2 = -\frac{b_1x}{a_1y}$$

solving, $x^2 = a, y^2 = b_1$

$$\therefore m_1 m_2 = \frac{bb_1a}{aa_1b_1} = \frac{b}{a_1} = -1$$

\therefore Condition for orthogonality is $a_1 + b = 0$.

$$14. \quad \text{T.P.} \quad \frac{\sin \theta}{\theta} < \frac{\sin(\sin \theta)}{\sin \theta}$$

$$\text{Let} \quad f(x) = \frac{\sin x}{x}$$

$$\therefore f'(x) = \frac{x \cos x - \sin x}{x^2}$$

$$\text{Let} \quad g(x) = x \cos x - \sin x$$

$$\therefore g'(x) = -x \sin x < 0 \quad \text{for } x \in \left(0, \frac{\pi}{2}\right)$$

$$g(0) = 0$$

$$\therefore g(x) < 0 \quad \text{for } x \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore f'(x) < 0 \quad \text{for } x \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore f(x) \text{ is M.D. in } \left(0, \frac{\pi}{2}\right)$$

Now $\theta > \sin \theta$ for $x \in \left(0, \frac{\pi}{2}\right)$ (as shown in Q.9)

$\therefore f(\theta) < f(\sin \theta)$ Hence proved.

$$15. \quad f(x) = \frac{1-x+x^2}{1+x+x^2}$$

$$\therefore f'(x) = \frac{(2x-1)(x^2+x+1) - (2x+1)(x^2-x+1)}{(x^2+x+1)^2}$$

$$= \frac{(2x^3+x^2+x-1) - (2x^3-x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{2(x+1)(x-1)}{(x^2+x+1)^2}$$

Now, $x^2+x+1 > 0 \quad \forall x \in \mathbb{R}$

$\therefore f(x)$ is MI in $(-\infty, -1) \cup (1, \infty)$ and MD in $(-1, 1)$.

16. $f(x) = (m+2)x^3 - 3mx^2 + 9mx - 1$

$\therefore f'(x) = 3(m+2)x^2 - 6mx + 9m < 0 \quad \forall x \in \mathbb{R}$

$\therefore (m+2)x^2 - 2mx + 3m < 0 \quad \forall x \in \mathbb{R}$

$\therefore m+2 < 0 \quad \Rightarrow \quad m < -2$

and $4m^2 - 4(m+2)(3m) < 0$

$\therefore m^2 - (m+2)(3m) < 0$

$\therefore -2m^2 - 6m < 0$

$\therefore m(m+3) > 0$

$\therefore m < -3 \quad \text{as} \quad m < -2$

$\therefore m \in (-\infty, -3)$

17. $\frac{\sin x}{x}$ is MD is proved in Q.10

Now $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ & $\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$

$\therefore \frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \frac{2x}{\pi} < \sin x < x$ for $x \in \left(0, \frac{\pi}{2}\right)$

18. Refer assignment Q.10

19. $h(x) = f(x) - [f(x)]^2 + [f(x)]^3$

$\therefore h'(x) = f'(x) - 2f(x)[f'(x)] + 3[f(x)]^2 f'(x) > 0$

$\therefore f'(x)(3f^2(x) - 2f(x) + 1) > 0$
 \downarrow
 > 0 for all $f(x) \in \mathbb{R}$

$\therefore f'(x) > 0$ for $h'(x) > 0$

$$20. \quad f(x) = \begin{cases} xe^{ax} & , x \leq 0 \\ x+ax^2-x^3 & , x > 0 \end{cases}, a > 0$$

$$f'(x) = e^{ax}(1+ax), x \leq 0$$

$$1+2ax-3x^2, x > 0$$

$$f'(x) > 0$$

For $x \leq 0$

$$f'(x) > 0 \text{ for } x \in \left(\frac{1}{a}, 0\right]$$

For $x > 0$,

$$3x^2 - 2ax - 1 < 0 \text{ for } f'(x) > 0$$

$$\therefore x \in \left[0, \frac{a + \sqrt{a^2 + 3}}{3}\right)$$

$$\therefore f(x) \text{ is monotonically increasing in } x \in \left(\frac{1}{a}, \frac{a + \sqrt{a^2 + 3}}{3}\right)$$

$$21. \quad f''(x) > 0 \quad \forall x \in \mathbb{R}$$

$$Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2)$$

$$\therefore Q(x) = 2x f'\left(\frac{x^2}{2}\right) - 2x f'(6-x^2) > 0$$

$$\therefore 2x \left(f'\left(\frac{x^2}{2}\right) - f'(6-x^2) \right) > 0$$

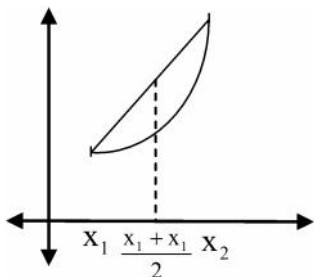
$$\therefore x \left(f'\left(\frac{x^2}{2}\right) - f'(6-x^2) \right) > 0$$

$$\frac{x^2}{2} = 6 - x^2 \quad \Rightarrow \quad x = \pm 2$$

$\therefore Q(x)$ is MI in $(-2, 0) \cup (2, \infty)$ & MD in $(-\infty, -2) \cup (0, 2)$

$$22. \quad f'(x) > 0, f''(x) > 0, x_1 < x_2$$

\therefore Graph of $f(x)$ is



By graph

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$$

$$23. \quad Q(x) = 2f\left(\frac{x}{2}\right) + f(2-x)$$

$$\therefore \quad Q'(x) = f'\left(\frac{x}{2}\right) - f'(2-x)$$

$$f''(x) < 0$$

$$\therefore \quad g'(x) > 0$$

$$\Rightarrow \quad f'\left(\frac{x}{2}\right) > f'(2-x)$$

$$\Rightarrow \quad \frac{x}{2} < 2-x$$

$$\therefore \quad x < \frac{4}{3}$$

$$\therefore \quad Q(x) \text{ increases in } \left(-\infty, \frac{4}{3}\right) \text{ \& decreases in } \left(\frac{4}{3}, \infty\right)$$

$$24. \quad f(x) = \sin^3 x + \lambda \sin^2 x$$

$$\therefore \quad f'(x) = 3 \sin^2 x \cos x + 2\lambda \sin x \cos x = 0$$

$$\therefore \quad \cos x (3 \sin^2 x + 2\lambda \sin x) = 0$$

$$\therefore \quad \cos x \neq 0 \text{ as } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \quad \sin x (3 \sin x + 2\lambda) = 0$$

$$\therefore \quad 3 \sin x + 2\lambda = 0 \text{ has a root}$$

$$\therefore \quad \sin x = -\frac{2\lambda}{3} \text{ has a root}$$

$$\therefore \quad \lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\} \text{ as } \sin x = 0 \text{ is already a root.}$$

$$25. \quad f(x) \text{ is } 6^{\text{th}} \text{ degree polynomial}$$

$$\lim_{x \rightarrow 0} \left\{ 1 + \frac{f(x)}{x^3} \right\}^{\frac{1}{x}} = r = e^2$$

$$\therefore \quad e^{\lim_{x \rightarrow 0} \frac{f(x)}{x^4}} = e^2 \quad f(x) = ax^6 + bx^5 + cx^4 \quad \& \quad c = 2$$

$$\therefore \quad \text{Roots are } 1 \text{ \& } 2$$

$$2 = \frac{8}{6a} \quad \therefore \quad a = \frac{2}{3}$$

$$\therefore \quad f'(x) = 6ax^5 + 5bx^4 + 8x^3 = 0$$

$$\therefore \quad 6ax^2 + 5bx + 8 = 0 \text{ has roots } 1 \text{ \& } 2$$

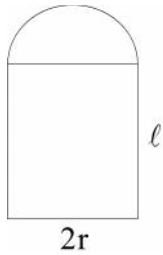
$$\therefore -\frac{5b}{6a} = 3 \text{ and } \frac{8}{6a} = 2$$

$$\therefore a = \frac{2}{3} \quad b = -3 \times 6 \times \frac{2}{3} \times \frac{1}{5}$$

$$b = -\frac{12}{5}$$

$$\therefore f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$$

26.



$$P = \pi r + 2r + 2l + 2r$$

$$= 4r + 2l + \pi r$$

$$L = 3 \times 2rl + \frac{\pi r^2}{2}$$

$$\therefore \frac{dL}{dr} = 0$$

$$\therefore 6l + 6r \left(\frac{4 + \pi}{2} \right) + \pi r = 0$$

$$\therefore 3l = 12r + 2\pi r$$

$$\frac{l}{r} = \frac{12 + 2\pi}{3}$$

$$\therefore \frac{dP}{dr} = 4 + \pi + \frac{2d\ell}{dr} = 0$$

$$\frac{d\ell}{dr} = -\frac{(4 + \pi)}{2}$$

27.

$$(a) \quad \frac{dx}{dt} = 6t^2 - 6t + 6$$

$$\frac{dy}{dt} = 6t^2 + 6t + 6$$

$$\therefore \frac{dy}{dx} = \frac{6t^2 + 6t + 6}{6t^2 - 6t + 6}$$

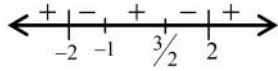
$$= \frac{t^2 + t + 1}{t^2 - t + 1}$$

Solving it is maximum at $t = 1$ and has max value = 3.

$$\therefore \text{Point on curve} = (5, 11)$$

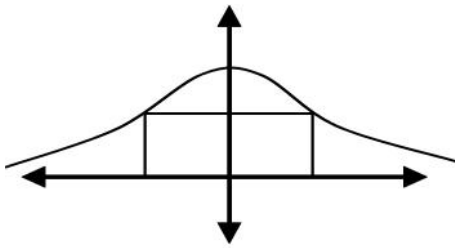
$$(b) \quad \frac{dy}{dx} = \frac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20} = \frac{6(2t^2 - t - 3)}{5(t^4 - 3t^2 - 4)}$$

$$= \frac{6(2t-3)(t+1)}{5(t+2)(t-2)}$$



$\therefore f(x)$ has minimum value at $t = -1$ and max at $t = \frac{3}{2}$

28.



By symmetry, points on x-axis will be $(-a, 0)$ and $(a, 0)$

$$A = 2ae^{-a^2}$$

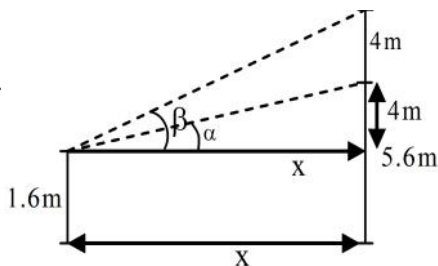
$$\therefore \frac{dA}{da} = 2e^{-a^2} - (2a)^2 e^{-a^2} = 0$$

$$\therefore 2a^2 = 1$$

$$\therefore a = \pm \frac{1}{\sqrt{2}}$$

$$\therefore A = \sqrt{2} e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}}$$

29.



$$\tan \alpha = \frac{4}{x}, \tan \beta = \frac{8}{x}$$

$$\therefore \beta - \alpha = \tan^{-1}\left(\frac{8}{x}\right) - \tan^{-1}\left(\frac{4}{x}\right) = \tan^{-1}\left(\frac{\frac{4}{x}}{1 + \frac{32}{x^2}}\right) = \tan^{-1}\left(\frac{4x}{x^2 + 32}\right)$$

$\tan^{-1} x$ is an increasing function

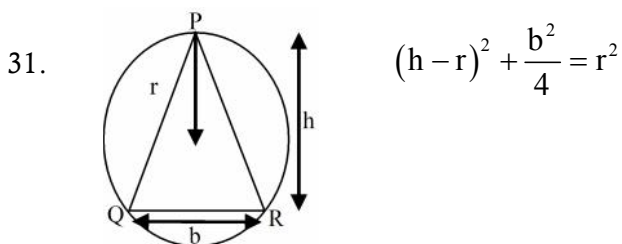
$$\therefore \text{max of } \frac{4x}{x^2 + 32}$$

$$\Rightarrow \text{max of } \tan^{-1}\left(\frac{4x}{x^2 + 32}\right)$$

$$f(x) = \frac{4x}{x^2 + 32}$$

$$\therefore f'(x) = \frac{4(x^2 + 32) - 8x^2}{(x^2 + 32)^2} = 0$$

$$\therefore 128 = 4x^2 \quad \therefore x = \sqrt{32} \text{ m} = 4\sqrt{2}$$



Solving as in (level 1) Q.12 We get the answer.

32. Q is (1,0)

Let equation of circle with centre be

$$(x-1)^2 + y^2 = r^2$$

$$x^2 + y^2 = 1$$

$$\therefore -2x + 1 = r^2 - 1$$

$$\therefore x = \frac{2-r^2}{2}$$

$$\therefore y^2 = 1 - \left(\frac{2-r^2}{2}\right)^2 = 1 - 1 + r^2 - \frac{r^4}{4}$$

$$\therefore y = r\sqrt{-\frac{r^2}{4} + 1}$$

$$\therefore \Delta = \frac{1}{4} r^2 \sqrt{4-r^2}$$

$$\therefore \frac{d\Delta}{dr} = \frac{r}{2} \sqrt{4-r^2} - \frac{r^2}{4} \times \frac{r}{\sqrt{4-r^2}} = 0$$

$$\therefore \frac{r}{2} \sqrt{4-r^2} = \frac{r^3}{4\sqrt{4-r^2}}$$

$$\therefore 2r(4-r^2) = r^3$$

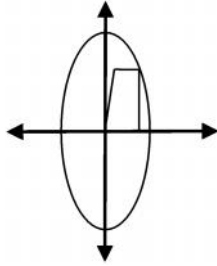
$$\therefore r = 0 \quad \text{or} \quad 8 - 2r^2 = r^2$$

$$\therefore r = \sqrt{\frac{8}{3}}$$

$$\therefore \sqrt{4 - r^2} = \frac{2}{\sqrt{3}}$$

$$\therefore \Delta = \frac{1}{4} \times \frac{8}{3} \times \frac{2}{\sqrt{3}} = \frac{4}{3\sqrt{3}}$$

33.



$$\Delta = \frac{1}{2} \times \beta \times (\alpha + \alpha - \beta)$$

$$= \frac{1}{2} \beta (2\alpha - \beta)$$

$$= \alpha\beta - \frac{\beta^2}{2}$$

$$4\alpha^2 + 3\beta^2 = 12$$

$$\therefore 8\alpha \frac{d\alpha}{d\beta} + 6\beta = 0$$

$$\therefore \frac{d\alpha}{d\beta} = -\frac{3\beta}{4\alpha}$$

$$\therefore \frac{d\Delta}{d\beta} = \alpha - \beta \left(\frac{3\beta}{4\alpha} \right) - \beta = 0 \quad \therefore 4\alpha^2 - 4\alpha\beta - 3\beta^2 = 0$$

$$\therefore (2\alpha + \beta)(2\alpha - 3\beta) = 0 \quad \text{But } \alpha\beta > 0$$

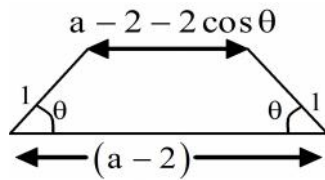
$$\therefore 2\alpha = 3\beta \quad \therefore 4\alpha^2 = 9\beta^2$$

$$4\alpha^2 + 3\beta^2 = 12$$

$$\therefore \beta = 1 \quad \text{and} \quad \alpha = \frac{3}{2} \quad \dots (\alpha, \beta > 0)$$

$$\text{Point is } \left(\frac{3}{2}, 1 \right)$$

34.



$$h = \sin \theta$$

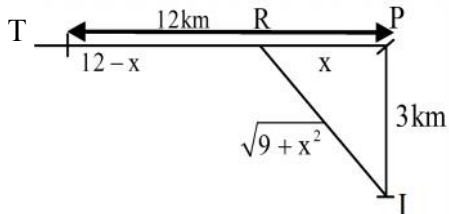
$$\begin{aligned} \therefore A &= \frac{1}{2} \sin \theta (2a - 4 - 2 \cos \theta) \\ &= a \sin \theta - 2 \sin \theta - \sin \theta \cos \theta \end{aligned}$$

$$\therefore \frac{dA}{d\theta} = a \cos \theta - 2 \cos \theta - \cos^2 \theta + \sin^2 \theta = 0$$

$$\therefore 2 \cos^2 \theta + (2 - a) \cos \theta - 1 = 0 \quad \theta \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \cos \theta = \frac{a - 2 + \sqrt{(a - 2)^2 + 8}}{4}$$

35.



$$\therefore \text{Rate of cable} = 4000\sqrt{9x^2} + 2000(12 - x)$$

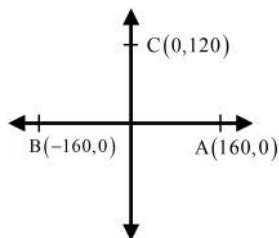
$$\therefore \frac{dR}{dx} = \frac{4000x}{\sqrt{9 + x^2}} - 2000 = 0$$

$$\therefore 2x = \sqrt{9 + x^2}$$

$$\therefore 4x^2 = x^2 + 9$$

$$\therefore x = \sqrt{3} \text{ km from P towards T.}$$

36. Solving we get $A \equiv (160, 0), B \equiv (-160, 0), C \equiv (0, 120)$



As ΔABC is an isosceles Δ , by symmetry, godown should be located on u-axis.

Let godown be located at $(0, y)$, $0 < y < 120$

\therefore Sum of distance from godown is

$$s = 120 - y + 2\sqrt{y^2 + 25600}$$

$$\therefore \frac{ds}{dy} = -1 + \frac{2y}{\sqrt{y^2 + 25600}}$$

$$\therefore 4y^2 = y^2 + 25600$$

$$\therefore y^2 = \frac{25600}{3}$$

$$\therefore y = \frac{160}{\sqrt{3}}$$

$$\therefore \text{Godown should be located at } \left(0, \frac{160}{\sqrt{3}}\right).$$

$$37. \quad P = (1200 - x)(725 + x)$$

$$\therefore \frac{dP}{dx} = -725 - x + 1200 - x = 0$$

$$\therefore x = \frac{475}{2} = 237.5$$

But as number of subscribers is an integer $x = 237$ or 238 (Values are same)

\therefore Subscribers = 962 or 963 .

$$38. \quad g(t) = 3t^4 - 8t^3 - 6t^2 + 24t$$

$$g'(t) = 12t^3 - 24t^2 - 12t + 24 = 0$$

$$\therefore t^3 - 2t^2 - t + 2 = 0$$

$$\therefore (t^2 - 1)(t - 2) = 0$$

By examining it has maxima at $t = 1$ and minima at $t = 2$

$$\therefore f(x) = 3x^4 - 8x^3 - 6x^2 + 24x ; 1 \leq x < 2$$

$$h(t) = 3t + \frac{1}{4} \sin^2 \pi t + 2$$

$$\therefore h'(t) = 3 + \frac{\pi}{4} \times \sin 2\pi t > 0 \quad \forall t$$

$\therefore f(x)$ has minima at $x = 2$

$$\therefore f(x) = 3x^4 - 8x^3 - 6x^2 + 24x ; 1 \leq x < 2$$

$$= 3x + \frac{1}{4} \sin^2 \pi x + 2 ; 2 \leq x \leq 4$$

$$f(1) = 3 - 8 - 6 + 24 = 13$$

$$\lim_{x \rightarrow 2^-} f(x) = 48 - 64 - 24 + 48 = 8$$

$$f(2) = 6 + 2 = 8$$

$$f(4) = 12 + 2 = 14$$

∴ Greatest value of $f(x) = 14$ and least value of $f(x) = 8$.

40. Refer Ex. level 1 (Q.21) for method

$$\text{We get } \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \geq 0$$

$$\therefore \frac{(b^2 + 1)(b - 1)}{(b + 1)(b + 2)} \geq 0$$

$$\therefore b \in (-2, -1) \cup [1, \infty).$$

41. $f(x) = x + \frac{a}{x^2} - 2 > 0 \quad \forall x \in \mathbb{R}$

$$\therefore f'(x) = 1 - \frac{2a}{x^3} = 0$$

$$\Rightarrow x = (2a)^{\frac{1}{3}}$$

$$\therefore (2a)^{\frac{1}{3}} + \frac{a^{\frac{1}{3}}}{2^{\frac{1}{3}}} - 2 > 0$$

$$\therefore \left(4^{\frac{1}{3}} + 1\right) a^{\frac{1}{3}} > 2^{\frac{4}{3}}$$

$$\therefore a > \frac{2^4}{\left(4^{\frac{1}{3}} + 1\right)^3}$$

$$\left(4^{\frac{1}{3}} + 1\right)^3 = 5 + 3\left(4^{\frac{2}{3}} + 4^{\frac{1}{3}}\right)$$

$$\therefore a > 1 \dots\dots$$

∴ Least natural number is 2.

42. $f'(c) = 2c - 2 = -1$

$$\therefore c = \frac{1}{2}$$

$$\therefore x = \frac{1}{2}, y = \frac{1}{4} - 1 + 3 = \frac{9}{4}$$

$$\therefore y - \frac{9}{4} = \frac{1}{2} - x$$

$$\therefore x + y = \frac{11}{4}$$

i.e. $4x + 4y = 11$

$$43. \quad \frac{f'(c)}{g'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)} = \frac{4}{2} = 2$$

$$44. \quad f(x) = x + e^x$$

$$\therefore f'(x) = 1 + e^x > 0 \quad \forall x \in \mathbb{R}$$

$$\text{Now, } f(-\infty) = -\infty, f(\infty) = \infty$$

$\therefore f(x)$ has only one real root.

$$45. \quad f(0) = 4, f(5) = -1$$

$$g(x) = \frac{f(x)}{x-1}$$

$$g(0) = -4, g(5) = -\frac{1}{4}$$

$$\therefore g'(c) = \frac{4 - \frac{1}{4}}{5} = \frac{3}{4}$$

46. $f(x)$ and $f'(x)$ should have a common root

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$\therefore f'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$$

Which do not have a common root.

$$47. \quad f(x) = x + \cos x - a$$

$$f'(x) = 1 - \sin x \geq 0 \quad \forall x \in \mathbb{R}$$

$\therefore f(x)$ is an increasing function

$$\text{If } a < 1, f(0) > 0$$

\therefore No positive root

$$\text{If } a > 1, f(0) < 0$$

\therefore one positive root.

$$48. \quad \text{Consider the function } f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

We have $f(0) = d$ and

$$f(1) = \frac{a}{3} + \frac{b}{2} + c + d = \frac{2a + 3b + 6c}{6} + d = 0 + d = d \quad (\because 2a + 3b + 6c = 0)$$

Therefore, 0 and 1 are the roots of the polynomial $f(x)$.

Consequently, there exists at least one root of the polynomial $f'(x) = ax^2 + bx + c$ lying between 0 and 1.