

CONTINUITY & DIFFERENTIABILITY
EXERCISE 1(A)

1. (d)
 L.H.L. at $x = 3$, $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x + \lambda) = \lim_{h \rightarrow 0} (3 - h + \lambda) = 3 + \lambda$ (i)
 R.H.L. at $x = 3$, $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (3x - 5) = \lim_{h \rightarrow 0} \{3(3 + h) - 5\} = 4$ (ii)
 Value of function $f(3) = 4$ (iii)
 For continuity at $x = 3$
 Limit of function = value of function $3 + \lambda = 4 \Rightarrow \lambda = 1$.

2. (c)
 If function is continuous at $x = 0$, then by the definition of continuity $f(0) = \lim_{x \rightarrow 0} f(x)$
 Since $f(0) = k$. Hence, $f(0) = k = \lim_{x \rightarrow 0} \left(\sin \frac{1}{x} \right)$
 $\Rightarrow k = 0$ (a finite quantity lies between -1 to 1)
 $\Rightarrow k = 0$.

3. (c)
 Since $f(x)$ is continuous at $x = 1$,
 $\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$ (i)
 Now $\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1 - h) = \lim_{h \rightarrow 0} 2(1 - h) + 1 = 3$ i.e., $\lim_{x \rightarrow 1^-} f(x) = 3$
 Similarly, $\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1 + h) = \lim_{h \rightarrow 0} 5(1 + h) - 2$ i.e., $\lim_{x \rightarrow 1^+} f(x) = 3$
 So according to equation (i), we have $k = 3$.

4. (d)
 We have $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sin \frac{1}{x} =$ An oscillating number which oscillates between -1 and 1 .
 Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.
 Consequently $f(x)$ cannot be continuous at $x = 0$ for any value of k .

5. (c)
 LHL = $\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} m(1 - h)^2 = m$
 RHL = $\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} 2(1 + h) = 2$ and $f(1) = m$
 Function is continuous at $x = 1$, \therefore LHL = RHL = $f(1)$
 Therefore $m = 2$.

6. (a)
 $\lim_{x \rightarrow 0} (\cos x)^{1/x} = k \Rightarrow \lim_{x \rightarrow 0} \frac{1}{x} \log(\cos x) = \log k$
 $\Rightarrow \lim_{x \rightarrow 0} \frac{1}{x} \lim_{x \rightarrow 0} \log \cos x = \log k$
 $\Rightarrow \lim_{x \rightarrow 0} \frac{1}{x} \times 0 = \log_e k \Rightarrow k = 1$.

7. (b)

Since f is continuous at $x = \frac{\pi}{4}$;

$$\therefore f\left(\frac{\pi}{4}\right) = \lim_{h \rightarrow 0} f\left(\frac{\pi}{4} + h\right) = \lim_{h \rightarrow 0} f\left(\frac{\pi}{4} - h\right)$$

$$\Rightarrow \frac{\pi}{4} + b = \frac{\pi}{4} + a^2 \Rightarrow b = a^2$$

Also as f is continuous at $x = \frac{\pi}{2}$;

$$\therefore f\left(\frac{\pi}{2}\right) = \lim_{h \rightarrow 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h \rightarrow 0} f\left(\frac{\pi}{2} - h\right)$$

$$\Rightarrow 2b + a = b \Rightarrow a = -b.$$

Hence $(-1, 1)$ & $(0, 0)$ satisfy the above relations.

8. (c)

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} \left[2 + \sin \frac{\pi}{2}(1-h) \right] = 3$$

$$\text{Similarly, } \lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} a(1+h) + b = a + b$$

$$\therefore f(x) \text{ is continuous at } x = 1 \text{ so } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$$

$$\Rightarrow a + b = 3 \quad \dots\dots(i)$$

$$\text{Again, } \lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} a(2-h) + b = 2a + b$$

$$\text{and } \lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} \tan \frac{\pi}{8}(2+h) = 1$$

$f(x)$ is continuous in $(-\infty, 6)$, so it is continuous at $x = 2$ also, so

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow 2a + b = 1 \quad \dots\dots(ii)$$

Solving (i) and (ii) $a = -2, b = 5$.

9. (a)

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \frac{\pi}{2}, \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = -\frac{\pi}{2}$$

$$\text{Since } \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) \neq \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) ,$$

\therefore Function is discontinuous at $x = \frac{\pi}{2}$

10. (b)

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{2 \sin^2 3x}{(3x)^2} \right) 3 = 6 \text{ and}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{9+\sqrt{x}}-3} = \lim_{x \rightarrow 0^+} \left(\sqrt{9+\sqrt{x}} + 3 \right) = 6$$

Hence $a = 6$.

11. (c)

The function $f(x) = \frac{1}{x^2 + x - 6}$ is discontinuous at 2 points.

The function $f(x) = \frac{1}{x^2 + x - 6}$ & $g(x) = \frac{1}{x-1} \Rightarrow g(f(x)) = \frac{1}{x^2 + x - 7}$

$g(f(x))$ is discontinuous at 4 points.

Hence, the composite $f(g(x))$ is discontinuous at three points $x = \frac{2}{3}, 1$ & $\frac{3}{2}$

12. (b)

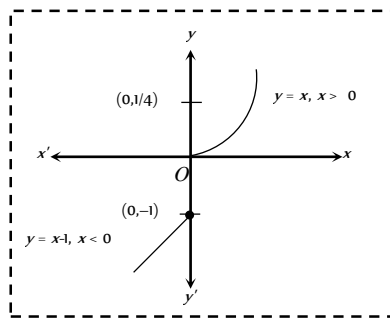
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln b \ln(a+x) - \ln a \ln(b-x)}{x} &= \lim_{x \rightarrow 0} \frac{\ln b (\ln(a+x) - \ln a) - \ln a (\ln a \ln(b-x) - \ln b)}{x} \\ &= \ln b \lim_{x \rightarrow 0} \frac{(\ln(a+x) - \ln a)}{x} + \ln a \lim_{x \rightarrow 0} \frac{(\ln(b-x) - \ln b)}{x} \\ &= \frac{\ln b}{a} \lim_{x \rightarrow 0} \frac{\ln\left(1 + \frac{x}{a}\right)}{\frac{x}{a}} + \frac{\ln a}{b} \lim_{x \rightarrow 0} \frac{\ln\left(1 + \frac{x}{b}\right)}{\frac{x}{b}} \\ &= \frac{\ln b}{a} + \frac{\ln a}{b} = \frac{\ln(b^b a^a)}{ab} \end{aligned}$$

13. (b)

$$f(2) = 2, f(2^+) = \lim_{x \rightarrow 2^+} \frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \rightarrow 2^+} \frac{(x-3)}{(x+2)} = -\frac{1}{4}$$

14. (c)

Clearly from curve drawn of the given function $f(x)$, it is discontinuous at $x = 0$.



15. (b)

$$f(x) = \begin{cases} (1 + |\tan x|)^{\frac{a}{3|\tan x|}}, & -\frac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\frac{\tan 6x}{\tan 3x}}, & 0 < x < \frac{\pi}{6} \end{cases}$$

For $f(x)$ to be continuous at $x = 0$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0} (1 + |\tan x|)^{\frac{a}{3|\tan x|}} = e^{\lim_{x \rightarrow 0} \left((1 + |\tan x| - 1) \frac{a}{3|\tan x|} \right)} = e^{a/3}$$

$$\text{Now, } \lim_{x \rightarrow 0^+} e^{\frac{\tan 6x}{\tan 3x}} = \lim_{x \rightarrow 0^+} e^{\left(\frac{\tan 6x}{6x} \cdot 6x \right) / \left(\frac{\tan 3x}{3x} \cdot 3x \right)} = e^2$$

$$\therefore e^{a/3} = b = e^2 \Rightarrow a = 6 \text{ and } b = e^2.$$

16. (d)

$$\text{Let } f(x) = \ln \frac{x}{4}$$

$$\lim_{x \rightarrow 4} x f(x) = \lim_{x \rightarrow 4} x \ln \frac{x}{4} = 0$$

17. (a)

Note that $[x+2] = 0$ if $0 \leq x+2 < 1$

i.e. $[x+2] = 0$ if $-2 \leq x < -1$.

Thus domain of f is $\mathbb{R} - [-2, -1)$

We have $\sin\left(\frac{\pi}{[x+2]}\right)$ is continuous at all points of $\mathbb{R} - [-2, -1)$ and $[x]$ is continuous on

$\mathbb{R} - \mathbb{I}$, where \mathbb{I} denotes the set of integers.

Thus the points where f can possibly be discontinuous are $\dots, -3, -2, -1, 0, 1, 2, \dots$. But for

$-1 \leq x < 0, [x+1] = 0$ and $\sin\left(\frac{\pi}{[x+2]}\right)$ is defined.

Therefore $f(x) = 0$ for $-1 \leq x < 0$.

Also $f(x)$ is not defined on $-2 \leq x < -1$.

Hence set of points of discontinuities of $f(x)$ is $\mathbb{I} - \{-1\}$.

18. (b)

$$f(x) = \lim_{x \rightarrow 0} \left(\frac{2x - \sin^{-1} x}{2x + \tan^{-1} x} \right) = f(0) \quad , \left(\frac{0}{0} \text{ form} \right)$$

$$\text{Applying L-Hospital's rule, } f(0) = \lim_{x \rightarrow 0} \frac{\left(2 - \frac{1}{\sqrt{1-x^2}} \right)}{\left(2 + \frac{1}{1+x^2} \right)} = \frac{2-1}{2+1} = \frac{1}{3}$$

19. (d)

For continuity at all $x \in \mathbb{R}$, we must have

$$f\left(-\frac{\pi}{2}\right) = \lim_{x \rightarrow (-\pi/2)^-} (4 \sin x) = \lim_{x \rightarrow (-\pi/2)^+} (a \sin x - b)$$

$$\Rightarrow 4 = -a - b$$

.....(i)

$$\text{and } f\left(\frac{\pi}{2}\right) = \lim_{x \rightarrow (\pi/2)^-} (a \sin x - b) = a - b = \lim_{x \rightarrow (\pi/2)^+} (\cos x) = 0$$

$$\Rightarrow 0 = a - b \quad \dots(\text{ii})$$

From (i) and (ii), $a = -2$ and $b = -2$.

20. (a)

$$f(5) = \lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x \rightarrow 5} \frac{(x-5)^2}{(x-2)(x-5)} = \frac{5-5}{5-2} = 0.$$

21. (c)

For continuity at 0, we must have $f(0) = \lim_{x \rightarrow 0} f(x)$

$$= \lim_{x \rightarrow 0} (x+1)^{\cot x} = \lim_{x \rightarrow 0} \left\{ (1+x)^{\frac{1}{x}} \right\}^{x \cot x} = \lim_{x \rightarrow 0} \left\{ (1+x)^{\frac{1}{x}} \right\}^{\lim_{x \rightarrow 0} \left(\frac{x}{\tan x} \right)} = e.$$

22. (a)

Conceptual question

23. (c)

$f(x)$ is continuous at $x = \frac{\pi}{3}$, then $\lim_{x \rightarrow \pi/3} f(x) = f(0)$ or

$$\lambda = \lim_{x \rightarrow \pi/3} \frac{1 - \sin \frac{3x}{2}}{\pi - 3x}, \left(\frac{0}{0} \text{ form} \right)$$

$$\text{Applying L-Hospital's rule, } \lambda = \lim_{x \rightarrow \pi/3} \frac{-\frac{3}{2} \cos \frac{3x}{2}}{-3} = 0$$

24. (d)

If $f(x)$ is continuous at $x = 0$ then,

$$f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{\sin 2x}, \left(\frac{0}{0} \text{ form} \right)$$

$$\text{Using L-Hospital's rule, } f(0) = \lim_{x \rightarrow 0} \frac{\left(-\frac{1}{2\sqrt{x+4}} \right)}{2 \cos 2x} = -\frac{1}{8}.$$

25. (d)

$$x^2 + 2 = 3x \Rightarrow x = 1, 2$$

$F(x)$ will be continuous only at $x = 1$ & 2 .

26. (b)

$$f(x) = \left[x^2 + e^{\frac{1}{2-x}} \right]^{-1} \text{ and } f(2) = k$$

If $f(x)$ is continuous from right at $x = 2$ then $\lim_{x \rightarrow 2^+} f(x) = f(2) = k$

$$\Rightarrow \lim_{x \rightarrow 2^+} \left[x^2 + e^{\frac{1}{2-x}} \right]^{-1} = k \Rightarrow k = \lim_{h \rightarrow 0} f(2+h) \Rightarrow k = \lim_{h \rightarrow 0} \left[(2+h)^2 + e^{\frac{1}{2-(2+h)}} \right]^{-1}$$

$$\Rightarrow k = \lim_{h \rightarrow 0} \left[4 + h^2 + 4h + e^{-1/h} \right]^{-1} \Rightarrow k = [4 + 0 + 0 + e^{-\infty}]^{-1} \Rightarrow k = \frac{1}{4}.$$

27. (c)

$$\lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} \frac{2 \cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} = \lim_{x \rightarrow \pi} \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} = \lim_{x \rightarrow \pi} \tan \left(\frac{\pi}{4} - \frac{x}{2} \right)$$

$$\therefore \text{At } x = \pi, f(\pi) = -\tan \frac{\pi}{4} = -1.$$

28. (c)

$$\text{L.H.L.} = \lim_{x \rightarrow 0^-} \frac{\sqrt{4+kx} - \sqrt{4-kx}}{x} = \lim_{x \rightarrow 0^-} \frac{2kx}{x} \times \frac{1}{\sqrt{4+kx} + \sqrt{4-kx}} = \frac{k}{2}$$

$$\text{R.H.L.} = \lim_{x \rightarrow 0^+} \frac{2x^2 + 3x}{\sin x} = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} (2x + 3) = 3$$

Since it is continuous, hence L.H.L = R.H.L $\Rightarrow k = 6$.

29. (c)

$|x|$ is continuous at $x = 0$ and $\frac{|x|}{x}$ is discontinuous at $x = 0$

$\therefore f(x) = |x| + \frac{|x|}{x}$ is discontinuous at $x = 0$.

30. (b)

$$\lim_{x \rightarrow 0^+} \frac{x(e^x - 1)}{|\tan x|} = \lim_{x \rightarrow 0^+} \frac{x(e^x - 1)}{\tan x} = 0$$

$$\lim_{x \rightarrow 0^-} \frac{x(e^x - 1)}{|\tan x|} = -\lim_{x \rightarrow 0^-} \frac{x(e^x - 1)}{\tan x} = 0$$

So $f(x)$ is continuous at $x = 0$.

$$\text{Now L.H.D.} = \lim_{x \rightarrow 0^-} \frac{x(e^x - 1)}{|\tan x|} = -\lim_{x \rightarrow 0^-} \frac{x}{\tan x} \times \frac{e^x - 1}{x} = -1$$

$$\text{R.H.D.} = \lim_{x \rightarrow 0^+} \frac{x(e^x - 1)}{|\tan x|} = \lim_{x \rightarrow 0^+} \frac{x}{\tan x} \times \frac{e^x - 1}{x} = 1$$

L.H.D. \neq R.H.D.

$F(x)$ is continuous but not differentiable at $x = 0$

31. (a)

$$\text{We have, } f(x) = \frac{x}{1+|x|} = \begin{cases} \frac{x}{1+x} & , x > 0 \\ 0 & , x = 0 ; \\ \frac{x}{1-x} & , x < 0 \end{cases}$$

$$\text{L.H.D.} = \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{\frac{-h}{1+h} - 0}{-h} = 1$$

$$\text{R.H.D.} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \frac{\lim_{h \rightarrow 0} \frac{h}{1+h} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{1+h} = 1$$

So, $f(x)$ is differentiable at $x = 0$; Also $f(x)$ is differentiable at all other points.

Hence, $f(x)$ is everywhere differentiable.

32. (b)

$$\text{Let } f(x) = |x-1| + |x-3| = \begin{cases} -(x-1) - (x-3) & , x < 1 \\ (x-1) - (x-3) & , 1 \leq x < 3 \\ (x-1) + (x-3) & , x \geq 3 \end{cases} = \begin{cases} -2x+4 & , x < 1 \\ 2 & , 1 \leq x < 3 \\ 2x-4 & , x \geq 3 \end{cases}$$

Since, $f(x) = 2$ for $1 \leq x < 3$. Therefore $f'(x) = 0$ for all $x \in (1, 3)$.

Hence, $f'(x) = 0$ at $x = 2$.

33. (a)

$$\text{We have, } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} \left(\frac{\sin x^2}{x^2} \right) x = 1 \times 0 = 0 = f(0)$$

So, $f(x)$ is continuous at $x = 0$,

$f(x)$ is also derivable at

$$x = 0, \text{ because } \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = 1$$

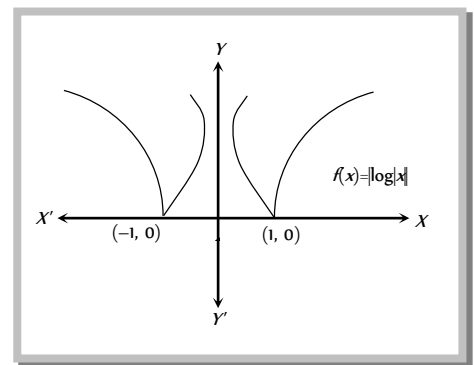
exists finitely.

34. (b)

It is evident from the graph of $f(x) = |\log|x||$ than

$f(x)$ is everywhere continuous but not differentiable

at $x = \pm 1$.



35. (a)

$$f(x) = [x] \sin(\pi x)$$

If x is just less than k , $[x] = k - 1$. $\therefore f(x) = (k - 1) \sin(\pi x)$, when $x < k \quad \forall k \in \mathbb{I}$

Now L.H.D. at $x = k$,

$$= \lim_{x \rightarrow k} \frac{(k-1) \sin(\pi x) - k \sin(\pi k)}{x - k} = \lim_{x \rightarrow k} \frac{(k-1) \sin(\pi x)}{(x - k)} \quad [\text{as } \sin(\pi k) = 0 \quad k \in \mathbb{I}]$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{(k-1) \sin(\pi(k-h))}{-h} \quad [\text{Let } x = (k-h)] \\
&= \lim_{h \rightarrow 0} \frac{(k-1)(-1)^{k-1} \sin h \pi}{-h} \\
&= \lim_{h \rightarrow 0} (k-1)(-1)^{k-1} \frac{\sin h \pi}{h \pi} \times (-\pi) \\
&= (k-1)(-1)^k \pi = (-1)^k (k-1) \pi .
\end{aligned}$$

36. (a)

$$\text{We have, } f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x < 0 \\ 1, & 0 \leq x < 1 \\ 2x-1, & x \geq 1 \end{cases}$$

Since, $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 1 = 1$, $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2x-1) = 1$ and $f(1) = 2 \times 1 - 1 = 1$

$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1)$. So, $f(x)$ is continuous at $x = 1$.

$$\text{Now, } \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{1-1}{-h} = 0 \text{ and}$$

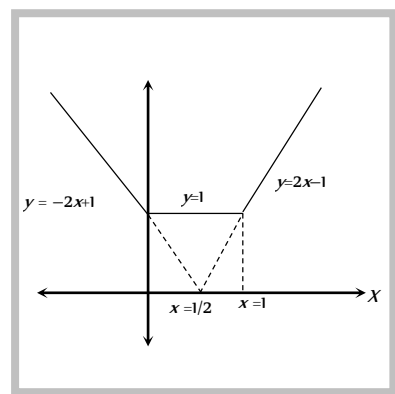
$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{2(1+h) - 1 - 1}{h} = 2 .$$

\therefore (LHD at $x = 1$) \neq (RHD at $x = 1$).

So, $f(x)$ is not differentiable at $x = 1$.

Alternately

By graph, it is clear that the function is not differentiable at $x = 0, 1$ as there it has sharp edges.



37. (c)

$$\text{Here } f(x) = |x-1| + |x+1| \Rightarrow f(x) = \begin{cases} 2x, & \text{when } x > 1 \\ 2, & \text{when } -1 \leq x \leq 1 \\ -2x, & \text{when } x < -1 \end{cases}$$

Alternate

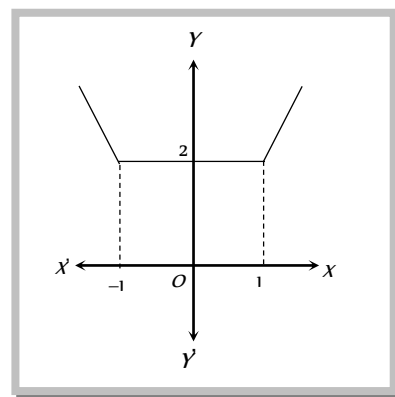
The graph of the function is shown alongside,

From the graph it is clear that the function is continuous at all real x , also differentiable at all real x except at $x = \pm 1$; Since sharp edges at $x = -1$ and $x = 1$.

At $x = 1$ we see that the slope from the right *i.e.*, R.H.D. = 2, while slope from the left *i.e.*, L.H.D. = 0

Similarly, at $x = -1$ it is clear that R.H.D. = 0 while L.H.D. = -2

$$\text{Here, } f'(x) = \begin{cases} -2, & x < -1 \\ 0, & -1 < x < 1 \text{ (No equality on } -1 \text{ and } +1) \\ 2, & x > 1 \end{cases}$$



Now, at $x=1$, $f'(1^+) = 2$ while $f'(1^-) = 0$ and

at $x=-1$, $f'(-1^+) = 0$ while $f'(-1^-) = -2$

Thus, $f(x)$ is not differentiable at $x = \pm 1$.

38. (d)

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (ax^2 + bx + 2) = a - b + 2 \text{ and}$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (bx^2 + ax + 4) = b - a + 4$$

For continuity $a - b + 2 = b - a + 4 \Rightarrow a - b = 1 \dots (i)$

$$\text{Now } f'(x) = \begin{cases} 2ax + b & , \quad x < -1 \\ 2bx + a & , \quad x > -1 \end{cases} \Rightarrow \text{R.H.D.} = -2a + b \text{ \& L.H.D.} = -2b + a$$

For differentiability $-2a + b = -2b + a \Rightarrow a = b \dots (ii)$

From (i) & (ii) no value of (a, b) is possible.

39. (b)

$$h(x) = e^{(f(x))^3 + (g(x))^3 + x} \Rightarrow h'(x) = e^{(f(x))^3 + (g(x))^3 + x} \left(3(f(x))^2 f'(x) + 3(g(x))^2 g'(x) + 1 \right)$$

$$\Rightarrow h'(x) = h(x) \left(3(f(x))^2 \frac{g'(x)}{f(x)} - 3(g(x))^2 \frac{f'(x)}{g(x)} + 1 \right)$$

$$\Rightarrow h'(x) = h(x) \Rightarrow h(x) = e^{x+c}$$

$$\text{Now } h(5) = e^6 \Rightarrow h(x) = e^{x+1}$$

$$\text{Hence } h(10) = e^{11}$$

40. (c)

$$[2+h] = 2, [2-h] = 1, [1+h] = 1, [1-h] = 0$$

At $x = 2$, we will check $\text{RHL} = \text{LHL} = f(2)$

$$\text{RHL} = \lim_{h \rightarrow 0} |4 + 2h - 3| [2+h] = 2, f(2) = 1.2 = 2$$

$$\text{LHL} = \lim_{h \rightarrow 0} |4 - 2h - 3| [2-h] = 1, \text{R} \neq \text{L}, \therefore \text{not continuous}$$

$$\text{At } x = 1, \text{RHL} = \lim_{h \rightarrow 0} |2 + 2h - 3| [1+h] = 1.1 = 1,$$

$$f(1) = |-1| [1] = 1$$

$$\text{LHL} = \lim_{h \rightarrow 0} \sin \frac{\pi}{2} (1-h) = 1$$

continuous at $x = 1$

$$\text{R.H.D.} = \lim_{h \rightarrow 0} \frac{|2 + 2h - 3| [1+h] - 1}{h} = \lim_{h \rightarrow 0} \frac{|-1|.1-1}{h} = \lim_{h \rightarrow 0} \frac{1-1}{h} = 0$$

$$\text{L.H.D.} = \lim_{h \rightarrow 0} \frac{|2 - 2h - 3| [1-h] - 1}{-h} = \lim_{h \rightarrow 0} \frac{1.0-1}{-h} = \lim_{h \rightarrow 0} \frac{1}{h} = \infty$$

Since $\text{R.H.D.} \neq \text{L.H.D.} \therefore \text{not differentiable. at } x = 1.$

41. (b)

Clearly, $f(x)$ is differentiable for all non-zero values of x ,

$$\text{For } x \neq 0, \text{ we have } f'(x) = \frac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$$

Now, (L.H.D. at $x = 0$)

$$\begin{aligned} &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{1-e^{-h^2}}}{-h} = \lim_{h \rightarrow 0} -\frac{\sqrt{1-e^{-h^2}}}{h} \\ &= -\lim_{h \rightarrow 0} \sqrt{\frac{e^{h^2}-1}{h^2}} \times \frac{1}{\sqrt{e^{h^2}}} = -1 \end{aligned}$$

$$\begin{aligned} \text{and, (RHD at } x = 0) &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \rightarrow 0} \frac{\sqrt{1-e^{-h^2}} - 0}{h} \\ &= \lim_{h \rightarrow 0} \sqrt{\frac{e^{h^2}-1}{h^2}} \times \frac{1}{\sqrt{e^{h^2}}} = 1. \end{aligned}$$

So, $f(x)$ is not differentiable at $x = 0$,

Hence, the points of differentiability of $f(x)$ are $(-\infty, 0) \cup (0, \infty)$

42. (a)

$$\text{We have, } f(x) = \begin{cases} e^{\sin x}, & -\frac{\pi}{2} \leq x < 0 \\ e^{-\sin x}, & 0 \leq x \leq \frac{\pi}{2} \end{cases}$$

Clearly, $f(x)$ is continuous and differentiable for all non-zero x .

$$\text{Now, } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} e^{\sin x} = 1 \text{ and } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} e^{-\sin x} = 1$$

$$\text{Also, } f(0) = e^0 = 1$$

So, $f(x)$ is continuous for all x .

$$\text{(LHD at } x = 0) = \left(\frac{d}{dx} (e^x) \right)_{x=0} = (e^x)_{x=0} = e^0 = 1$$

$$\text{(RHD at } x = 0) = \left(\frac{d}{dx} (e^{-x}) \right)_{x=0} = (-e^{-x})_{x=0} = -1$$

So, $f(x)$ is not differentiable at $x = 0$.

43. (b)

We have, $f(x) = \sqrt{1-\sqrt{1-x^2}}$. The domain of definition of $f(x)$ is $[-1, 1]$.

$$\text{For } x \neq 0, x \neq 1, x \neq -1 \text{ we have } f'(x) = \frac{1}{\sqrt{1-\sqrt{1-x^2}}} \times \frac{x}{\sqrt{1-x^2}}$$

Since $f(x)$ is not defined on the right side of $x = 1$ and on the left side of $x = -1$.

Also, $f'(x) \rightarrow \infty$ when $x \rightarrow -1^+$ or $x \rightarrow 1^-$.

So, we check the differentiability at $x = 0$.

$$\begin{aligned} \text{Now, (LHD at } x = 0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{h \rightarrow 0} \frac{f(0 - h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{1 - \sqrt{1 - h^2}} - 0}{-h} = -\lim_{h \rightarrow 0} \frac{\sqrt{1 - \{1 - (1/2)h^2 + (3/8)h^4 + \dots\}}}{h} \\ &= -\lim_{h \rightarrow 0} \sqrt{\frac{1}{2} - \frac{3}{8}h^2 + \dots} = -\frac{1}{\sqrt{2}} \end{aligned}$$

$$\text{Similarly, (RHD at } x = 0) = \frac{1}{\sqrt{2}}$$

Hence, $f(x)$ is not differentiable at $x = 0$.

44. (d) Since $f(x)$ is differentiable at $x = c$, therefore it is continuous at $x = c$.

$$\text{Hence, } \lim_{x \rightarrow c} f(x) = f(c).$$

45. (c)

$$(x^2 - 3x + 2) = (x - 1)(x - 2) > 0 \text{ When } x < 1 \text{ or } > 2,$$

$$\text{And } (x^2 - 3x + 2) = (x - 1)(x - 2) < 0 \text{ when } 1 \leq x \leq 2$$

$$\text{Also } \cos |x| = \cos x$$

$$\therefore f(x) = -(x^2 - 4)(x^2 - 3x + 2) + \cos x, \quad 1 \leq x \leq 2$$

$$\text{and } f(x) = (x^2 - 4)(x^2 - 3x + 2) + \cos x, \quad x < 1 \text{ or } x > 2$$

Evidently $f(x)$ is not differentiable at $x = 1$.

46. (b)

$$f(0) = 0 \text{ and } f(x) = x^2 e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}$$

$$\text{R.H.L.} = \lim_{h \rightarrow 0} (0 + h)^2 e^{-2/h} = \lim_{h \rightarrow 0} \frac{h^2}{e^{2/h}} = 0$$

$$\text{L.H.L.} = \lim_{h \rightarrow 0} (0 - h)^2 e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} = 0$$

$\therefore f(x)$ is continuous at $x = 0$.

$$\text{R.H.D. at } (x = 0) = \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 e^{-2/h}}{h} = h e^{-2/h} = 0$$

$$\text{L.H.D. at } (x = 0) = \lim_{h \rightarrow 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{h^2 e^{-\left(\frac{1}{h} - \frac{1}{h}\right)}}{-h} = \lim_{h \rightarrow 0} (-h) = 0$$

$F(x)$ is differentiable at $x = 0$

47. (d)

$$\lim_{x \rightarrow 0} f(x) = x^3 \sin^2 \left(\frac{1}{x} \right) = 0 \text{ as } 0 \leq \sin^2 \left(\frac{1}{x} \right) \leq 1 \text{ and } x \rightarrow 0$$

Therefore $f(x)$ is continuous at $x = 0$.

Also, the function $f(x) = x^3 \sin^2 \frac{1}{x}$ is differentiable because

$$\text{RHD} = \lim_{h \rightarrow 0} \frac{h^3 \sin^2 \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h^2 \sin^2 \frac{1}{h} = 0, \text{ LHD} = \lim_{h \rightarrow 0} \frac{h^3 \sin^2 \left(\frac{1}{-h} \right)}{-h} = 0.$$

48. (b)

49. (d)

50. (c)

$$\lim_{h \rightarrow 0^-} 1 + (2 - h) = 3, \quad \lim_{h \rightarrow 0^+} 5 - (2 + h) = 3, \quad f(2) = 3$$

Hence, f is continuous at $x = 2$

$$\text{Now RHD} = \lim_{h \rightarrow 0} \frac{5 - (2 + h) - 3}{h} = -1$$

$$\text{LHD} = \lim_{h \rightarrow 0} \frac{1 + (2 - h) - 3}{-h} = 1$$

$\therefore f(x)$ is not differentiable at $x = 2$.

51. (c)

$g(x) = |f(|x|)| \geq 0$. So $g(x)$ cannot be onto.

If $f(x)$ is one-one and $f(x_1) = -f(x_2)$ then $g(x_1) = g(x_2)$.

So, ' $f(x)$ is one-one' does not ensure that $g(x)$ is one-one.

If $f(x)$ is continuous for $x \in \mathbf{R}$, $|f(|x|)|$ is also continuous for $x \in \mathbf{R}$.

So the answer (c) is correct.

The fourth answer (d) is not correct as $f(x)$ being differentiable does not ensure $|f(x)|$ being differentiable.

52. (b)

Given $f(4) = 6, f'(4) = 1$

$$\therefore \lim_{x \rightarrow 4} \frac{xf(4) - 4f(x)}{x - 4} = \lim_{x \rightarrow 4} \frac{xf(4) - 4f(4) + 4f(4) - 4f(x)}{x - 4}$$

$$= \lim_{x \rightarrow 4} \frac{(x - 4)f(4)}{x - 4} - 4 \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4}$$

$$= f(4) - 2f'(4) = 4$$

53. (c)

$f(x + 2y) = 2f(x)f(y) \Rightarrow 2f'(x + 2y) = 2f(x)f'(y)$ {partially differentiating w.r.to y }

For $x = 5$ & $y = 0$, $f'(5) = f(5)f'(0) \Rightarrow f'(5) = 6$

54. (c)

By L'hospital's rule

$$\lim_{x \rightarrow 2} \frac{g^2(x)f^2(2) - f^2(x)g^2(2)}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{g(x)g'(x)f^2(2) - f(x)f'(x)g^2(2)}{x}$$

$$= \frac{(-1) \times 4 \times 9 - 3 \times (-2) \times 1}{2} = -15$$

55. (b)

$$\text{Given } 5f(2x) + 3f\left(\frac{2}{x}\right) = 2x + 2 \quad \dots\dots(i)$$

$$\text{Replacing } x \text{ by } \frac{1}{x} \text{ in (i), } 5f\left(\frac{2}{x}\right) + 3f(2x) = \frac{2}{x} + 2 \quad \dots\dots(ii)$$

$$\text{On solving equation (i) and (ii), we get, } 8f(2x) = 5x - \frac{3}{x} + 2,$$

$$\Rightarrow 8f(x) = \frac{5x}{2} - \frac{6}{x} + 2$$

$$\therefore 8f'(x) = \frac{5}{2} + \frac{6}{x^2}$$

$$\because y = xf(x) \Rightarrow \frac{dy}{dx} = f(x) + xf'(x)$$

$$= \frac{1}{8} \left(\frac{5x}{2} - \frac{6}{x} + 2 \right) + \frac{x}{8} \left(\frac{5}{2} + \frac{6}{x^2} \right)$$

$$\text{at } x = 1, \frac{dy}{dx} = \frac{1}{8} \left(\frac{5}{2} - 6 + 2 \right) + \frac{1}{8} \left(\frac{5}{2} + 6 \right) = \frac{7}{8}$$

56. (d)

$$f(x) = \begin{cases} x^3 - 1, & x \geq 1 \\ 1 - x^3, & x < 1 \end{cases} \quad \text{and} \quad f'(x) = \begin{cases} 3x^2, & x \geq 1 \\ -3x^2, & x < 1 \end{cases}$$

$$f'(1^+) = 3, f'(1^-) = -3$$

57. (b)

$$f(x) = \sin 2x \cdot \cos 2x \cdot \cos 3x + \log_2 2^{x+3},$$

$$\Rightarrow f(x) = \frac{1}{2} \sin 4x \cos 3x + (x+3) \log_2 2,$$

$$\Rightarrow f(x) = \frac{1}{4} [\sin 7x + \sin x] + x + 3$$

Differentiate w.r.t. x ,

$$f'(x) = \frac{1}{4} [7 \cos 7x + \cos x] + 1,$$

$$\Rightarrow f'(\pi) = -2 + 1 = -1.$$

58. (b) In neighborhood of $x = \frac{3\pi}{4}$, $|\cos^3 x| = -\cos^3 x$ and $|\sin^3 x| = \sin^3 x$

$$\therefore y = -\cos^3 x + \sin^3 x$$

$$\therefore \frac{dy}{dx} = 3 \cos^2 x \sin x + 3 \sin^2 x \cos x$$

$$\text{At } x = \frac{3\pi}{4}, \frac{dy}{dx} = 3 \cos^2 \frac{3\pi}{4} \sin \frac{3\pi}{4} + 3 \sin^2 \frac{3\pi}{4} \cos \frac{3\pi}{4} = 0.$$

59. (b)

$$f(x) = \log_x(\log x) = \frac{\log(\log x)}{\log x}$$

$$\Rightarrow f'(x) = \frac{\frac{1}{x} - \frac{1}{x} \log(\log x)}{(\log x)^2}$$

$$\Rightarrow f'(e) = \frac{\frac{1}{e} - 0}{1} = \frac{1}{e}$$

60. (d)

$$f(x) = |\log x| = \begin{cases} -\log x, & \text{if } 0 < x < 1 \\ \log x, & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{1}{x}, & \text{if } 0 < x < 1 \\ \frac{1}{x}, & \text{if } x > 1 \end{cases}.$$

Clearly $f'(1^-) = -1$ and $f'(1^+) = 1$,

$\therefore f'(x)$ does not exist at $x = 1$

61. (c)

$$\text{Let } y = \left[\log \left\{ e^x \left(\frac{x-1}{x+1} \right) \right\} \right] = \log e^x + \log \left(\frac{x-1}{x+1} \right)$$

$$\Rightarrow y = x + [\log(x-1) - \log(x+1)]$$

$$\Rightarrow \frac{dy}{dx} = 1 + \left[\frac{1}{x-1} - \frac{1}{x+1} \right] = 1 + \frac{2}{x^2-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+1}{x^2-1}.$$

62. (a)

$$x = \exp \left\{ \tan^{-1} \left(\frac{y-x}{x} \right) \right\} \Rightarrow \log x = \tan^{-1} \left(\frac{y-x}{x} \right)$$

$$\Rightarrow \frac{y-x}{x} = \tan(\log x) \Rightarrow y = x \tan(\log x) + x$$

$$\Rightarrow \frac{dy}{dx} = \tan(\log x) + x \frac{\sec^2(\log x)}{x} + 1$$

$$\Rightarrow \frac{dy}{dx} = \tan(\log x) + \sec^2(\log x) + 1$$

$$\text{At } x = 1, \frac{dy}{dx} = 2.$$

63. (a)

$$\begin{aligned}
 y &= \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) \\
 &= \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{\pi}{2} \\
 \Rightarrow \frac{dy}{dx} &= 0
 \end{aligned}$$

64. (d)

$$\frac{d}{dx} \tan^{-1}\left[\frac{\cos x - \sin x}{\cos x + \sin x}\right] = \frac{d}{dx} \tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right] = -1.$$

65. (b)

$$\text{Let } y = \sin^2\left(\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$$

$$\text{Put } x = \cos \theta \Rightarrow \theta = \cos^{-1} x$$

$$\Rightarrow y = \sin^2\left(\cot^{-1}\sqrt{\frac{1-\cos \theta}{1+\cos \theta}}\right) = \sin^2\left(\cot^{-1}\left(\tan \frac{\theta}{2}\right)\right)$$

$$\Rightarrow y = \sin^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos^2 \frac{\theta}{2} = \frac{1}{2}(1 + \cos \theta) = \frac{1}{2}(1 + x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

66. (a)

$$\text{Let } \cos \alpha = \frac{5}{13}. \text{ Then } \sin \alpha = \frac{12}{13}. \text{ So, } y = \cos^{-1}\{\cos \alpha \cdot \cos x - \sin \alpha \cdot \sin x\}$$

$$\therefore y = \cos^{-1}\{\cos(x + \alpha)\} = x + \alpha \quad (\because x + \alpha \text{ is in the first or the second quadrant})$$

$$\therefore \frac{dy}{dx} = 1.$$

67. (c)

$$y \left(\frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x} \right) \cot 3x = \left(\frac{\tan 2x - \tan x}{1 + \tan 2x \tan x} \right) \left(\frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \right) \cot 3x$$

$$\Rightarrow y = \tan x \tan 3x \cot 3x = \tan x$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 x$$

68. (a)

$$f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$$

$$\text{Put } x^x = \tan \theta, \quad \therefore y = f(x) = \cot^{-1}\left(\frac{\tan^2 \theta - 1}{2 \tan \theta}\right)$$

$$= \cot^{-1}(-\cot 2\theta) = \pi - \cot^{-1}(\cot 2\theta)$$

$$\Rightarrow y = \pi - 2\theta = \pi - 2 \tan^{-1}(x^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^{2x}} \cdot x^x(1+\log x)$$

$$\Rightarrow f'(1) = -1.$$

69. (a)

$$y = \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)}{1-x} = \frac{1-x^{16}}{1-x}$$

$$\therefore \frac{dy}{dx} = \frac{-16x^{15}(1-x) + 1 - x^{16}}{(1-x)^2}, \therefore \text{At } x=0, \frac{dy}{dx} = 1.$$

70. (c)

$$f(x) = \frac{2 \sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x}{2 \sin x} = \frac{\sin 8x}{8 \sin x}$$

$$\therefore f'(x) = \frac{1}{8} \cdot \frac{8 \cos 8x \cdot \sin x - \cos x \cdot \sin 8x}{\sin^2 x}$$

$$\therefore f'\left(\frac{\pi}{4}\right) = 0.$$

71. (a)

$$xe^{x+y} = y + 2 \sin x \Rightarrow e^{x+y} + xe^{x+y}(1+y') = y' + 2 \cos x$$

$$\text{Now } x=0 \text{ gives } y=0, \text{ hence } \frac{dy}{dx} = -1.$$

72. (a)

$$\sin(3x-2y) = \log(3x-2y) \Rightarrow \left(3-2\frac{dy}{dx}\right) \cos(3x-2y) = \left(3-2\frac{dy}{dx}\right) \frac{1}{3x-2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}$$

73. (c)

$$x^4 y^5 = 2(x+y)^9 \Rightarrow 4x^3 y^5 + 5x^4 y^4 \frac{dy}{dx} = 18(x+y)^8 \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow 4 \frac{2(x+y)^9}{x} + 5 \frac{2(x+y)^9}{y} \frac{dy}{dx} = 18(x+y)^8 \left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{4}{x} - \frac{9}{x+y} = \left(\frac{9}{x+y} - \frac{5}{y}\right) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

74. (b)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{a[\cos \theta - \theta(-\sin \theta) - \cos \theta]}{a[-\sin \theta + \theta \cos \theta + \sin \theta]} = \frac{\theta \sin \theta}{\theta \cos \theta} = \tan \theta.$$

75. (d)

Obviously $x = \cos^{-1} \frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1} \frac{t}{\sqrt{1+t^2}}$

$$\Rightarrow x = \tan^{-1} t \text{ and } y = \tan^{-1} t$$

$$\Rightarrow y = x \Rightarrow \frac{dy}{dx} = 1.$$

76. (c)

$$x = \frac{1-t^2}{1+t^2} \text{ and } y = \frac{2t}{1+t^2}$$

Put $t = \tan \theta$ in both the equations to get

$$x = \frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta \text{ and } y = \frac{2 \tan \theta}{1+\tan^2 \theta} = \sin 2\theta.$$

Differentiating both the equations, we get $\frac{dx}{d\theta} = -2 \sin 2\theta$ and $\frac{dy}{d\theta} = 2 \cos 2\theta$.

$$\text{Therefore } \frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\frac{x}{y}.$$

77. (d)

$$y = \sqrt{x+1 + \sqrt{x+1 + \sqrt{x+1 \dots \text{to } \infty}}} \Rightarrow y = \sqrt{x+1+y}$$

$$\Rightarrow y^2 = x+y+1 \Rightarrow 2y \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} (2y-1) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

78. (b)

$$y = (x+1)^{(x+1)^{(x+1) \dots \infty}} \Rightarrow y = (x+1)^y$$

$$\Rightarrow \log_e y = y \log_e (x+1)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{y}{(x+1)} + \ln(x+1) \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \ln(x+1) \right) \frac{dy}{dx} = \frac{y}{x+1}$$

$$\Rightarrow (x+1)(1 - \ln y) \frac{dy}{dx} = y^2$$

79. (a)

$$y = x^2 + \frac{2}{y} \Rightarrow y^2 = x^2 y + 2$$

$$\Rightarrow 2y \frac{dy}{dx} = y \cdot 2x + x^2 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{2y-x^2}$$

80. (c)

$$x = e^{2y+x}$$

Taking log both sides, $\log x = (2y + x) \log e = 2y + x$

$$\Rightarrow 2y + x = \log x \Rightarrow 2 \frac{dy}{dx} + 1 = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1-x}{2x}$$

CONTINUITY & DIFFERENTIABILITY
EXERCISE 1(B)

1. $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (2x - 1) = -3$

$$f(-1) = (4x + 1)_{x=-1} = -3$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (4x + 1) = -3$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (4x + 1) = 5$$

$$f(1) = (4x + 1)_{x=1} = 5$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3 - 2x) = 1$$

As $\lim_{x \rightarrow -1^-} f(x) = f(-1) = \lim_{x \rightarrow -1^+} f(x) = -3$ Hence $f(x)$ is cont. at $x = -1$.

Further $\lim_{x \rightarrow 1^-} f(x) = f(1) \neq \lim_{x \rightarrow 1^+} f(x)$ hence $f(x)$ is discont. at $x = 1$.

Being all linear functions $f(x)$ is cont. everywhere else.

Ans.[B]

2.
$$f(x) = \begin{cases} \frac{|\tan x|}{x}, & x < 0 \\ \frac{1 - e^{\sin x}}{\sin x}, & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|\tan x|}{x} = \lim_{x \rightarrow 0^-} \frac{-\tan x}{x} = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1 - e^{\sin x}}{\sin x} = 1$$

As $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ hence $f(x)$ is discont. at $x = 0$ and discontinuity is not removable.

Ans.[C]

3. As codomain of $f(x)$ is open interval hence $f(x)$ can approach ± 2 as limit but cant acquire these values. If $f(x)$ is onto then at the points where it approaches ± 2 it will be discontinuous but as $f(x)$ is given cont. hence it must be an into function.

Ans.[B]

4. As codomain of $f(x)$ contains only natural numbers and no two natural numbers are adjacent on number line (between every two consecutive natural numbers there will be infinitely many non integral numbers) hence to be continuous $f(x)$ must be a constant function.

Ans.[B]

5. $f(g(x))$ may not be continuous at $x = a$ as if $g(a) = b$ and $f(x)$ is discont. at $x = b$, then $f(g(x))$ will be discontinuous but it is not necessary.

Ans.[D]

6. $f(-1-0) = -1, f(-1) = -(-1) = 1$

$$\Rightarrow f(-1-0) \neq f(-1)$$

$$\Rightarrow f(x) \text{ is not continuous at } x = -1$$

Further, $f(1) = -1$

$$f(1+0) = 1 \quad \Rightarrow f(1) \neq f(1+0)$$

$$\Rightarrow f(x) \text{ is not continuous at } x = 1.$$

Ans.[D]

7. Since $f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

but $f(0) = 0$ (given)

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^k \cos(1/x) = 0, \text{ if } k > 0.$$

Ans.[B]

8. Obviously function $f(x)$ is discontinuous at $x = 0$ and $x = 1$ because the function is not defined, when $x < 0$ and $x > 1$, therefore $f(0-0)$ and $f(1+0)$ do not exist. Again

$$f\left(\frac{1}{2}+0\right) = \lim_{x \rightarrow 1/2} \left(\frac{3}{2} - x\right) = 1$$

$$f\left(\frac{1}{2}-0\right) = \lim_{x \rightarrow 1/2} \left(\frac{1}{2} - x\right) = 0$$

$$\therefore f\left(\frac{1}{2}+0\right) \neq f\left(\frac{1}{2}-0\right)$$

function $f(x)$ is discontinuous at $x = \frac{1}{2}$

Ans.[B]

9. $\therefore f(x)$ is continuous at $x = 2$

$$\therefore f(2-0) = f(2+0) = f(2) = k$$

But $f(2+0)$

$$= \lim_{h \rightarrow 0} \frac{(2+h)^3 + (2+h)^2 - 16(2+h) + 20}{(2+h-2)^2}$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + 7h^2}{h^2} = 7$$

Ans. [C]

10. Since $f(x)$ is continuous at $x = 2$

$$\therefore f(2) = \lim_{x \rightarrow 2^+} f(x)$$

$$\Rightarrow 1 = \lim_{x \rightarrow 2^+} (ax + b)$$

$$\therefore 1 = 2a + b \quad \dots(1)$$

Again $f(x)$ is continuous at $x = 4$,

$$\therefore f(4) = \lim_{x \rightarrow 4^-} f(x)$$

$$\Rightarrow 7 = \lim_{x \rightarrow 4^-} (ax + b)$$

$$\therefore 7 = 4a + b \quad \dots(2)$$

Solving (1) and (2), we get $a = 3$, $b = -5$.

Ans.[B]

11. Let us first examine continuity at $x = 0$.

$$f(0) = 0 \quad (\because 0 \in \mathbb{Q})$$

$$= f(0-0) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} \{ -h \text{ or } h \text{ according as } -h \in \mathbb{Q} \text{ or } -h \notin \mathbb{Q} \}$$

$$= 0$$

$$f(0+0) = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} \{ h \text{ or } -h \} = 0$$

$$f(0) = f(0-0) = f(0+0)$$

$\Rightarrow f(x)$ is continuous at $x = 0$.

Now let $a \in \mathbb{R}$, $a \neq 0$, then

$$\begin{aligned} f(a-0) &= \lim_{h \rightarrow 0} f(a-h) \\ &= \lim_{h \rightarrow 0} \{(a-h) \text{ or } -(a-h)\} \\ &= a \text{ or } -a, \text{ which is not unique.} \end{aligned}$$

$\Rightarrow f(a-0)$ does not exist

$\Rightarrow f(x)$ is not continuous at $a \in \mathbb{R}_0$.

Hence $f(x)$ is continuous only at $x = 0$.

Ans.[B]

12. We know that $[x]$ is discontinuous at every integer. Therefore it is continuous only at $x = 1/2$, while the function x is continuous at all points $x = 0, -1, 1, 1/2$. Thus the given function is continuous only at $x = 1/2$.

Ans.[D]

13.
$$f\left(\frac{\pi}{2}-0\right) = \lim_{h \rightarrow 0} \frac{1 - \sin^3\left(\frac{\pi}{2}-h\right)}{3 \cos^2\left(\frac{\pi}{2}-h\right)}$$
$$= \lim_{h \rightarrow 0} \frac{1 - \cos^3 h}{3 \sin^2 h}$$
$$= \lim_{h \rightarrow 0} \frac{(1 - \cosh)(1 + \cosh + \cos^2 h)}{3(1 - \cosh)(1 + \cosh)} = 1/2$$

$$f\left(\frac{\pi}{2}+0\right) = \lim_{h \rightarrow 0} \frac{b \left[1 - \sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi - 2\left(\frac{\pi}{2}+h\right)\right]}$$
$$= \lim_{h \rightarrow 0} \frac{b(1 - \cosh)}{4h^2}$$
$$= \lim_{h \rightarrow 0} \frac{2b \sin^2 h / 2}{4h^2} = \frac{b}{8}$$

Now $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\Rightarrow f\left(\frac{\pi}{2}-0\right) = f\left(\frac{\pi}{2}+0\right) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{1}{2} = \frac{b}{8} = a$$

$$\therefore a = 1/2, b = 4$$

Ans.[C]

14. Obviously the function $f(x)$ is continuous at $x = 1$ and 3 . Therefore $\lim_{x \rightarrow 1^+} f(x) = f(1)$

$$\Rightarrow a + b = 2 \quad \dots(1)$$

$$\text{and } \lim_{x \rightarrow 3^-} f(x) = f(3)$$

$$\Rightarrow 3a + b = 6 \quad \dots(2)$$

Solving (1) and (2), we get $a = 2, b = 0$.

Ans.[C]

15.
$$f(0-0) = \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2} = \frac{2 \sin^2 2x}{x^2} = 8$$

$$f(0+0) = \lim_{x \rightarrow 0} \frac{\sqrt{x}}{(\sqrt{16+\sqrt{x}}-4)} \times \frac{\sqrt{16+\sqrt{x}}+4}{\sqrt{16+\sqrt{x}}+4}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{x}(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16} = 8$$

$$\therefore f(0+0) = f(0-0)$$

$\therefore f(x)$ can be continuous at $x = 0$, if

$$f(0) = a = 8.$$

Ans.[B]

16. As given $f(0-0) = f(0+0) = k$

$$\text{Now } f(0-0) = \lim_{h \rightarrow 0} \frac{\cos \frac{(-h)}{2[-h]}}{[-h]}$$

$$= \lim_{h \rightarrow 0} \frac{\cos \left(\frac{-h}{2(-1)} \right)}{-1} = -1$$

$$f(0+0) = \lim_{h \rightarrow 0} \frac{\sin[h]}{[h]+1} = \lim_{h \rightarrow 0} \frac{\sin 0}{0+1} = 0$$

$\therefore f(0-0) \neq f(0+0)$, so k is indeterminate.

Ans.[D]

$$17. \quad f(0-0) = \lim_{h \rightarrow 0} (1 + |\sin(-h)|)^{a/|\sin(-h)|}$$

$$= (1 + \sin h)^{a/\sin h} = e^a$$

$$f(0+0) = \lim_{h \rightarrow 0} \frac{\tan 2h}{e^{\tan 3h}} = e^{\lim_{h \rightarrow 0} \left(\frac{\tan 2h}{\tan 3h} \right)}$$

$$= \frac{\lim_{h \rightarrow 0} 2 \sec^2 2h}{3 \sec^2 3h} = e^{2/3}$$

Now $f(x)$ is continuous at $x = 0$

$$\Rightarrow f(0-0) = f(0+0) = f(0)$$

$$\Rightarrow e^a = e^{2/3} = b$$

$$\therefore a = 2/3, b = e^{2/3}$$

Ans.[A]

$$18. \quad f(x) = |x-1| + |x-2| + |x-3| \Rightarrow f(x) = \begin{cases} 6-3x & x < 1 \\ 4-x & 1 \leq x < 2 \\ x & 2 \leq x < 3 \\ 3x-6 & x \geq 3 \end{cases}$$

It can be easily verified that $f(x)$ is cont. everywhere, now

$$f'(x) = \begin{cases} -3, & x < 1 \\ -1, & 1 < x < 2 \\ 1, & 2 < x < 3 \\ 3, & x > 3 \end{cases}$$

Hence $f(x)$ is not diff. at $x = 1, 2$ & 3

Ans.[B]

19. In neighborhood of $x = 1$, $\sin\left(\frac{1}{x-1}\right)$ will not have a limiting value but it will be a finite number

lying in $[-1, 1]$. Further $\lim_{x \rightarrow 1} (\ln x) = 0$, hence $\lim_{x \rightarrow 1} (\ln x)^k = 0$ if $k > 0$.

$$\therefore \lim_{x \rightarrow 1} (\ln x)^k \sin\left(\frac{1}{x-1}\right) = 0 = f(1) \text{ for } k > 0.$$

$$\begin{aligned} \text{Now } f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{(\ln x)^k \sin\left(\frac{1}{x-1}\right)}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{\ln(1+y)}{y} \times \lim_{x \rightarrow 1} (\ln(1+y))^{k-1} \sin\left(\frac{1}{y}\right). \end{aligned}$$

By the same argument used above, k must be greater than 1 for $\lim_{x \rightarrow 1} (\ln(1+y))^{k-1} \sin\left(\frac{1}{y}\right)$.

Hence $f(x)$ will be diff. if $k > 1$.

Ans.[B]

20. $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (ax^2 + b \cos \pi x) = a - b$ &

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^4 = 1 = f(1)$$

For $f(x)$ to be cont. at $x = 1$, $a - b = 1$.

Now further

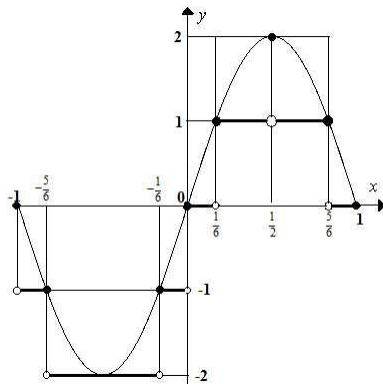
$$f'(x) = \begin{cases} 2ax - \pi b \sin \pi x, & x < 1 \\ 4x^3, & x > 1 \end{cases}$$

$$\text{LHD}_{x=1} = 2a \text{ \& \text{RHD}_{x=1} = 4}$$

For $f(x)$ to be diff. at $x = 1$, $2a = 4$ i.e. $a = 2$ & $b = 1$.

Ans.[B]

21. Refer the adjoining graph.



Ans.[C]

22. $f(x) = \sin^{-1} \frac{2x}{1+x^2} = \begin{cases} -\pi - 2 \tan^{-1} x, & x < -1 \\ 2 \tan^{-1} x, & -1 \leq x \leq 1 \\ \pi - 2 \tan^{-1} x, & x > 1 \end{cases}$

Hence $f(x)$ is cont. everywhere and not diff. at $x = \pm 1$.

Ans.[B]

23. at $x = 0$:

$$f'(0-0) = \lim_{h \rightarrow 0} \frac{|0-h|-0}{-h} = -1$$

$$f'(0+0) = \lim_{h \rightarrow 0} \frac{|0+h|-0}{h} = 1$$

Now, since $f'(0-0) \neq f'(0+0)$
 $\Rightarrow f(x)$ is not differentiable at $x = 0$.

Ans.[B]

24. Differentiability at $x = 0$

$$R[f'(0)] = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(0+h)^2 - 0}{h} = \lim_{h \rightarrow 0} h = 0$$

$$L[f'(0)] = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-(0-h) - 0}{-h} = -1$$

$$\therefore R[f'(0)] \neq L[f'(0)]$$

$\therefore f(x)$ is not differentiable at $x = 0$

Differentiability at $x = 1$

$$R[f'(1)] = \lim_{h \rightarrow 0} \frac{f(1+h)^3 - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^3 - (1+h) + 1 - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2h + 3h^2 + h^3}{h} = 2$$

$$L[f'(1)] = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{(1-h) - 1}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h + h^2}{-h} = 2$$

Thus $R[f'(1)] = L[f'(1)]$

\therefore function $f(x)$ is differentiable at $x = 1$

Ans.[B]

25. Since $f(1-0) = \lim_{x \rightarrow 1} 3^x = 3$

$$f(1+0) = \lim_{x \rightarrow 1} (4-x) = 3$$

$$\text{and } f(1) = 3^1 = 3$$

$$f(1-0) = f(1+0) = f(1)$$

$\therefore f(x)$ is continuous at $x = 1$

$$\Rightarrow \text{Again } f'(1+0) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{3^x - 3}{x - 1}$$

$$= \lim_{h \rightarrow 0} \frac{3^{1+h} - 3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3^{1+h} - 3}{h}$$

$$= 3 \lim_{h \rightarrow 0} \frac{3^h - 1}{h}$$

$$= 3 \log 3$$

$$\text{and } f'(1+0) = \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{4 - x - 3}{x - 1} = -1$$

$$\therefore f'(1+0) \neq f'(1-0)$$

$\neq f(x)$ is not differentiable at $x = 1$.

Ans.[A]

26. When $x < 0$, $f(x) = \frac{x}{1-x}$

$$f'(x) = \frac{1}{(1-x)^2} \quad \dots(1)$$

which exists finitely for all $x < 0$

Also when $x > 0$, $f(x) = \frac{1}{1+x}$

$$\Rightarrow f'(x) = \frac{1}{(1+x)^2} \quad \dots(2)$$

which exists finitely for all $x > 0$. Also from (1) and (2) we have

$$\begin{cases} f'(0-0) = 1 \\ f'(0+0) = 1 \end{cases} \Rightarrow f'(0) = 1$$

Hence $f(x)$ is differentiable $\square x \in \mathbb{R}$

Ans.[A]

27. When $x \neq 0$

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$$

$$= 2x \sin \frac{1}{x} - \cos \left(\frac{1}{x}\right)$$

which exists finitely for all $x \neq 0$

$$\text{and } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin 1/x}{x} = 0$$

$\therefore f$ is also derivable at $x = 0$. Thus

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{Also } \lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$$

$$= 2 - \lim_{x \rightarrow 0} \cos \frac{1}{x}$$

But $\lim_{x \rightarrow 0} \cos \frac{1}{x}$ does not exist, so $\lim_{x \rightarrow 0} f'(x)$ does not exist. Hence f' is not continuous (so not derivable) at $x = 0$.

Ans.[B]

28. $\sqrt{1-x} + \sqrt{1-y} = 1 \Rightarrow y = x + 2\sqrt{1-x} - 1$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{\sqrt{1-x}} = \frac{1 - \sqrt{1-x}}{\sqrt{1-x}}$$

Ans.[B]

29. $x = \ln(y + \sqrt{1+y^2}) \Rightarrow \sqrt{1+y^2} + y = e^x \text{ \& \ } \sqrt{1+y^2} - y = e^{-x}$

$$\Rightarrow y = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}$$

Ans.[C]

30. for $f(x)$, $y = x + \ln x$

then for $f^{-1}(x)$, $x = y + \ln y$

$$\Rightarrow \frac{dx}{dy} = 1 + \frac{1}{y} \text{ or } \frac{dy}{dx} = \frac{y}{1+y}$$

Further $x + \ln x = 1 \Rightarrow x = 1$, hence for $f^{-1}(x)$, $y = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+1} = \frac{1}{2}$$

Ans.[B]

31. for $x > \frac{1}{2}$, $\sin^{-1}(3x - 4x^3) = \pi - 3\sin^{-1} x$

Now $y = \pi - 3\sin^{-1} x \Rightarrow \frac{dy}{dx} = -\frac{3}{\sqrt{1-x^2}}$

Ans.[B]

32. $x = a \sec \theta \Rightarrow \frac{dx}{d\theta} = a \sec \theta \tan \theta$, $y = b \tan \theta \Rightarrow \frac{dy}{d\theta} = b \sec^2 \theta$

$$\Rightarrow \frac{dy}{dx} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b^2 x}{a^2 y}$$

Ans.[A]

33. $\frac{dx}{d\theta} = a(1 + \cos \theta)$, $\frac{dy}{d\theta} = a \sin \theta$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 + \cos \theta)} = \tan \frac{1}{2} \theta \quad \text{Ans.[C]}$$

34. $y = \log e^x - \log(e^x + 1)$
 $= x - \log(e^x + 1)$

$$\therefore \frac{dy}{dx} = 1 - \frac{e^x}{e^x + 1} = \frac{1}{e^x + 1}$$

Ans.[A]

35. $\frac{dy}{dx} = \frac{-2x}{(x^2 - a^2)^2} \Rightarrow \frac{d^2y}{dx^2}$

$$= -\frac{(x^2 - a^2)^2 \cdot 2 - 2x \cdot 2(x^2 - a^2) \cdot 2x}{(x^2 - a^2)^4}$$

$$= \frac{2(3x^2 + a^2)}{(x^2 - a^2)^3}$$

Ans.[C]

36. $y = \frac{\sec x - \tan x}{\sec x + \tan x} \cdot \frac{\sec x - \tan x}{\sec x - \tan x}$
 $= (\sec x - \tan x)^2 / 1$

$\therefore \frac{dy}{dx} = 2(\sec x - \tan x) (\sec x \tan x - \sec^2 x)$
 $= -2 \sec x (\sec x - \tan x)^2$

Ans.[B]

37. Let us first express y in terms of x because all alternatives are in terms of x. So

$x\sqrt{1+y} = -y\sqrt{1+x}$
 $\Rightarrow x^2(1+y) = y^2(1+x)$
 $\Rightarrow x^2 - y^2 + x^2y - y^2x = 0$
 $\Rightarrow (x-y)(x+y+xy) = 0$
 $\Rightarrow x+y+xy = 0 \quad (\because x \neq y)$

$\Rightarrow y = -\frac{x}{1-x}$

$\therefore \frac{dy}{dx} = -\frac{(1+x)1-x \cdot 1}{(1+x)^2} = -\frac{1}{(1+x)^2}$

Ans.[B]

38. Taking log on both sides, we have

$y \log x + x \log y = 0$

Now using partial derivatives, we have

$\frac{dy}{dx} = -\frac{y/x + \log y}{\log x + x/y} = -\frac{y(y+x \log y)}{x(x+y \log x)}$

Ans [D]

39. Here $y = \sqrt{\sin x + y} \Rightarrow y^2 = \sin x + y$

$\therefore 2y \frac{dy}{dx} = \cos x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$

Ans.[B]

40. $y = e^{x+y}$

$\Rightarrow \log y = x + y \Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{1-y}$ **Ans.[C]**

CONTINUITY & DIFFERENTIABILITY
EXERCISE 1(C)

$$1. \quad \begin{aligned} \lim_{h \rightarrow 0} g(n+h) &= \lim_{h \rightarrow 0} \frac{e^h - \cos 2h - h}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{e^h - h - 1}{h^2} + \lim_{h \rightarrow 0} \frac{(1 - \cos 2h)}{4h^2} \cdot 4 = \frac{1}{2} + 2 = \frac{5}{2} \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} g(n-h) &= \frac{e^{1-\{n-h\}} - \cos 2(1-\{n-h\}) - (1-\{n-h\})}{(1-\{n-h\})^2} \\ &= \lim_{h \rightarrow 0} \frac{e^h - \cos 2h - h}{h^2} \quad (\{n-h\} = \{-h\} = 1-h) = \frac{5}{2} \end{aligned}$$

$g(n) = \frac{5}{2}$. Hence $g(x)$ is continuous at $\forall x \in I$.

Hence $g(x)$ is continuous $\forall x \in \mathbb{R}$]

2. By theorem, if g and h are continuous functions on the open interval (a, b) , then g/h is also continuous at all x in the open interval (a, b) where $h(x)$ is not equal to zero.

$$3. \quad h(x) = \begin{cases} \frac{2 \cos x - \sin 2x}{(\pi - 2x)^2} & x < \frac{\pi}{2} \\ \frac{e^{-\cos x} - 1}{8x - 4\pi} & x > \frac{\pi}{2} \end{cases}$$

LHL at $x = \pi/2$

$$\lim_{h \rightarrow 0} \frac{2 \sin h - \sin 2h}{4h^2} = \lim_{h \rightarrow 0} \frac{2 \sin h(1 - \cosh h)}{4h^2} = 0$$

$$\text{RHL: } \lim_{h \rightarrow 0} \frac{e^{\sin h} - 1}{((\pi/2) + h) - 4\pi} = \lim_{h \rightarrow 0} \frac{e^{\sin h} - 1}{8h} \cdot \frac{\sin h}{\sin h} = \frac{1}{8}$$

\Rightarrow $h(x)$ is discontinuous at $x = \pi/2$.

Irremovable discontinuity at $x = \pi/2$.

$$f\left(\frac{\pi^+}{2}\right) = 0 \text{ and } g\left(\frac{\pi^-}{2}\right) = \frac{1}{8} \quad \Rightarrow \quad f\left(\frac{\pi^+}{2}\right) \neq g\left(\frac{\pi^-}{2}\right)]$$

$$4. \quad \lim_{x \rightarrow 0} \frac{x - e^x + 1 - (1 - \cos 2x)}{x^2} = -\frac{1}{2} - 2 = -\frac{5}{2}; \quad \text{hence for continuity } f(0) = -\frac{5}{2}$$

$$\therefore [f(0)] = -3; \{f(0)\} = \left\{-\frac{5}{2}\right\} = \frac{1}{2}; \text{ hence } [f(0)] \{f(0)\} = -\frac{3}{2} = -1.5$$

$$5. \quad g'(0^+) = \lim_{h \rightarrow 0} \frac{\cosh h - 1}{h} = 0$$

$$g'(0^-) = \lim_{h \rightarrow 0} \frac{-h + b - 1}{-h} \text{ for existence of limit } b = 1 \text{ thus } g'(0^-) = 1$$

Hence g can not be made differentiable for any value of b .]

$$6. \quad \left[\begin{array}{l} x - 2k\pi \text{ for } 2k\pi - \frac{\pi}{2} \leq x \leq 2k\pi + \frac{\pi}{2} \\ (2k+1)\pi - x \text{ for } 2k\pi + \frac{\pi}{2} \leq x < 2k\pi + \frac{3\pi}{2} \end{array} \right.$$

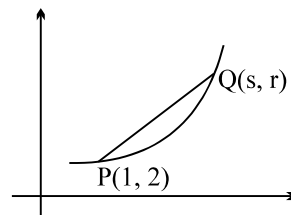
$$7. \quad \begin{array}{l} f(1^+) = f(1^-) = f(1) = 2 \\ f(0) = 1, \quad f(2) = 2 \\ f(2^-) = 1; \quad f(2) = 2 \end{array} \Rightarrow f \text{ is not continuous at } x = 2$$

$$8. \quad x^{1/3} \text{ is not differentiable at } x = 0]$$

$$9. \quad f(2^+) = 8 ; f(2^-) = 16$$

10. **I** By definition $f'(1)$ is the limit of the slope of the secant line when $s \rightarrow 1$.

$$\begin{aligned} \text{Thus } f'(1) &= \lim_{s \rightarrow 1} \frac{s^2 + 2s - 3}{s - 1} \\ &= \lim_{s \rightarrow 1} \frac{(s-1)(s+3)}{s-1} \\ &= \lim_{s \rightarrow 1} (s+3) = 4 \end{aligned} \Rightarrow \text{(D)}$$



II By substituting $x = s$ into the equation of the secant line, and cancelling by $s - 1$ again, we get $y = s^2 + 2s - 1$. This is $f(s)$, and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.]

$$11. \quad g(x) = x - [x] = \{x\}$$

f is continuous with $f(0) = f(1)$

$$h(x) = f(g(x)) = f(\{x\})$$

Let the graph of f is as shown in the figure satisfying

$$f(0) = f(1)$$

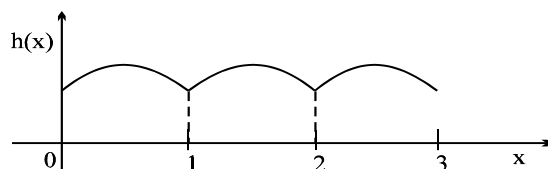
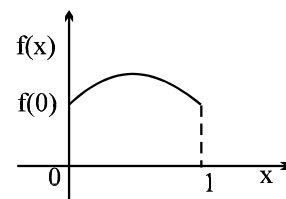
$$\text{now } h(0) = f(\{0\}) = f(0) = f(1)$$

$$h(0.2) = f(\{0.2\}) = f(0.2)$$

$$h(1.5) = f(\{1.5\}) = f(0.5) \text{ etc.}$$

Hence the graph of $h(x)$ will be periodic graph as shown

$\Rightarrow h$ is continuous in $\mathbb{R} \Rightarrow C$



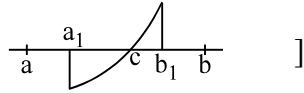
]

12. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(h) + |x|h + xh^2}{h}$ where $x = h$ and $y = x$

$\therefore f(0) = 0$; hence $f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(h) - f(0)}{h} + |x| + xh \right)$

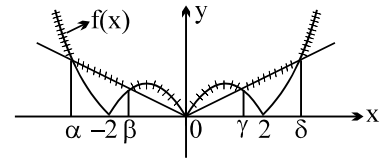
$f'(x) = f'(0) + |x| = |x|$

13. I and II are false. The function $f(x) = 1/x, 0 < x < 1$, is a counter example. Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$



14. $f(x)$ is non differentiable at $x = \alpha, \beta, 0, \gamma, \delta$

and $g(x)$ is non differentiable at $x = \alpha, \beta, 0, -2, 2 \Rightarrow$ (B)



15. $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \geq 1 \end{cases}$

for differentiability at $x = 1, g'(1^+) = g'(1^-)$

$$a = 6x - \frac{4}{2\sqrt{x}} \Rightarrow a = 6 - 2 = 4$$

for continuity at $x = 1, g(1^+) = g(1^-)$

$$a + b = 3 - 4 + 1 \Rightarrow a + b = 0 \Rightarrow b = -4$$

$a = 4$, and $b = -4$ Ans.]

16. $g(x) = \frac{\sin \frac{\pi[x]}{4}}{[x]}$

obv. cont. at $x = 3/2$

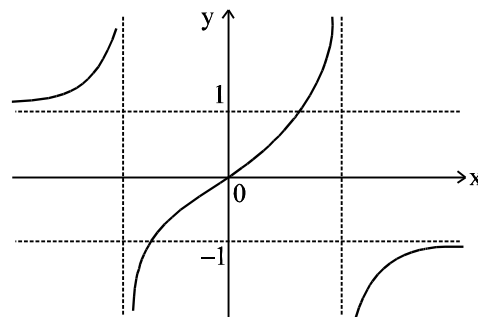
$$\text{at } x = 2 f(2^-) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f(2) = \frac{\sin \frac{\pi}{2}}{2} = \frac{1}{2}$$

Hence discontinuous at $x = 2$

17. not derivable at $x = 0$ and 2

18. $f(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \geq 0, x \neq 1 \\ \frac{x}{1+x} & \text{if } x < 0, x \neq -1 \end{cases}$



$$\text{and } f'(x) = \left[\begin{array}{l} \frac{1}{(1-x)^2} \text{ if } x > 0, x \neq 1 \\ \frac{1}{(1+x)^2} \text{ if } x < 0, x \neq -1 \end{array} \right]$$

19. $f(-1) = b(1-1) + 1 = 1$

and $\lim_{h \rightarrow 0} f(-1+h) = 1$

$$\lim_{h \rightarrow 0} f(-1-h) = \sin((-1+h+a)\pi) = -\sin \pi a$$

for continuity $\sin \pi a = -1 = \sin\left(2n\pi + \frac{3\pi}{2}\right) \Rightarrow \pi a = 2n\pi + \frac{3\pi}{2} \Rightarrow a = 2n + \frac{3}{2}$

hence $a = 2n + \frac{3}{2}$, $n \in I$ and $b \in R$

20. $f(0) = \lim_{x \rightarrow 0} \frac{\ln(e^{x^2} + 2\sqrt{x})}{\sqrt{x}} = \lim_{x \rightarrow 0} \ln(e^{x^2} + 2\sqrt{x})^{\frac{1}{\sqrt{x}}}$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} (e^{x^2} + 2\sqrt{x} - 1) = \lim_{x \rightarrow 0} \left(\frac{e^{x^2} - 1}{\sqrt{x}} + 2 \right) = 2$$

21. $f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 1$ also $f(0) = -c$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) + f(h) + c - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = f'(0) = 1$$

$\therefore f'(x) = 1$

22. Differentiate column wise, where $\Delta_1 = -4$; $\Delta_2 = 0$ and $\Delta_3 = 8$

23. $D^*f(x) = 2f(x) \cdot f'(x)$

$$D^*(x \ln x) = 2x \ln x (1 + \ln x)$$

24. In the immediate neighborhood of $x = \pi/2$, $\sin x > \sin^3 x \Rightarrow |\sin x - \sin^3 x| = \sin x - \sin^3 x$

$$\text{Hence for } x \neq \pi/2, \quad f(x) = \left[\frac{2(\sin x - \sin^3 x) + \sin x - \sin^3 x}{2(\sin x - \sin^3 x) - \sin x + \sin^3 x} \right] = \frac{3 \sin x - 3 \sin^3 x}{\sin x - \sin^3 x} = 3$$

Hence f is continuous and diff. at $x = \pi/2$

25. $y = (A + Bx)e^{mx} + (m-1)^{-2} \cdot e^x$

$$y \cdot e^{-mx} = (A + Bx) + (m-1)^{-2} \cdot e^{(1-x)x}$$

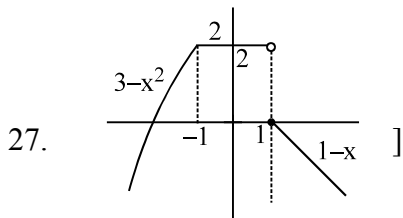
$$e^{-mx} \cdot y_1 - my + e^{-mx} = B - (m-1)^{-1} \cdot e^{-(m-1)x}$$

$$e^{-mx} \cdot y_2 - y_1 e^{-mx} \cdot m - m[e^{-mx} \cdot y_1 - y e^{-mx} \cdot m] = e^{-(m-1)x}$$

$$e^{-mx} \cdot y_1 - m_2 y_1 e^{-mx} + m y \cdot e^{-mx} = e^{-(m-1)x}$$

$$y_2 - 2m y_1 + m y = e^x \text{ Ans.]}$$

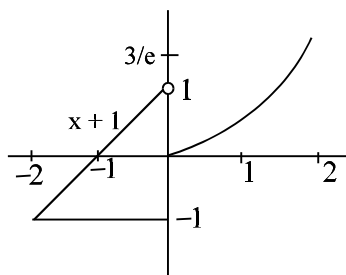
26. Let $f(x) = px^2 + qx + r$
 $f(1) = f(-1)$ gives $p + q + r = p - q + r$
hence $q = 0$
Hence $f(x) = px^2 + r$
 $f'(x) = 2px$ (1)
Given a, b, c are in A.P.
hence $2pa, 2pb, 2pc$ will also be in A.P.
or $f'(a), f'(b), f'(c)$ will also be in A.P. \Rightarrow (D)]



28. $2x + 2yy' = 0$
 $x + yy' = 0 \Rightarrow y' = -\frac{x}{y}$ (1)
 $1 + yy'' + (y')^2 = 0$
 $y'' = -\frac{1 + (y')^2}{y}$

$$\text{now } k = \frac{y''}{(1 + (y')^2)^{3/2}} = -\frac{1 + (y')^2}{y(1 + (y')^2)^{3/2}} = -\frac{1}{y\sqrt{1 + (y')^2}} = -\frac{1}{y\sqrt{1 + \frac{x^2}{y^2}}} = -\frac{1}{\sqrt{y^2 + x^2}} = -\frac{1}{R}$$

29. $f(x) = \begin{cases} (x+1)e^{-2/x} & \text{if } x > 0 \\ x+1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$



the graph of $f(x)$ is
hence f can assume all values for $f(-2)$ to $f(2)$

30. Put $\cos \phi = \frac{2}{\sqrt{13}}$; $\sin \phi = \frac{3}{\sqrt{13}}$; $\tan \phi = \frac{3}{2}$
 $y = \cos^{-1}\{\cos(x + \phi)\} + \sin^{-1}\{\cos(x - \phi)\}$
 $= \cos^{-1}\{\cos(x + \phi)\} + \frac{\pi}{2} - \cos^{-1}\{\cos(\phi - x)\}$ (think !)
 $= x + \phi + \frac{\pi}{2} - \phi + x$
 $y = 2x + \frac{\pi}{2}$; $z = \sqrt{1 + x^2}$

now compute $\frac{dy}{dz}$

31. We have $f(x) = \frac{x^2 - x}{x^2 + 4x}$

$f(x)$ is not defined at $x = 0, -4$

\therefore domain of $f = \mathbb{R} - \{0, -4\}$

For all $x \in$ domain of f , we have

$$f(x) = \frac{x^2 - x}{x^2 + 4x} = \frac{x - 1}{x + 4} = 1 - \frac{5}{x + 4}$$

$$f(f^{-1}(x)) = x$$

$$1 - \frac{5}{f^{-1}(x) + 4} = x \quad \text{or} \quad 1 - x = \frac{5}{f^{-1}(x) + 4}$$

$$\therefore f^{-1}(x) = \frac{5}{1 - x} - 4$$

$$\frac{d}{dx}(f^{-1}(x)) = \frac{5}{(1 - x)^2}; \quad \therefore \frac{d}{dx}(f^{-1}(x))_{\text{at } x=2} = \frac{5}{(1 - 2)^2} = 5 \text{ Ans.]}$$

32. $f(x) = \sqrt{x + 2\sqrt{2x - 4}} + \sqrt{x - 2\sqrt{2x - 4}}$

$$\therefore f(x) = \sqrt{(\sqrt{x - 2} + \sqrt{2})^2} + \sqrt{(\sqrt{x - 2} - \sqrt{2})^2} = |\sqrt{x - 2} + \sqrt{2}| + |\sqrt{x - 2} - \sqrt{2}|$$

for $\sqrt{x - 2}$ to exist $x \geq 2$

Also, $\sqrt{x - 2} + \sqrt{2} > 0$ (always true, think ! why?)

but $\sqrt{x - 2} - \sqrt{2} \geq 0$ only if $x \geq 4$
 < 0 only if $x < 4$

\therefore now $f(x)$ becomes

$$f(x) = \sqrt{x - 2} + \sqrt{2} - \sqrt{x - 2} + \sqrt{2} \quad \text{for } 2 \leq x < 4$$

$$= \sqrt{x - 2} + \sqrt{2} + \sqrt{x - 2} - \sqrt{2} \quad \text{for } x \geq 4$$

$$\therefore f(x) = 2\sqrt{2}, \quad \text{for } 2 \leq x < 4$$

$$= 2\sqrt{x - 2}, \quad \text{for } 4 \leq x < \infty$$

\therefore f is continuous $[2, 4) \cup [4, \infty)$ (verify)

$$\therefore f'(x) = 0, \quad 2 \leq x < 4$$

$$= \frac{1}{\sqrt{x - 2}}, \quad 4 \leq x < \infty$$

$$\therefore f'(102^+) = \frac{1}{\sqrt{102 - 2}} = \frac{1}{10}$$

$$\therefore 10 f'(102^+) = 1$$

33. $A : 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$$\therefore y'(\sqrt{2}) = -1 = A$$

$$B : \cos y \cdot y' + \cos x = \sin x \cdot \cos y \cdot y' + \sin y \cdot \cos x$$

when $x = y = \pi$

$$-y' - 1 = 0 + 0 \quad \Rightarrow \quad y'(\pi) = -1$$

$$C : 2e^{xy}(xy' + y) + e^x e^y y' + e^y e^x - e^x - e^y y' = e \cdot e^{xy}(xy' + y)$$

at $x = 1, y = 1$

$$2e(y' + 1) + e^2 y' + e^2 - e - e y' = e^2(y' + 1)$$

$$e y' + e = 0 \quad \Rightarrow \quad y' = -1$$

$$\text{hence } A + B + C = -3$$

$$34. \quad C_2 \rightarrow C_2 - xC_3 \\ \Rightarrow f(x) = x^2(\tan x - \cos x) \\ \Rightarrow f'(x) = (\tan x - \cos x)2x - x^2(\sec^2 x + \sin x)$$

$$35. \quad \frac{x+a}{2} = b \cot^{-1}(b \ln y); \quad \cot\left(\frac{x+a}{2b}\right) = b \ln y$$

$$\therefore -\operatorname{cosec}^2\left(\frac{x+a}{2b}\right) \frac{1}{2b} = \frac{b}{y} y'; \quad \therefore -\frac{1}{2b^2} \left(1 + \cot^2\left(\frac{x+a}{2b}\right)\right) = \frac{y'}{y}$$

$$\therefore -\frac{1}{2b^2} \left(1 + (b \ln y)^2\right) = \frac{y'}{y}; \quad \therefore -\frac{1}{2b^2} \left(2(b \ln y) \frac{b}{y} y'\right) = \frac{yy'' - y'^2}{y^2}$$

$$\therefore -\ln y y' = y y'' - y'^2; \quad \therefore y y'' = y'^2 - y' y \ln y$$

$$\therefore y y'' + y y' \ln y = y'^2 - y' y \ln y + y y' \ln y = y'^2$$

$$36. \quad \lim_{x \rightarrow 0} x^x = 1; \quad \text{let } l = x^{x^x} \text{ hence as } x \rightarrow 0, x^x \rightarrow 1$$

$$\therefore L = (0)' - 1 = -1 \quad \Rightarrow \quad (C)$$

$$37. \quad 2 \frac{d}{dx} (y^3 y_2) = 2 (y^3 \cdot y_3 + 3 y^2 y_1 y_2).$$

Now differentiate $y^2 = P(x)$ thrice)

$$38. \quad \text{for continuity} \quad \lim_{x \rightarrow 0} \frac{1-e^x}{x} = f(0); \quad \text{hence} \quad f(0) = -\lim_{h \rightarrow 0} \frac{e^x - 1}{-x} = -1$$

$$f'(0^+) = \lim_{h \rightarrow 0} \frac{\frac{1-e^h}{h} + 1}{h} = \lim_{h \rightarrow 0} \frac{1-e^h+h}{h^2} = \frac{1-h - \left[1 + \frac{h}{1!} + \frac{h^2}{2!} + \dots\right]}{h^2} = -\frac{1}{2}$$

$$f'(0^-) = \lim_{h \rightarrow 0} \frac{\frac{1-e^{-h}}{-h} + 1}{-h} = \lim_{h \rightarrow 0} \frac{1-e^{-h}-h}{h^2} = \frac{1-h - \left[1 - \frac{h}{1!} + \frac{h^2}{2!} - \dots\right]}{h^2} = -\frac{1}{2}$$

$$\text{hence } f(x) = \begin{cases} \frac{1-e^{-x}}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

$$\begin{aligned}
39. \quad & (a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0 \\
\Rightarrow & (a^2 - 2a - 15) = 0 \quad \text{and} \quad b^2 - 2b - 15 = 0 \\
\Rightarrow & (a - 5)(a + 3) = 0 \quad \text{and} \quad (b - 5)(b + 3) = 0 \\
\Rightarrow & a = 5 \text{ or } -3 \quad \text{and} \quad b = 5 \text{ or } -3 \\
\therefore & a \neq b \text{ hence } a = 5 \quad \text{and} \quad b = -3 \\
& \quad \quad \quad \text{or} \quad a = -3 \quad \text{and} \quad b = 5 \\
\Rightarrow & ab = -15 \text{ Ans.]}
\end{aligned}$$

40. When $x = 0$, $e^y = e \Rightarrow y = 1$
Differentiating w.r.t. x , we get

$$e^y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0 \quad \dots\dots\dots(1)$$

$$e^y \frac{d^2y}{dx^2} = e^y \left(\frac{dy}{dx} \right)^2 + \frac{dy}{dx} + \frac{dy}{dx} + x \frac{d^2y}{dx^2} = 0 \quad \dots\dots\dots(2)$$

When $x = 0$, $y = 1 \quad \therefore$ From (1) $\frac{dy}{dx} = -\frac{1}{e}$

Putting the data in (2), we get

$$e \cdot \frac{d^2y}{dx^2} + e \cdot \frac{1}{e^2} - \frac{2}{e} = 0 \quad \therefore \frac{d^2y}{dx^2} = -\frac{1}{e^2} \quad]$$

$$\begin{aligned}
41. \quad N^r &= \cos 6x + (1+5) \cos 4x + (5+10) \cos 2x + 10 \\
&= \cos 6x + \cos 4x + 5(\cos 4x + \cos 2x) + 10(1 + \cos 2x) \\
&= 2 \cos 5x \cos x + 10 \cos 3x \cos x + 20 \cos^2 x \\
&= 2 \cos x [\cos 5x + 5 \cos 3x + 10 \cos x] \\
&\quad \text{-----Denominator-----}
\end{aligned}$$

$$\therefore y = \frac{N^r}{D^r} = 2 \cos x \quad \therefore \frac{dy}{dx} = -2 \sin x \Rightarrow (C)$$

$$42. \quad y^4 = x^2 - 6$$

$$4y^3 \frac{dy}{dx} = 2x \Rightarrow y^3 \frac{dy}{dx} = \frac{x}{2} \Rightarrow y^3 \frac{d^2y}{dx^2} + 3y^2 \left(\frac{dy}{dx} \right)^2 = \frac{1}{2}$$

$$y^3 \frac{d^2y}{dx^2} + 3y^2 \left(\frac{x}{2y^3} \right)^2 = \frac{1}{2} \Rightarrow y^3 \frac{d^2y}{dx^2} + 3y^2 \frac{x^2}{4y^6} = \frac{1}{2} \Rightarrow y^3 \frac{d^2y}{dx^2} + \frac{3x^2}{4y^4} = \frac{1}{2}$$

$$\Rightarrow y^3 \frac{d^2y}{dx^2} = \frac{1}{2} - \frac{3x^2}{4y^4} \Rightarrow y^3 \frac{d^2y}{dx^2} = \frac{2y^4 - 3x^2}{4y^4} \Rightarrow \frac{d^2y}{dx^2} = \frac{2y^4 - 3x^2}{4y^7}$$

$$43. \quad \frac{d}{dx} \{ [f(x)]^2 - [\phi(x)]^2 \} = 2 [f(x) \cdot f'(x) - \phi(x) \cdot \phi'(x)] = 2 [f(x) \cdot \phi(x) - \phi(x) \cdot f(x)] = 0$$

[$\therefore f'(x) = \phi(x)$ and $\phi'(x) = f(x)$]

$$\Rightarrow [f(x)]^2 - [\phi(x)]^2 = \text{constant}$$

$$\therefore [f(10)]^2 - [\phi(10)]^2 = [f(3)]^2 - [\phi(3)]^2 = [f(3)]^2 - [f'(3)]^2 = 25 - 16 = 9$$

$$44. \quad (x + y) \left(\frac{\cos^4 \alpha}{x} + \frac{\sin^4 \alpha}{y} \right) = 1 = (\cos^2 \alpha + \sin^2 \alpha)^4$$

$$\therefore \frac{y}{x} \cos^4 \alpha + \frac{x}{y} \sin^4 \alpha - 2 \sin^2 \alpha \cos^2 \alpha = 0$$

$$\text{or} \quad \left(\sqrt{\frac{y}{x}} \cos^2 \alpha - \sqrt{\frac{x}{y}} \sin^2 \alpha \right)^2 = 0$$

$$\therefore \tan^2 \alpha = \frac{y}{x} \text{ or } y = x \tan^2 \alpha \quad \therefore \frac{dy}{dx} = \tan^2 \alpha$$

45. $y = e^x + x$; diff. w.r.t y ,

$$1 = (e^x + 1) \frac{dx}{dy} ; \frac{dx}{dy} = \frac{1}{e^x + 1}$$

$$\Rightarrow \left. \frac{dx}{dy} \right|_{x=\ln 2} = \frac{1}{e^{\ln 2} + 1} = \frac{1}{3}$$

$$46. \quad \frac{dy}{dx} = \frac{x}{1+x^2} + \tan^{-1} x + \frac{1}{\left| \frac{1}{x} \right| \sqrt{\frac{1}{x^2} - 1}} \left(-\frac{1}{x^2} \right)$$

$$= \frac{x}{1+x^2} + \tan^{-1} x + \frac{|x|^2}{\sqrt{1-x^2}} \left(-\frac{1}{x^2} \right)$$

$$\text{as } x \rightarrow 0, \quad \frac{dy}{dx} = -1 \text{ Ans.}$$

Alternatively :

$$f'(0^+) = \lim_{h \rightarrow 0} \frac{h \tan^{-1}(h) + \sec^{-1}(1/h) - \pi/2}{h} = \lim_{h \rightarrow 0} \frac{\cos^{-1}(h) - \pi/2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-\sin^{-1}(h)}{h} = -1$$

$$\text{Similarly } f'(0^-) = -1$$

$$\text{Hence } f'(0) = -1$$

$$47. \quad y = \frac{\sin x}{1 + \frac{\cos x}{1 + \frac{\sin x}{1 + \dots}}}; \quad y = \frac{\sin x}{1 + \frac{\cos x}{1+y}} = \frac{\sin x(1+y)}{1+y+\cos x}; \quad y(0) = 0$$

$$y(1+y+\cos x) = \sin x(1+y)$$

$$y' + 2yy' + \cos x y' - y \sin x = \cos x(1+y) + \sin xy'$$

$$y'(0) [1 + 2y + \cos x] - 0 = 1 + 0$$

$$2y'(0) = 1 \quad \Rightarrow \quad y'(0) = \frac{1}{2}$$

$$48. \quad f'(0) = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h} = \lim_{h \rightarrow 0} \frac{1/h}{e^{1/h^2}} = \frac{-1/h^2}{-e^{1/h^2} \cdot \frac{2}{h^3}} = \lim_{h \rightarrow 0} \frac{2h^3}{h^2 e^{1/h^2}} = 0$$

Hence f is differentiable at $x=0$. Also $\lim_{x \rightarrow \pm\infty} e^{-\frac{1}{x^2}} \rightarrow 1 \Rightarrow C$

Alternatively:

check concavity by finding $\frac{d^2y}{dx^2}$ and eliminate D .

49. Given $\frac{d^2y}{dx^2} + y \frac{dy}{dx} = 0$

now $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{1}{\frac{dx}{dy}} \right) = \frac{d}{dy} \left(\frac{1}{\frac{dx}{dy}} \right) \cdot \frac{dy}{dx} = - \frac{1}{\left(\frac{dx}{dy} \right)^2} \cdot \frac{d^2x}{dy^2} \cdot \frac{1}{\frac{dx}{dy}}$

$$\frac{d^2y}{dx^2} = - \frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^3} \quad (\text{putting in (1)})$$

$$- \frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^3} + y \frac{dy}{dx} = 0 \Rightarrow y \left(\frac{dy}{dx} \right)^2 - \frac{d^2x}{dy^2} = 0 \Rightarrow C$$

50. Let $f(x) = y \Rightarrow x = f^{-1}(y) = g(y) \Rightarrow x = e^{e^y}$

$$\Rightarrow \frac{dx}{dy} = e^{e^y} \cdot e^y = e^{e^y+y} = g'(y)$$

hence $g'(x) = e^{e^x+x}$

CONTINUITY & DIFFERENTIABILITY EXERCISE 2(A)

More than one options may be correct

Q.1 If $f(x) = \begin{cases} \frac{x \cdot \ln(\cos x)}{\ln(1+x^2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$ then :

- (A) f is continuous at $x = 0$ (B) f is continuous at $x = 0$ but not differentiable at $x = 0$
 (C) f is differentiable at $x = 0$ (D) f is not continuous at $x = 0$

Sol. [A, C]

$$\Rightarrow f'(0^+) = \lim_{h \rightarrow 0} \frac{h \ln(\cos h)}{h \ln(1+h^2)} = \lim_{h \rightarrow 0} \frac{\ln(\cos h) \frac{1}{h^2}}{\frac{\ln(1+h^2)}{h^2}}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{h^2} (\cos h - 1) = -\frac{1}{2}$$

$$\Rightarrow \text{Paralally } f'(0^-) = -\frac{1}{2}$$

Hence f is continuous and derivable at $x = 0$

Q.2 Given that the derivative $f'(a)$ exists. Indicate which of the following statement(s) is/are always true.

- (A) $f'(x) = \lim_{h \rightarrow a} \frac{f(h) - f(a)}{h - a}$ (B) $f'(a) = \lim_{h \rightarrow 0} \frac{f(a) - f(a-h)}{h}$
 (C) $f'(a) = \lim_{t \rightarrow 0} \frac{f(a+2t) - f(a)}{t}$ (D) $f'(a) = \lim_{t \rightarrow 0} \frac{f(a+2t) - f(a+t)}{2t}$

Sol. [A, B]

\Rightarrow (C) is false and is True only if $f'(a) = 0$ limit is $2f'(a)$. In (D) same logic limit is $\frac{1}{2}f'(a)$

Q.3 Let $[x]$ denote the greatest integer less than or equal to x . If $f(x) = [x \sin \pi x]$, then $f(x)$ is:

- (A) continuous at $x = 0$ (B) continuous in $(-1, 0)$
 (C) differentiable at $x = 1$ (D) differentiable in $(-1, 1)$

Sol. [A, B, D]

$$\Rightarrow f(x) = \begin{cases} 0 & 0 < x < 1 \\ 0 & x = 0 \text{ or } 1 \text{ or } -1 \\ 0 & -1 < x < 0 \end{cases}$$

$\Rightarrow f(x) = 0$ for all in $[-1, 1]$

Q.4 The function, $f(x) = [x] - [x]$ where $[x]$ denotes greatest integer function

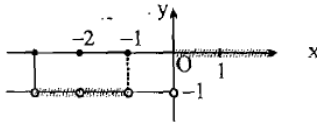
- (A) is continuous for all positive integers
- (B) is continuous for all non positive integers
- (C) has finite number of elements in its range
- (D) is such that its graph does not lie above the $x - axis$.

Sol. [A, B, C, D]

$$\Rightarrow \lceil |x| \rceil - \lfloor x \rfloor = \begin{cases} 0 & x = -1 \\ -1 & -1 < x < 0 \\ 0 & 0 \leq x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}$$

\Rightarrow range is $\{0, -1\}$

The graph is



Q.5 Let $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then:

- (A) $f(x)$ must be continuous $\forall x \in \mathbb{R}$
- (B) $f(x)$ may be continuous $\forall x \in \mathbb{R}$
- (C) $f(x)$ must be discontinuous $\forall x \in \mathbb{R}$
- (D) $f(x)$ may be discontinuous $\forall x \in \mathbb{R}$

Sol. [B, D]

$$\Rightarrow \lim_{h \rightarrow 0} f(x+h) = \lim_{h \rightarrow 0} f(x) + f(h)$$

$$\Rightarrow f(x) + \lim_{h \rightarrow 0} f(h)$$

Hence if $h \rightarrow 0$

$$\Rightarrow f(h) = 0$$

\Rightarrow 'f' is continuous otherwise discontinuous

Q.6 The function $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$

- (A) has its domain $-1 \leq x \leq 1$.
- (B) has finite one sided derivatives at the point $x = 0$.
- (C) is continuous and differentiable at $x = 0$.
- (D) is continuous but not differentiable at $x = 0$.

Sol. [A, B, D]

$$\Rightarrow f'(0^+) = \frac{1}{\sqrt{2}}; f'(0^-) = -\frac{1}{\sqrt{2}}; f(x) = \frac{\sqrt{x^2}}{\sqrt{1 + \sqrt{1 - x^2}}} = \frac{|x|}{\sqrt{1 + \sqrt{1 - x^2}}}$$

Q.7 Consider the function $f(x) = |x^3 + 1|$ then

- (A) Domain of f is \mathbb{R}
- (B) Range of f is \mathbb{R}^+
- (C) f has no inverse.
- (D) f is continuous and differentiable for every $x \in \mathbb{R}$.

Sol. [A, C]

Range is $\mathbb{R}^+ \cup \{0\} \Rightarrow B$ is not correct

f is not differentiable at $x = -1$

$$\Rightarrow \text{as } f(x) = \begin{cases} x^3 + 1 & \text{if } x \geq -1 \\ -(x^3 + 1) & \text{if } x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 3x^2 & \text{if } x > -1 \\ -3x^2 & \text{if } x < -1 \end{cases}$$

$$\Rightarrow f'(-1^+) = 3;$$

$$\Rightarrow f'(-1^-) = -3$$

f is not differentiable at $x = -1$

also since f is not bijective hence it has no inverse

Q.8 Let $f(x) = \frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1} \cdot x$ then:

- (A) $f'(10) = 1$ (B) $f'\left(\frac{3}{2}\right) = -1$
(C) domain of $f(x)$ is $x \geq 1$ (D) none

Sol. [A, B]

$$\Rightarrow f(x) = \frac{\sqrt{(\sqrt{x-1})^2 + 1 - 2\sqrt{x-1}}}{\sqrt{x-1}-1} \cdot x = \frac{|\sqrt{x-1}-1|}{\sqrt{x-1}-1} \cdot x = \begin{cases} -x & \text{if } x \in [1, 2) \\ x & \text{if } x \in (2, \infty) \end{cases}$$

Q.9 f is a continuous function in $[a, b]$; g is a continuous function in $[b, c]$
A function $h(x)$ is defined as

$$\begin{aligned} h(x) &= f(x) && \text{for } x \in [a, b) \\ &= g(x) && \text{for } x \in (b, c] \end{aligned}$$

If $f(b) = g(b)$, then

- (A) $h(x)$ has a removable discontinuity at $x=b$.
(B) $h(x)$ may or may not be continuous in $[a, c]$
(C) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$
(D) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

Sol. [A, C]

Given f is continuous in $[b, c]$ (1)

g is continuous in $[b, c]$ (2)

$f(b) = g(b)$ (3)

$$\Rightarrow \left. \begin{aligned} h(x) &= f(x) && \text{for } x \in [a, b) \\ &= f(b) = g(b) && \text{for } x = b \\ &= g(x) && \text{for } x \in (b, c] \end{aligned} \right\} \text{(4)}$$

$\Rightarrow h(x)$ is continuous in $[a, b) \cup (b, c]$ [using (1), (2)]

also $f(b^-) = f(b)$; $g(b^+) = g(b)$ (5) [using (1), (2)]

$$\Rightarrow \therefore h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$$

\Rightarrow now, verify each alternative. Of course! $g(b^-)$ and $f(b^+)$ are undefined.

$$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$$

$$\Rightarrow \text{and } h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$$

$$\Rightarrow \text{hence } h(b^-) = h(b^+) = f(b) = g(b)$$

\Rightarrow and $h(b)$ is not defined

Q.10 The function $f(x) = \begin{cases} |x-3| & , x \geq 1 \\ \left(\frac{x^2}{4}\right) - \left(\frac{3x}{2}\right) + \left(\frac{13}{4}\right) & , x < 1 \end{cases}$ is:

(A) continuous at $x = 1$

(B) differentiable at $x = 1$

(C) continuous at $x = 3$

(D) differentiable at $x = 3$

Sol. [A, B, C]

$$\Rightarrow f(x) = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } 1 \leq x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & \text{if } x < 1 \end{cases}$$

$$\Rightarrow f'(1^+) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{3 - (1+h) - 2}{h} = -1$$

$$\Rightarrow f'(1^-) = \lim_{h \rightarrow 0} \frac{\frac{(1-h)^2}{4} - \frac{3}{2}(1-h) + \frac{13}{4} - 2}{-h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{(1-h)^2 - 6(1-h) + 5}{-4h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{h^2 - 2h + 6h}{-4h} = -1$$

$\Rightarrow f'$ is continuous at $x = 1$

Q.11 Which of the following statements are true?

(A) If $xe^{xy} = y + \sin-x$, then at y $I(0) = 1$.

(B) If $f(x) = a_0 x^{2m+1} + a_1 x^{2m} + a_2 x^{2m-1} + \dots + a_{2m+1} = 0$ ($a_0 \neq 0$) is a polynomial equation with rational co-efficients then the equation $f'(x) = 0$ must have a real root. ($m \in \mathbb{N}$).

(C) If $(x-r)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0$ repeated m times where $1 \leq m \leq n$ then r is a root of the equation $f'(x) = 0$ repeated $(m-1)$ times.

(D) If $y = \sin^{-1}(\cos \sin^{-1} x) + \cos^{-1}(\sin \cos^{-1} x)$ then $\frac{dy}{dx}$ is independent on x .

Sol. [A, C, D]

[D] Let $\sin^{-1} x = t$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - t$$

$$\Rightarrow y = \sin^{-1}(\cos t) + \cos^{-1}\left(\sin\left(\frac{\pi}{2} - t\right)\right) = \sin^{-1}(\cos t) + \cos^{-1}(\cos t)$$

$$\Rightarrow y = \frac{\pi}{2}$$

$$\Rightarrow \frac{dy}{dx} = 0$$

Q.12 Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$ then $\frac{dy}{dx} =$

(A) $\frac{1}{2y-1}$

(B) $\frac{x}{x+2y}$

(C) $\frac{1}{\sqrt{1+4x}}$

(D) $\frac{y}{2x+y}$

Sol. [A, C, D]

$$\Rightarrow y^2 = x + y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

also $y = \frac{x}{y} + 1$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x+y}$$

make a quadratic in y to get explicit function

Q.13 If $\sqrt{y+x} + \sqrt{y-x} = c$ (where $c \neq 0$), then $\frac{dy}{dx}$ has the value equal to

(A) $\frac{2x}{c^2}$

(B) $\frac{x}{y + \sqrt{y^2 - x^2}}$

(C) $\frac{y\sqrt{y^2 - x^2}}{x}$

(D) $\frac{c^2}{2y}$

Sol. [A, B, C]

\Rightarrow Square both sides, differentiate and rationalize

Q.14 If $f(x) = \cos\left[\frac{\pi}{x}\right] \cos\left(\frac{\pi}{2}(x-1)\right)$; where $[x]$ is the greatest integer function of x , then $f(x)$ is

continuous at

(A) $x = 0$

(B) $x = 1$

(C) $x = 2$

(D) none of these

Sol. [B, C]

\Rightarrow (A) = Not defined at $x = 0$;

\Rightarrow (B) = $f(1) = \cos 3$; $f(2) = 0$ and both the limits exist

Q.15 Select the correct statements.

- (A) The function f defined by $f(x) = \begin{cases} 2x^2 + 3 & \text{for } x \leq 1 \\ 3x + 2 & \text{for } x > 1 \end{cases}$ is neither differentiable nor continuous at $x = 1$
- (B) The function $f(x) = x^2|x|$ is twice differentiable at $x = 0$.
- (C) If f is continuous at $x = 5$ and $f(5) = 2$ then $\lim_{x \rightarrow 2} f(4x^2 - 11)$ exists.
- (D) If $\lim_{x \rightarrow a} (f(x) + g(x)) = 2$ and $\lim_{x \rightarrow a} (f(x) - g(x)) = 1$ then $\lim_{x \rightarrow a} f(x) \cdot g(x)$ need not exist.

Sol. [B, C]

Q.16 Which of the following functions has/have removable discontinuity at $x = 1$.

- (A) $f(x) = \frac{1}{\ln|x|}$
- (B) $f(x) = \frac{x^2 - 1}{x^3 - 1}$
- (C) $f(x) = 2^{-2\left(\frac{1}{1-x}\right)}$
- (D) $f(x) = \frac{\sqrt{x+1} - \sqrt{2x}}{x^2 - x}$

Sol. [B, D]

- (A) $\lim_{x \rightarrow 1} f(x)$ does not exist
- (B) $\lim_{x \rightarrow 1} f(x) = \frac{2}{3} \therefore f(x)$ has removable discontinuity at $x = 1$
- (C) $\lim_{x \rightarrow 1} f(x)$ does not exist
- (D) $\lim_{x \rightarrow 1} f(x) = \frac{-1}{2\sqrt{2}} \therefore f(x)$ has removable discontinuity at $x = 1$

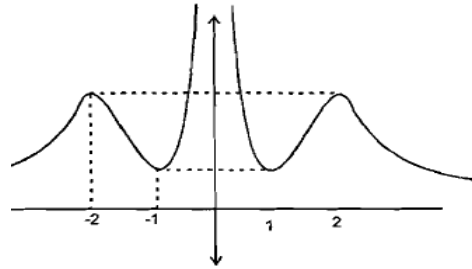
Q.17 $f(x)$ is an even function, $x = 1$ is a point of minima and $x = 2$ is a point of maxima for $y = f(x)$. Further $\lim_{x \rightarrow \infty} f(x) = 0$, and $\lim_{x \rightarrow \infty} f(x) = \infty$. $f(x)$ is increasing in $(1, -2)$ & decreasing everywhere in

$(0, 1) \cup (2, \infty)$. Also $f(1) = 3$ & $f(2) = 5$ Then

- (A) $f(x) = 0$ has no real roots
- (B) $y = f(x)$ and $y = |f(x)|$ are identical functions
- (C) $f'(x) = 0$ has exactly four real roots whose sum is zero
- (D) $f'(x) = 0$ has exactly four real roots whose sum is 6

Sol. [A, B, C]

$$\lim_{x \rightarrow 0} f(x) = \infty, \quad \lim_{x \rightarrow \infty} f(x) = 0$$



$\Rightarrow f(x)$ is increasing in $(1, 2)$ and decreasing in $(0, 1) \cup (2, \infty)$ from the graph

Q.18

Q.19

Q.20

PASSAGE 1

A curve is represented parametrically by the equations $x = f(t) = a^{\ln(b^t)}$ and $y = g(t) = b^{-\ln(a^t)}$, $a, b > 0$ and $a \neq 1, b \neq 1$ where $t \in \mathbb{R}$.

Q.21 Which of the following is not a correct expression for $\frac{dy}{dx}$?

- (A) $\frac{-1}{f(t)^2}$ (B) $-(g(t))^2$ (C) $\frac{-g(t)}{f(t)}$ (D) $\frac{-f(t)}{g(t)}$

Sol. [D]

Q.22 The value of $\frac{d^2y}{dx^2}$ at the point where $f(t) = g(t)$ is

- (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

Sol. [D]

Q.23 The value of $\frac{f(t)}{f'(t)} \cdot \frac{f'(-t)}{f'(-t)} + \frac{f(-t)}{f'(-t)} \cdot \frac{f''(t)}{f'(t)}$ $\forall t \in \mathbb{R}$, is equal to

- (A) -2 (B) 2 (C) -4 (D) 4

Sol. [B]

$$\Rightarrow x = f(t) = a^{\ln(b^t)} = a^{t \ln b} \dots\dots(1)$$

$$\Rightarrow y = g(t) = b^{-\ln(a^t)} = (b^{\ln a})^{-t} = (a^{\ln b})^{-t} = a^{-t \ln b}$$

$$\Rightarrow \therefore y = g(t) = a^{\ln(b^{-1})} = f(-t) \dots\dots(2)$$

From equation (1) and (2)

$$\Rightarrow xy = 1$$

(i) $\therefore y = \frac{1}{x}$

$$\Rightarrow \therefore \frac{dy}{dx} = -\frac{1}{x^2} = -\frac{1}{f^2(t)} \quad \text{(A) is correct}$$

$$\Rightarrow \text{Also } xy = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} = -\frac{y^2}{1} = -g^2(t) \quad \text{(B) is correct}$$

$$\Rightarrow \text{Again } xy = 1 \quad \frac{dy}{dx} = -\frac{y}{x} = -\frac{g(t)}{f(t)} \quad \text{(C) is correct}$$

(D) is incorrect

$$\text{(ii) } f(t) = g(t) \Rightarrow f(t) = f(-t) \Rightarrow t = 0$$

{ \because $f(t)$ is one-one function}

$$\text{At } t = 0, x = y = 1$$

$$\Rightarrow \therefore \frac{dy}{dx} = \frac{-1}{x^2} \text{ and } \frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$$\Rightarrow \text{At } x = 1, \frac{d^2y}{dx^2} = 2$$

$$\text{(iii) } \therefore xy = 1 \quad \therefore fg = 1 \quad \therefore fg' + g'f' = 0$$

$$\Rightarrow \therefore fg'' + g'f'' + g'f'' + g'f'' = 0$$

$$\Rightarrow fg'' + g'f'' + 2g'f'' = 0$$

$$\Rightarrow \frac{fg''}{f'g'} + \frac{gf''}{g'f'} = -2 \quad \dots\dots(3)$$

from equation (2)

$$\Rightarrow g(t) = f(-t)$$

$$\Rightarrow \therefore g'(t) = -f'(-t)$$

$$\text{and } g''(t) = f''(-t)$$

substituting in equation (3)

$$\Rightarrow \frac{f(t)}{f'(t)} \cdot \frac{f''(-t)}{-f'(-t)} + \frac{f(-t)}{-f'(-t)} \cdot \frac{f''(t)}{f'(t)} = -2$$

$$\Rightarrow \frac{f(t)}{f'(t)} \cdot \frac{f''(-t)}{f'(-t)} + \frac{f(-t)}{f'(-t)} \cdot \frac{f''(t)}{f'(t)} = 2$$

$$\Rightarrow 1$$

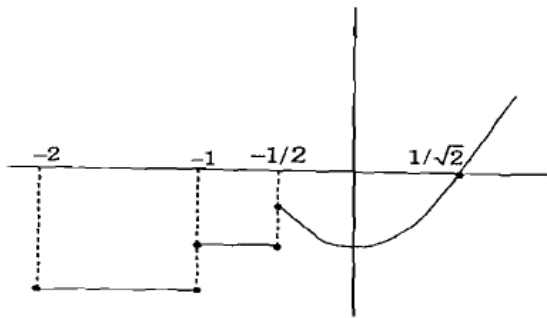
PASSAGE 2

Let a function be defined as $f(x) = \begin{cases} [x], & -2 \leq x \leq -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \leq 2 \end{cases}$, where $[.]$ denotes greatest integer function.

Answer the following question by using the above information.

Q.24 The number of points of discontinuity of $f(x)$ is
 (A) 1 (B) 2 (C) 3 (D) 0

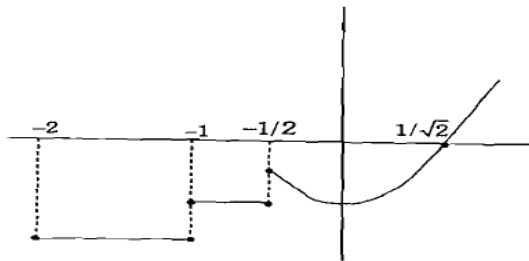
Sol. [B]



\Rightarrow Two points of discontinuity $-1, -\frac{1}{2}$

Q.25 The function $f(x-1)$ is discontinuous at the points
 (A) $-1, -\frac{1}{2}$ (B) $-\frac{1}{2}, 1$ (C) $0, \frac{1}{2}$ (D) $0, 1$

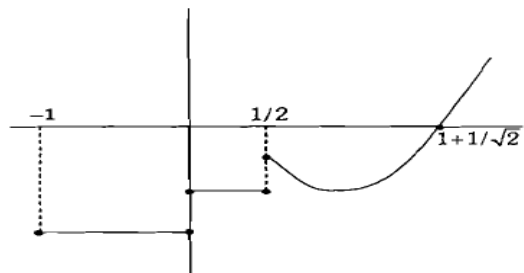
Sol. [C]



\Rightarrow Discontinuous at $1, \frac{1}{2}$

Q.26 Number of points where $|f(x)|$ is not differentiable is
 (A) 1 (B) 2 (C) 3 (D) 4

Sol. [C]



\Rightarrow at $-1, -\frac{1}{2}, \frac{1}{\sqrt{2}}$ the function is not differentiable.

PASSAGE 3

Two students, A & B are asked to solve two different problem. A is asked to evaluate

$$\lim_{x \rightarrow 0} \frac{1 - \cos(\ln(1+x))}{x^2} \text{ \& B is asked to evaluate } \lim_{x \rightarrow \infty} \left(\frac{\sqrt{n}}{\sqrt{n^3+1}} + \frac{\sqrt{n}}{\sqrt{n^3+1}} + \dots + \frac{\sqrt{n}}{\sqrt{n^3+2n}} \right), n \in \mathbb{N}. \text{ A}$$

provides the following solution

$$\text{Let } h = \lim_{x \rightarrow 0} \frac{1 - \cos\left(\frac{\ln(1+x)}{x} \cdot x\right)}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \left(\text{As } \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \right)$$

$$l_1 = \frac{1}{2}$$

B provides the following solution

$$\text{Let } l_2 = \lim_{n \rightarrow \infty} \left\{ \sum_{r=1}^{2n} \frac{\sqrt{n}}{\sqrt{n^3+r}} \right\} = \lim_{n \rightarrow \infty} \left\{ \sum_{r=1}^{2n} \frac{1}{n} \frac{\sqrt{n}}{\sqrt{n + \frac{r}{n^2}}} \right\}$$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \left\{ \sqrt{\frac{n}{n + \frac{1}{n^2}}} + \sqrt{\frac{n}{n + \frac{2}{n^2}}} + \dots + \sqrt{\frac{n}{n + \frac{2n}{n^2}}} \right\} \right]$$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n} \left(\underbrace{1+1+\dots+1}_{2n \text{ times}} \right) \right] = \lim_{n \rightarrow \infty} \frac{2n}{n} = 2$$

Q.27 Identify the correct statement

- (A) both of them get the correct answer
- (B) both of them get the incorrect answer
- (C) A gets the correct answer while B gets the incorrect answer.
- (D) B gets the correct answer while A gets the incorrect answer.

Sol. [A]

Q.28 Who has solved the problem correctly.

- (A) A
- (B) B
- (C) both of them
- (D) no one

Sol. [D]

Q.29 $f(x) = \begin{cases} 4l_1 \left(\frac{\tan x - \sin x}{x^3} \right) & x < 0 \\ k & x = 0 \text{ where } l_1 \text{ and } l_2 \text{ are correct values of the corresponding limits, if is} \\ l_2 \left(\frac{e^x - x - 1}{1 - \cos x} \right) & x > 0 \end{cases}$

continuous at $x = 0$ the K is equal to:

- (A) 1 (B) 2 (C) 3 (D) no value of K

Sol. [D]

$$\Rightarrow l_1 = \lim_{x \rightarrow 0} \frac{1 - \cos(\ln(1+x))}{\ln^2(1+x)} \cdot \left(\frac{\ln(1+x)}{x} \right)^2 = \frac{1}{2}$$

A & B have made the same mistake, they used the notion of limit partly in the problem, where as once the limiting notion has been used the resulting expression must be free from the variable on which the limit has been imposed

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{2n\sqrt{n}}{\sqrt{n^3+1}} < l_2 < \lim_{n \rightarrow \infty} \frac{2n\sqrt{2}}{\sqrt{n^3+1}}$$

Hence $l_2 = 2$ (sandwich theorem)

\Rightarrow **Sol.1** Hence (A)

\Rightarrow **Sol.2** Hence (D)

$$\Rightarrow \text{Sol.3} \quad \lim_{x \rightarrow 0} 4 \cdot \frac{1}{2} \left(\frac{\tan x - \sin x}{x^3} \right) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} l_2 \left(\frac{e^x - x - 1}{x^2} \cdot \frac{x^2}{1 - \cos x} \right) = 2(2 \cdot 2) = 8$$

\Rightarrow for no value if K

Hence (D)

PASSAGE 4

Q.30

Q.31

Q.32

Matrix match type

Q.33

Q.34

Column – I

Column – II

(A) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}$, at $x=0$ is

(P) continuous

(B) For every $x \in \mathbb{R}$ the function

(Q) differentiability

$$g(x) = \frac{\sin(\pi[x - \pi])}{1 + [x]^2} \quad (\text{R}) \text{ discontinuous}$$

where $[x]$ denotes the greatest integer function is (S) non derivable

(C) $h(x) = \sqrt{\{x\}^2}$ where $\{x\}$ denotes fractional part function for all $x \in I$, is

(D) $k(x) = \begin{cases} x^{\frac{1}{\ln x}} & \text{if } x \neq 1 \\ e & \text{if } x = 1 \end{cases}$ at $x = 1$ is

Sol. (A) \Rightarrow P, S; (B) \Rightarrow P, Q; (C) \Rightarrow R, S; (D) \Rightarrow P, Q

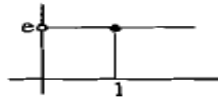
(A) $f'(0) = \lim_{h \rightarrow 0} \frac{\cos h - 0}{h}$ does not exist. Obviously $f(0) = f(0^+) = 1$

Hence continuous and not derivable

(B) $g(x) = 0$ for all x , hence continuous and derivable

(C) as $0 \leq \{f(x)\} < 1$, hence $h(x) = \sqrt{\{x\}^2} = \{x\}$ which is discontinuous hence non derivable all $x \in I$

(D) $\lim_{x \rightarrow 1} x^{\frac{1}{\ln x}} = \lim_{x \rightarrow 1} x^{\log_x e} = e = f(1)$



\Rightarrow Hence $k(x)$ is constant for all $x > 0$ hence continuous and differentiable at $x = 1$.

Q.35

Column – I

Column – II

- | | |
|--|-------|
| (A) Number of points of discontinuity of $f(x) = \tan^2 x - \sec^2 x$ in $(0, 2\pi)$ is | (p) 1 |
| (B) Number of points at which $f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x$ is non-differentiable in $(-1, 1)$ is | (q) 2 |
| (C) Number of points of discontinuity of $y = [\sin x]$, $x \in [0, 2\pi)$ where $[\cdot]$ represents greatest integer function | (r) 0 |
| (D) Number of points where $y = (x-1)^3 + (x-2)^5 + x-3 $ is non-differentiable | (s) 3 |

Sol. (A) \Rightarrow q; (B) \Rightarrow r; (C) \Rightarrow q; (D) \Rightarrow p

(A) $\tan^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \sec^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

\Rightarrow Number of discontinuities = 2

(B) Since $f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x = \sin^{-1} x + \frac{\pi}{2}$

$\Rightarrow \therefore f(x)$ is differentiable in $(-1, 1)$

\Rightarrow number of points of non-differentiable = 0

$$(C) \quad y = [\sin x] = \begin{cases} 0 & , 0 \leq x < \frac{\pi}{2} \\ 1 & , x = \frac{\pi}{2} \\ 0 & , \frac{\pi}{2} < x \leq \pi \\ -1 & , \pi < x < 2\pi \\ 0 & , x = 2\pi \end{cases} \quad 7t$$

$\Rightarrow \therefore$ Points of discontinuity are $\frac{\pi}{2}, \pi$

(D) $y = |(x-1)^3| + |(x-2)^5| + |x-3|$ is non differentiable at $x = 3$ only.

CONTINUITY & DIFFERENTIABILITY EXERCISE 2(B)

Q.1 If the function $f(x) = \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$ is continuous at $x = -2$. Find the value of $a + f(-2)$.

Sol. [14]

Since the function is conti then

$$\Rightarrow V.F. \Big|_{x=-2} = RHL \Big|_{x=-2} = LHL \Big|_{x=-2}$$

$$\Rightarrow f(-2) = \lim_{x \rightarrow -2} f(x)$$

$$\Rightarrow \lim_{x \rightarrow -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}; \left(\frac{15 - a}{0} \right) \text{ form}$$

this limit will exist when

$$\Rightarrow 15 - a = 0$$

$$\Rightarrow a = 15 \quad \dots\dots(1)$$

$$\Rightarrow \lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2}$$

$$\Rightarrow \lim_{x \rightarrow -2} \frac{3(x+2)(x+3)}{x^2 + x - 2}$$

$$\Rightarrow \lim_{x \rightarrow -2} \frac{3(x+3)}{(x-1)}$$

$$\Rightarrow \frac{3}{-3} = -1$$

$$\Rightarrow \text{hence } f(-2) = -1$$

Q.2 Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \frac{f(x)}{x-3}, & x \neq 3 \\ K, & x = 3 \end{cases}$ then

find the value of K that makes h continuous at $x = 3$

Sol. [5]

$$\Rightarrow \lim_{x \rightarrow 3^+} \frac{x^3 - 3x^2 - 4x + 12}{x - 3} = \lim_{x \rightarrow 3^+} \frac{(x-3)(x^2 - 4)}{(x-3)} = \lim_{x \rightarrow 3^+} (x^2 - 4) = 5$$

$$\Rightarrow \text{hence } k = 5$$

Q.3 Determine the value of product of values of a & b so that f is continuous at $x = \frac{\pi}{2}$. $f(x)$

$$= \begin{cases} \frac{1 - \sin^2 x}{3 \cos^2 x} & \text{if } x < \frac{\pi}{2} \\ a & \text{if } x = \frac{\pi}{2} \\ \frac{b(1 - \sin x)}{(x - 2x)^2} & \text{if } x > \frac{\pi}{2} \end{cases}$$

Sol. [2]

$$\Rightarrow \text{V.F.} \Big|_{x=\frac{\pi}{2}} = a$$

$$\Rightarrow \text{LHL} \Big|_{x=\frac{\pi}{2}} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} f(x)$$

$$\Rightarrow \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(\frac{1 - \sin^3 x}{3 \cos^2 x} \right); \frac{0}{0} \text{ form}$$

$$\Rightarrow \text{put } x = \frac{\pi}{2} - h$$

$$\Rightarrow \lim_{h \rightarrow 0} \left(\frac{1 - \sin^3 \left(\frac{\pi}{2} - h \right)}{3 \cos^2 \left(\frac{\pi}{2} - h \right)} \right) = \left(\lim_{h \rightarrow 0} \frac{1 - \cos^3 h}{3 \sin^2 h} \right) = \lim_{h \rightarrow 0} \frac{(1 - \cos h)(1 + \cos^2 h + \cos h)}{3(1 - \cos h)(1 + \cos h)} = \frac{1}{2}$$

$$\Rightarrow \text{RHL} \Big|_{x=\frac{\pi}{2}} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{b(1 - \sin x)}{(\pi - 2x)^2}$$

$$\Rightarrow \text{put } x = \frac{\pi}{2} + h$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{b(1 - \cosh)}{4h^2} = \frac{b}{8}$$

since function is conti

$$\Rightarrow \text{V.F.} \Big|_{x=\frac{\pi}{2}} = \text{LHL} \Big|_{x=\frac{\pi}{2}} = \text{RHL} \Big|_{x=\frac{\pi}{2}}$$

$$\Rightarrow a = \frac{1}{2} = \frac{b}{8}$$

$$\Rightarrow a = \frac{1}{2}, b = 4$$

Q.4 If $f(x) = \frac{\sin 3x + A \sin 2x + B \sin x}{x^5}$ ($x \neq 0$) is cont. at $x = 0$. Find the value of $A + B + f(0)$

Sol. [2]

Since the function is conti

$$\Rightarrow VF|_{x=0} = RHL|_{x=0} = LHL|_{x=0}$$

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{\sin 3x + A \sin 2x + B \sin x}{x^5} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{3 \cos 3x + 2A \cos 2x + B \cos x}{5x^4} \right); \left(\frac{3 + 2A + B}{0} \right) \text{ form}$$

$$\Rightarrow 3 + 2A + B = 0 \quad \dots(1)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{-9 \sin 3x + 4A \sin 2x - B \sin x}{20x^3} \right); \left(\frac{0}{0} \text{ form} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{-27 \cos 3x - 8A \cos 2x - B \cos x}{60x^2} \right); \left(\frac{-27 - 8A - B}{0} \right) \text{ form}$$

$$\Rightarrow 27 + 8A + B = 0 \quad \dots(2)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{81 \sin 3x + 16A \sin 2x + B \sin x}{120x} \right); \left(\frac{0}{0} \right) \text{ form}$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{243 \cos 3x + 32A \cos 2x + B \cos x}{120} \right)$$

$$\Rightarrow f(0) = \frac{243 + 32A + B}{120} \quad \dots(3)$$

using (1) & (2)

$$\Rightarrow A = -4, B = 5$$

$$\Rightarrow \text{then } f(0) = 1$$

Q.5 Let $g(x) = \lim_{n \rightarrow \infty} \frac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$, $x \neq 1$ and $g(1) = \lim_{x \rightarrow 1} \frac{\sin^2(\pi \cdot 2^x)}{\ln(\sec(\pi \cdot 2^x))}$ be a continuous function at

$x = 1$, find the value of $4g(1) + 2f(1) - h(1)$. Assume that $f(x)$ and $h(x)$ are continuous at $x = 1$.

Sol. [5]

$$g(x) = \lim_{x \rightarrow \infty} \frac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3} \quad \text{since } g \text{ is conti}$$

$$\& \quad g(1) = 2$$

$$\Rightarrow g(1) = \lim_{x \rightarrow 1} \frac{\sin^2(\pi \cdot 2^x)}{\ln(\sec(\pi \cdot 2^x))}$$

$$\Rightarrow \arg = 1 \text{ so } \arg - 1$$

$$\Rightarrow g(1) = \lim_{x \rightarrow 1} \frac{\sin^2(\pi \cdot 2^x)}{(\sec(\pi \cdot 2^x) - 1)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\sin^2(\pi \cdot 2^x) \cdot \cos(\pi \cdot 2^x)}{1 - \cos(\pi \cdot 2^x)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{(1 - \cos \pi 2^x)(1 + \cos \pi 2^x)}{1 - \cos(\pi 2^x)}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{(1 - \cos \pi 2^x)(1 - \cos \pi 2^x) \cos(\pi 2^x)}{1 - \cos(\pi 2^x)}$$

$$\Rightarrow g(1) = 2 \quad \dots\dots\dots(1)$$

$$\Rightarrow g(x) = \lim_{x \rightarrow \infty} \frac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}; x \neq 1$$

$$\Rightarrow g(x) = \begin{cases} \frac{h(x)+1}{3(x+1)}, & 0 < x < 1 \\ 2, & x = 1 \\ \frac{f(x)}{2}, & x > 1 \end{cases} \quad \text{using (1)}$$

$$\text{Hint } \lim_{x \rightarrow \infty} x^n = \begin{cases} 0, & 0 < x < 1 \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$$

when $x < 1$ then $x^n = 0$

$$\Rightarrow g(x) = \frac{h(x)+1}{3(x+1)}$$

$$\Rightarrow g(x) = \lim_{x \rightarrow \infty} \left(\frac{f(x) + \left(\frac{h(x)+1}{x^n} \right)}{2 + \left(\frac{3x+3}{x^n} \right)} \right)$$

$$\Rightarrow g(x) = \frac{f(x)}{2}$$

$$\Rightarrow h(1) = 1$$

$$\Rightarrow f(1) = 4$$

therefore

$$\Rightarrow 4g(1) + 2f(1) - h(1)$$

$$\Rightarrow 4 \cdot 2 + 2 \cdot 4 - 11$$

$$\Rightarrow 8 + 8 - 11 = 5$$

Q.6 The function $f(x) = \left(\frac{2 + \cos x}{x^3 \sin x} - \frac{3}{x^4} \right)$ is not defined at $x = 0$.

Let L be the value of the function at $x = 0$ so that it is continuous at $x = 0$, then find the value of L^{-1}

Sol. [60]

Since the function is cont. At $x = 0$

then,

$$\Rightarrow \text{V.F.}|_{x=0} = \text{LHL}|_{x=0} = \text{RHL}|_{x=0}$$

$$\Rightarrow f(x) = \left(\frac{2 + \cos x}{x^3 \sin x} - \frac{3}{x^4} \right)$$

$$\Rightarrow L = f(0) = \lim_{x \rightarrow 0} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{2x + x \cos x - 3 \sin x}{x^4 \left(\frac{\sin x}{x} \right) \cdot x} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{2x + x \cos x - 3 \sin x}{x^5} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{1}{x^5} \left[2x + x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \right) - 3 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots \right) \right]$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^5}{x^5} \left[\left(\frac{1}{4!} - \frac{3}{5!} \right) + \dots \right]$$

$$\Rightarrow \frac{1}{4} - \frac{3}{120} = \frac{1}{60}$$

$$\Rightarrow \therefore L^{-1} = 60$$

Q.7 Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \frac{f(x)}{g(x)}$ where h is a rational function such that (a) it is

continuous everywhere except when $x = -1$, (b) $\lim_{x \rightarrow \infty} h(x) = \infty$ and (c) $\lim_{x \rightarrow -1} h(x) = \frac{1}{2}$. If

$\lim_{x \rightarrow 0} (3h(x) + f(x) - 2g(x)) = -\frac{p}{q}$ where p and q are coprime then find the value of $p + q$

Sol. [43]

$$\Rightarrow h(x) = \frac{f(x)}{g(x)}$$

$$\Rightarrow h(x) = \frac{x^3 - x^2 - 3x - 1}{(x+1)a}$$

$\Rightarrow \therefore x = -1$, function will be discontinuous; and for $x = -1$, $h(x) = \infty$

$$\text{Now } \lim_{x \rightarrow -1} h(x) = \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{x^3 - x^2 - 3x - 1}{(x+1)a} = \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{(x^2 - 2x - 1)(x + 1)}{(x + 1)a} = \frac{1}{2}$$

$$\Rightarrow \frac{2}{a} = \frac{1}{2}$$

$$\Rightarrow a = 4$$

put in (2)

$$\Rightarrow h(x) = \frac{x^3 - x^2 - 3x - 1}{(x + 1)4}$$

$$\Rightarrow g(x) = 4(x + 1)$$

using (1)

$$\Rightarrow \lim_{x \rightarrow 0} \left(3 \cdot \left(\frac{x^3 - x^2 - 3x - 1}{(x + 1)4} \right) + (x^3 - x^2 - 3x - 1) - 2(4(x + 1)) \right)$$

$$\Rightarrow \frac{3(-1)}{4} - 1 - 8$$

$$\Rightarrow -\frac{39}{4} = -\frac{p}{q}$$

Q.8 Suppose f and g are two functions such that $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \ln(1 + \sqrt{1 + x^2})$ and

$g(x) = \ln(x + \sqrt{1 + x^2})$ then find the value of $e^{g(x)} \left(f\left(\frac{1}{x}\right) \right)' + g'(x)$ at $x = 1$.

Sol. [0]

$$g(x) = \ln(x + \sqrt{1 + x^2})$$

$$\Rightarrow \therefore g'(x) = \frac{1}{\sqrt{1 + x^2}} \quad \dots\dots(1)$$

$$\text{also } \left(f\left(\frac{1}{x}\right) \right)' = f'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$\text{now } f'(x) = \frac{1}{1 + \sqrt{1 + x^2}} \cdot \frac{x}{\sqrt{1 + x^2}}$$

$$\Rightarrow f'\left(\frac{1}{x}\right) = \frac{x}{x + \sqrt{1 + x^2}} \cdot \frac{1}{x} \cdot \frac{x}{\sqrt{1 + x^2}}$$

$$\Rightarrow f'\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x\sqrt{1 + x^2}(x + \sqrt{1 + x^2})}$$

$$\text{also } e^{g(x)} = e^{\ln(x + \sqrt{1 + x^2})} = x + \sqrt{1 + x^2}$$

$$\Rightarrow x e^{g(x)} = x(x + \sqrt{1 + x^2})$$

$$\Rightarrow \text{Hence, } e^{g(x)} \cdot x \cdot \left(f\left(\frac{1}{x}\right) \right)' = -\frac{1}{\sqrt{1+x^2}} \quad \dots(2)$$

$$\Rightarrow \text{From (1) and (2), } g'(x) + x e^{g(x)} \left(f\left(\frac{1}{x}\right) \right)' = 0$$

Q.9 If $f(x)$ is derivable at $x = 3$ & $f'(3) = 2$, then find the value of $\lim_{x \rightarrow 0} \frac{f(3+h^2) - f(3-h^2)}{2h^2}$

Sol. [2]

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(3+h^2) - f(3-h^2)}{2h^2}$$

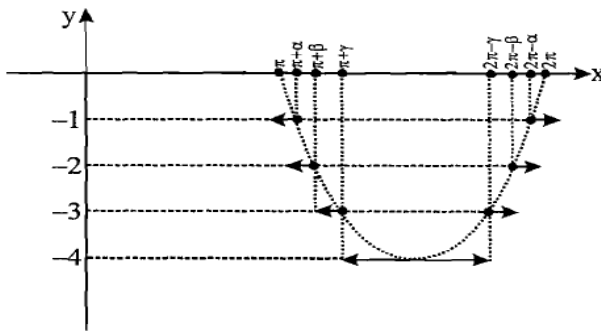
$$\Rightarrow \lim_{x \rightarrow 0} \frac{2hf(3+h^2) + 2hf'(3-h^2)}{4h}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f'(3+h^2) + f'(3-h^2)}{2} \text{ by L' hospital}$$

$$\Rightarrow \frac{f'(3) + f'(3)}{2} = \frac{4}{2} = 2$$

Q.10 Let $f(x) = [3 + 4 \sin x]$ (where $[]$ denotes the greatest integer function). If sum of all the values of 'x' in $[\pi, 2\pi]$ where $f(x)$ fails to be differentiable, is $\frac{k\pi}{2}$, then find the value of k.

Sol. [24]



$$\Rightarrow f(x) = [3 + 4 \sin x]$$

$$\Rightarrow + [4 \sin x]$$

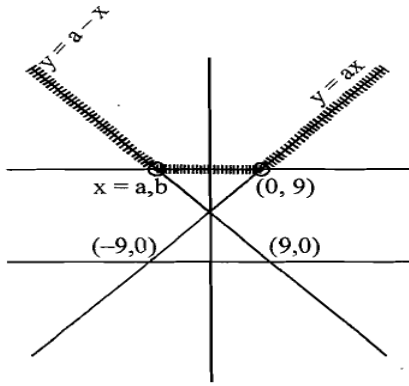
$$\text{Sum of all } x = \pi + (\pi - \alpha) + (\pi + \beta) + (\pi + \gamma) + (2\pi - \beta) + (2\pi - \alpha)$$

$$\Rightarrow 12\pi$$

$$\Rightarrow \therefore k = 24$$

Q.11 The number of points at which the function $f(x) = \max. \{a - x, a + x, b\}$, $-\infty < x < \infty, 0 < a < b$ cannot be differentiable is

Sol. [2]



Q.12 Let f , g and h are differentiable functions. If $f(0) = 1$; $g(0) = 2$; $h(0) = 3$ and the derivatives of their pair wise products at $x = 0$ are $(fg)'(0) = 6$; $(gh)'(0) = 4$ and $(hf)'(0) = 5$, then compute the value of $(fgh)'(0)$.

Sol. [16]

$$\begin{aligned} \Rightarrow (fgh)'(0) &= f g h' + g h f' + h f g' = \frac{(fg)'h + (gh)'f + (hf)'g}{2} \\ \Rightarrow \therefore (fgh)'(0) &= \frac{(fg)'(0) \cdot h(0) + (gh)'(0) \cdot f(0) + (hf)'(0) \cdot g(0)}{2} \\ \Rightarrow \frac{6 \cdot 3 + 4 \cdot 1 + 5 \cdot 2}{2} &= 16 \end{aligned}$$

Q.13 Let $f(x) = x + \frac{1}{2x} + \frac{1}{2x} + \frac{1}{2x} + \dots \infty$. Compute the value of $f(10) \cdot f'(10)$

Sol. [10]

$$\Rightarrow f(x) = x + \frac{1}{x + x + \frac{1}{2x + \frac{1}{2x + \dots \infty}}} = x + \frac{1}{x + f(x)}$$

$$\text{hence } f(x) - x = \frac{1}{x + f(x)}$$

$$\Rightarrow \therefore f^2(x) - x^2 = 1$$

Differentiating w.r.t. x

$$\Rightarrow 2f(x) \cdot f'(x) - 2x = 0 \quad \text{or} \quad f(x) \cdot f'(x) = x$$

$$\Rightarrow \text{Hence } f(10) \cdot f'(10) = 10$$

Q.14 If the value of the expression $y^3 \frac{d^2y}{dx^2}$ for the ellipse $3x^2 + 4y^2 = 12$, is $-\frac{p}{q}$ where p and q are co-prime then the value of $p + q$

Sol. [13]

Differentiating implicitly we have

$$\Rightarrow 6x + 8yy' = 0 \text{ and hence } y' = -\frac{3x}{4y}; 4[yy'' + (y')^2] = -3$$

Differentiating again and substitute for y' we have

$$\Rightarrow 3 + 4(y')^2 + 4yy'' = 0 \text{ and hence } 3 + \frac{9x^2}{4y^2} + 4yy'' = 0$$

multiplying by y^2 , $3y^2 + \frac{9x^2}{4} + 4y^3 \frac{d^2y}{dx^2} = 0$

$$\Rightarrow \frac{3y^2}{4} + \frac{9x^2}{16} + y^3 y''' = 0$$

$$\Rightarrow \frac{3}{16}(3x^2 + 4y^2) + y^3 y''' = 0$$

$$\Rightarrow \text{but } 3x^2 + 4y^2 = 12 \text{ and hence } y^3 y''' = -\frac{9}{4} = -\frac{p}{q} \text{ at every point on the ellipse}$$

Q.15 If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in \mathbb{R}$, then find the value of $f(2) - f(1) + f(0)$.

Sol. [0]

Given that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3) \dots(1)$

Putting $x = 0$ and $x = 1$ in (1), we get

$$f(0) = f'''(3) \text{ and } f(1) = 1 + f'(1) + f''(2) + f'''(3)$$

$$\therefore f(1) - f(0) = 1 + f'(1) + f''(2) \dots(2)$$

Differentiating both sides of (1) w.r.t. x , we get

$$\Rightarrow f'(x) = 3x^2 + 2x f'(1) + f''(2) \dots(3)$$

$$\Rightarrow \text{and } f''(x) = 6x + 2f'(1) \dots(4)$$

$$\Rightarrow \text{also } f'''(x) = 6 \dots(5)$$

Putting $x = 1, 2, 3$ in (3), (4), (5) respectively then

$$\Rightarrow f'(1) = 3 + 2f'(1) + f''(2) \dots(6)$$

$$\text{or } f'(1) = f''(2) = -3 \dots(7)$$

$$\Rightarrow f''(2) = 12 + 2f'(1) \text{ or } 2f'(1) - f''(2) = -12 \dots(8)$$

$$\Rightarrow \text{and } f'''(3) = 6 \dots(8)$$

Solving (6) and (7), we get $f'(1) = -5$ and $f''(2) = 2$

Hence $f(1) - f(0) = 1 - 5 + 2 = -2$

Also from (1), $f(2) = 8 + 4f'(1) + 2f''(2) + f'''(3)$

$$\Rightarrow 8 - 20 + 4 + 6 = -2 \dots(10)$$

Hence from (9) and (10), we get $f(2) = f(1) - f(0)$.

$$\Rightarrow \therefore f(2) - f(1) + f(0) = 0$$

Q.16 Let $y = x \sin kx$. Find the sum of possible values of k for which the differential equation

$$\frac{d^2y}{dx^2} + y = 2k \cos kx \text{ holds true for all } x \in \mathbb{R}$$

Sol. [0]

$$y = x \sin kx \dots(1)$$

$$\Rightarrow y_1 = kx \cos kx + \sin kx$$

$$\Rightarrow y_2 = k [\cos kx - kx \sin kx] + k \cos kx$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2k \cos kx - k^2x \sin kx \quad \dots\dots\dots(2)$$

$$\Rightarrow \text{Now, given } \frac{d^2y}{dx^2} + y = 2 \cos kx \quad (\text{substituting the values of } y \text{ and } \frac{d^2y}{dx^2})$$

$$\Rightarrow \text{Hence, } 2k \cos kx - k^2x \sin kx + x \sin kx = 2 \cos kx$$

$$\Rightarrow x \sin kx [1 - k^2] = 0,$$

$$\Rightarrow \text{hence } k = 1, -1 \text{ or } 0$$

Q.17 The function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$ for all real x . Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

Sol. [6]

$$\text{Given, } f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$$

Put $x = 1$ in the given relation

$$\Rightarrow f(1) \cdot f''(1) = (f'(1))^2$$

$$\text{let } f'(1) = a \quad \dots\dots(1) \quad (\text{as } f(1) = 1)$$

to find $(a + b) = ?$

differentiating the given relation

$$\text{we get } f(x^2) \cdot f'''(x) + f''(x) \cdot 2x + f'(x^2) \cdot f''(x)$$

put $x = 1$

$$\Rightarrow 8 + 2ba = 2ab + ab$$

$$\Rightarrow ab = 8 \quad \dots\dots(2)$$

from (1) and (2)

$$\Rightarrow a^3 = 8$$

$$\Rightarrow a = 2 \text{ and } b = 4$$

$$\Rightarrow a + b = 2 + 4 = 6$$

Q.18 A polynomial function $f(x)$ is such that $f(2x) = f'(x) f''(x)$, then find the value of $f(3)$

Sol. [12]

Suppose degree of $f(x) = n$.

$$\Rightarrow \text{Then } \deg f' = n - 1 \text{ and } \deg f'' = n - 2, \text{ so } n = n - 1 + n - 2.$$

Hence $n = 3$.

$$\Rightarrow \text{So put } f(x) = ax^3 + bx^2 + cx + d. (a \neq 0). \text{ Now using } f(2x) = f'(x) \cdot f''(x)$$

$$\text{Then we have } 8ax^3 + 4bx^2 + 2cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$$

$$18a^2x^3 + 18abx^2 + (6ac + 4b^2)x + 2bc.$$

$$\Rightarrow \text{Comparing } x^3, \quad 18a^2 = 8a \quad \dots(1) \Rightarrow a = \frac{4}{9}$$

$$\Rightarrow \text{Comparing } x^2, \quad 18ab = 4b \quad \dots(2) \Rightarrow b = 0$$

$$\Rightarrow \text{Comparing } x, \quad 2c = 6ac + 4b^2 \quad \dots(3) \Rightarrow c = 0$$

$$\Rightarrow \text{Comparing constant term, } a = 2bc \quad \Rightarrow d = 0$$

$$\Rightarrow f(x) = \frac{4x^3}{9} \therefore f(3) = 12.$$

Q.19

Sol. [6]

$$f(x + y^n) = f(x) + \{f(y)\}^n, \forall x, y \in \mathbb{R}$$

Differentiating w.r.to x gives

$$f'(x + y^n) = f'(x)$$

Hence $f'(x) = \text{constant}$

$$\Rightarrow f(x) = ax + b$$

for $x = 1, y = 0$

$$f(1) = f(1) + \{f(0)\}^n \Rightarrow f(0) = 0$$

for $x = 0, y = 1$

$$f(1) = f(0) + \{f(1)\}^n \Rightarrow f(1) = 0 \text{ or } 1$$

But $f(1) = 0 \Rightarrow f(x) = 0$ for all x.

Hence $f(x) = x$

$$\Rightarrow f'(10) + f(5) = 6$$

Q.20

$$f(x) = \begin{cases} \frac{1 - a^x + x a^x \ln a}{a^x \cdot x^2}, & x < 0 \\ \frac{2^x a^x - x \ln 2 - x \ln a - 1}{x^2}, & x > 0 \end{cases}$$

$$\Rightarrow \ell = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1 - a^x + x a^x \ln a}{a^x x^2}$$

Replace x with 2x to get $\ell = \lim_{x \rightarrow 0^-} \frac{1 - a^{2x} + 2x a^{2x} \ln a}{4a^{2x} x^2}$

$$\Rightarrow \ell = \lim_{x \rightarrow 0^-} \frac{(1 - a^x + x a^x \ln a)(a^x - x a^x \ln a + 1) + x^2 a^{2x} (\ln a)^2}{4a^{2x} x^2}$$

$$\Rightarrow \ell = \frac{(\ln a)^2}{4} + \frac{\ell}{4} \lim_{x \rightarrow 0^-} \frac{(a^x - x a^x \ln a + 1)}{a^x}$$

$$\Rightarrow \ell = \frac{(\ln a)^2}{4} + \frac{\ell}{2}$$

$$\Rightarrow \ell = \frac{(\ln a)^2}{2}$$

Further $r = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{(2a)^x - x \ln 2a - 1}{x^2}$

Replace x with 2x to get

$$\Rightarrow r = \lim_{x \rightarrow 0^-} \frac{(2a)^{2x} - 2x \ln 2a - 1}{4x^2}$$

$$\Rightarrow r = \lim_{x \rightarrow 0^-} \frac{\left((2a)^x - x \ln 2a - 1\right)\left((2a)^x + x \ln 2a + 1\right) + x^2 (\ln 2a)^2}{4x^2}$$

$$\Rightarrow r = \frac{r}{4} \times \lim_{x \rightarrow 0^-} \left((2a)^x + x \ln 2a + 1\right) + \frac{(\ln 2a)^2}{4}$$

$$\Rightarrow r = \frac{(\ln 2a)^2}{4} + \frac{r}{2}$$

$$\Rightarrow r = \frac{(\ln 2a)^2}{2}$$

For $f(x)$ to be continuous,

$$\ell = r = f(0)$$

CONTINUITY & DIFFERENTIABILITY
EXERCISE 3

1 Let $f(x) = \begin{cases} \frac{\ln \cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \frac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define $f(0)$ to make the function continuous at $x = 0$. If yes what is the value of $f(0)$, if not then indicate the nature of discontinuity.

Sol. $\text{LHL}|_{x=0} = \lim_{x \rightarrow 0^-} \frac{e^{\sin 4x}-1}{\ln(1+\tan 2x)}$

put $x = 0 - h$

$$= \lim_{x \rightarrow 0} \frac{e^{-\sin 4x}-1}{\ln(1-\tan 2h)}$$

$$= \lim_{h \rightarrow 0} \frac{e^{-\sin 4h}-1}{-\sin 4h} \left(\frac{-\sin 4h}{4h} \right) \cdot 4h \left(\frac{1}{\frac{\ln(1-\tan 2h)}{(-\tan 2h)} \left(\frac{-\tan 2h}{2h} \right) \cdot 2h} \right)$$

$f(0^-) = 2$

$$\text{RHL}|_{x=0} = \lim_{x \rightarrow 0^+} \left(\frac{\ln \cos x}{\sqrt[4]{(1+x^2)}-1} \right)$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{\cos x - 1}{1 + \frac{1}{4}x^2 - 1} \right)$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{1 - \cos x}{x^2} \right) (-4)$$

$f(0^+) = -2$

hence $f(0)$ can not define.

and $\therefore f(0^-)$ & $f(0^+)$ are finite hence there non-removable type disconti.

2 Let $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim_{n \rightarrow \infty} y_n(x)$

Discuss the continuity of $y_n(x)$ ($n \in \mathbb{N}$) and $y(x)$ at $x = 0$

Sol. $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^{n-1}}$

$$y_n(x) = x^2 \frac{\left(1 - \left(\frac{1}{1+x^2}\right)^n\right)}{1 - \frac{1}{1+x^2}}$$

$$= x^2 \frac{\left\{1 - \left(\frac{1}{1+x^2}\right)^n\right\}}{\frac{1+x^2-1}{1+x^2}}$$

$$y_n(x) = (1+x^2)^3 \{1 - (1+x^2)^{-n}\}$$

- 3 Let $f(x) = \begin{cases} \frac{1 - \sin \pi x}{1 + \cos 2\pi x}, & x < \frac{1}{2} \\ p, & x = \frac{1}{2} \\ \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}-2}}, & x > \frac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at $x = 1/2$.

Sol. V.F. $\Big|_{x=\frac{1}{2}} = p \quad \dots(1)$

$$\text{LHL} \Big|_{x=\frac{1}{2}} = \lim_{x \rightarrow \frac{1}{2}^-} f(x)$$

$$= \lim_{x \rightarrow \frac{1}{2}^-} \frac{1 - \sin \pi x}{1 + \cos(2\pi x)}$$

$$\text{put } x = \frac{1}{2} - h$$

$$= \lim_{h \rightarrow 0} \frac{1 - \sin\left(\frac{\pi}{2} - \pi h\right)}{1 + \cos(\pi - 2\pi h)}$$

$$= \lim_{h \rightarrow 0} \left(\frac{1 - \cos \pi h}{(\pi h)^2} \right) \left(\frac{1}{\frac{1 - \cos(2\pi h)}{(2\pi h)^2}} \right) \left(\frac{\pi^2 h^2}{4\pi^2 h^2} \right)$$

$$\text{LHL} \Big|_{x=\frac{1}{2}} = \frac{1}{4} \quad \dots(2)$$

$$\begin{aligned} \text{RHL}|_{x=\frac{1}{2}} &= \lim_{x \rightarrow \left(\frac{1}{2}\right)^+} f(x) \\ &= \lim_{x \rightarrow \left(\frac{1}{2}\right)^+} \left(\frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2} \right) \\ &= \lim_{x \rightarrow \frac{1}{2}^+} \left(\frac{\sqrt{2x-1}}{4+\sqrt{2x-1}-4} \right) (\sqrt{4+\sqrt{2x-1}}+2) \end{aligned}$$

$$\text{RHL}|_{x=\frac{1}{2}} = 4$$

$$\therefore \text{LHL}|_{x=\frac{1}{2}} \neq \text{RHL}|_{x=\frac{1}{2}}$$

so the value of function cannot determine & the function is discontinuous.

4 Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate $h(g(2))$ (b) If $f(x) = \begin{cases} g(x), & x \leq 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

Sol. (i) $h(g(2)) =$

$$g(2) = \sqrt{6-4} = \sqrt{2}$$

$$h(x) = 2x^2 - 3x + a$$

$$h(\sqrt{2}) = 4 - 3\sqrt{2} + a \quad \text{Ans}$$

$$(ii) f(x) = \begin{cases} g(x) & ; x \leq 1 \\ h(x) & ; x > 1 \end{cases}$$

$$f(x) = \begin{cases} \sqrt{6-2x} & ; x \leq 1 \\ 2x^2 - 3x + a & ; x > 1 \end{cases}$$

$$\text{V.F.}|_{x=1} = 2 \quad \dots(1)$$

$$\begin{aligned} \text{R.H.L.}|_{x=1} &= \lim_{x \rightarrow 1^+} f(x) \\ &= \lim_{x \rightarrow 1^+} (2x^2 - 3x + a) \end{aligned}$$

$$\text{R.H.L.}|_{x=1} = a - 1 \quad \dots(2)$$

$$\begin{aligned} \text{L.H.L.}|_{x=1} &= \lim_{x \rightarrow 1^-} f(x) \\ &= \lim_{x \rightarrow 1^-} \sqrt{6-2x} \\ &= 2 \end{aligned}$$

since function is conti

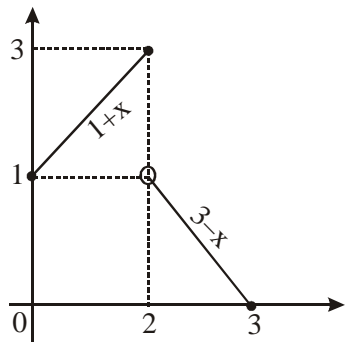
$$\text{L.H.L.}|_{x=1} = \text{R.H.L.}|_{x=1} = \text{VF}|_{x=1}$$

$$2 = a - 1 = 2$$

$$a - 1 = 2 \Rightarrow \boxed{a = 3}$$

- 5 Let $f(x) = \begin{cases} 1+x & , 0 \leq x \leq 2 \\ 3-x & , 2 < x \leq 3 \end{cases}$. Determine the form of $g(x) = f[f(x)]$ & hence find the point of discontinuity of g , if any.

Sol. $f(x) = \begin{cases} 1+x & ; 0 \leq x \leq 2 \\ 3-x & ; 2 < x \leq 3 \end{cases}$

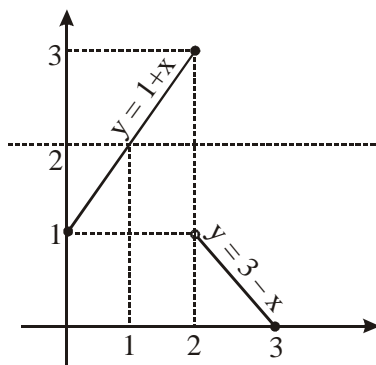


$$g(x) = f(f(x))$$

$$= \begin{cases} 1+f(x) & ; 0 \leq f(x) \leq 2 \\ 3-f(x) & ; 2 < f(x) \leq 3 \end{cases}$$

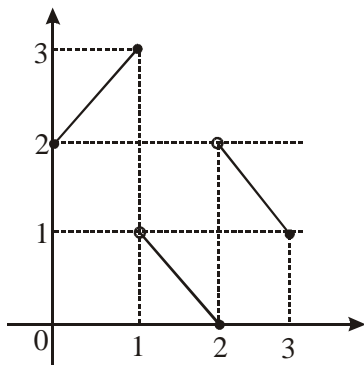
let $f(x) = y$

$$f(y) = \begin{cases} 1+y & ; 0 \leq y \leq 2 \\ 3-y & ; 2 < y \leq 3 \end{cases}$$



$$= \begin{cases} 1+(1+x) & ; 0 \leq x \leq 1 \\ 1+(3-x) & ; 2 < x \leq 3 \\ 3-(1+x) & ; 1 < x \leq 2 \end{cases}$$

$$= \begin{cases} 2+x & ; 0 \leq x \leq 1 \\ 2-x & ; 1 < x \leq 2 \\ 4-x & ; 2 \leq x \leq 3 \end{cases}$$



so the point of discontinuity

1, 2 **Ans**

Or

$$F.V.|_x = LHL = RHL$$

- 6 Let $[x]$ denote the greatest integer function & $f(x)$ be defined in a neighbourhood of 2 by

$$f(x) = \begin{cases} \frac{(\exp\{(x+2)\ln 4\})^{\frac{[x+1]}{4}} - 16}{4^x - 16} & , x < 2 \\ A \frac{1 - \cos(x-2)}{(x-2)\tan(x-2)} & , x > 2 \end{cases}$$

Find the values of A & $f(2)$ in order that $f(x)$ may be continuous at $x = 2$.

$$\text{Sol. } RHL|_{x=2} = \lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} \frac{4^2 \cdot 4^{\frac{-h}{2}} - 16}{4^2 \cdot 4^{-h} - 16}$$

$$= \lim_{x \rightarrow 2^+} \frac{A(1 - \cos(x-2))}{(x-2) \cdot \tan(x-2)} = \lim_{h \rightarrow 0} \frac{4^{-n/2} - 1}{4^{-h} - 1}$$

put $x = 2 + h$

$$= \lim_{h \rightarrow 0} \frac{A(1 - \cosh)}{h \tan h} = \lim_{h \rightarrow 0} \left(\frac{4^{-h/2} - 1}{-\frac{h}{2}} \right) \cdot \left(-\frac{h}{2} \right) \frac{1}{\left(\frac{4^{-h} - 1}{-h} \right) (-h)}$$

$$= \lim_{h \rightarrow 0} A \left(\frac{1 - \cosh}{h^2} \right) \frac{1}{\left(\frac{\tan h}{h} \right)} = \ln 4 \cdot \frac{1}{2} \cdot \frac{1}{\ln 4} = \frac{1}{2}$$

$$\text{RHL}|_{x=2} = \frac{A}{2}$$

since the function is contin.

$$\text{VF}|_{x=2} = \text{RHL}|_{x=2} = \text{LHL}|_{x=2}$$

$$\text{LHL}|_{x=2} \Rightarrow \lim_{x \rightarrow 2^-} f(x)$$

$$\text{V.F.}|_{x=2} = \frac{A}{2} = \frac{1}{2}$$

$$= \lim_{x \rightarrow 2^-} \frac{(e^{(x+2)^{[x]+1}}) - 16^{\frac{[x+1]}{4}}}{4^x - 16}$$

$$\boxed{\text{V.F.}|_{x=2} = \frac{1}{2}} \text{ Ans}$$

$$= \lim_{x \rightarrow 2^-} \frac{4^{\frac{(x+2)([x]+1)}{4}} - 16}{4^x - 16}$$

$$\boxed{A = 1} \text{ Ans}$$

$$= \lim_{x \rightarrow 2^-} \frac{4^{\left(\frac{x+2}{2}\right)} - 16}{4^x - 16}$$

$$\text{put } x = 2 - h$$

$$= \lim_{x \rightarrow 0} \frac{4^{\frac{4-h}{2}} - 16}{4^{2-h} - 16}$$

7 The function $f(x) = \begin{cases} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \frac{\pi}{2} \\ b+2 & \text{if } x = \frac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$.

Sol. $\text{V.F.}|_{x=\frac{\pi}{2}} = b + 2 \quad \dots(1)$

$$\text{LHL}|_{x=\frac{\pi}{2}} = \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}$$

$$\text{put } x = \frac{\pi}{2} - h$$

$$\text{LHL}|_{x=\frac{\pi}{2}} = \lim_{h \rightarrow 0} \left(\frac{6}{5}\right)^{\frac{\tan(3\pi - 6h)}{\tan(5\pi/2 - 5h)}} = \lim_{h \rightarrow 0} \left(\frac{6}{5}\right)^{\frac{\tan 6h}{\cot 5h}} = 1$$

$$\text{RHL}|_{x=\frac{\pi}{2}} = \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^+} f(x)$$

$$= \lim_{x \rightarrow \left(\frac{\pi}{2}\right)^+} (1 - \cos x)^{\frac{a}{b} \tan x}$$

$$\text{put } x = \frac{\pi}{2} + h$$

$$= \lim_{h \rightarrow 0} (1 + \sinh)^{\frac{a}{b} \cot h} ; 1^\infty \text{ form}$$

$$= \lim_{h \rightarrow 0} (\sinh)^{\frac{a}{b} \cot h}$$

$$= e^{\lim_{h \rightarrow 0} \frac{a}{b} \cosh} = e^{\frac{a}{b}}$$

since the function is conti so

$$\text{LHL}|_{x=\frac{\pi}{2}} = \text{RHL}|_{x=\frac{\pi}{2}} = \text{V.F.}|_{x=\frac{\pi}{2}}$$

$$1 = e^{\frac{a}{b}} = b + 2$$

$$\boxed{a = 0, b = -1}$$

$$8 \quad \text{Let } f(x) = \begin{cases} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1 - \{x\}^2)\right) \sin^{-1}(1 - \{x\})}{\sqrt{2}(\{x\} - \{x\}^3)} & ; x \neq 0 \\ \frac{\pi}{2} & ; x = 0 \end{cases}$$

where $\{x\}$ is the fractional part of x . Consider another function $g(x)$; such that

$$g(x) = f(x); x \geq 0$$

$$= 2\sqrt{2}f(x); x < 0$$

Discuss the continuity of the function $f(x)$ & $g(x)$ at $x = 0$.

$$\text{Sol. } \text{RHL}|_{x=0} = \lim_{x \rightarrow 0^+} f(x) \\ = \lim_{x \rightarrow 0^+} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1 - (x - [x])^2)\right) \sin^{-1}(1 - x + [x])}{\sqrt{2}(x - [x] - (x - [x])^3)}$$

$$= \lim_{x \rightarrow 0^+} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1 - x^2)\right) \sin^{-1}(1 - x)}{\sqrt{2}x(1 - x^2)}$$

$$= \lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1 - x^2) \cdot \sin^{-1}(1 - x)}{\sqrt{2}x(1 - x^2)}$$

$$= \frac{\pi}{2\sqrt{2}} \lim_{x \rightarrow 0^+} \frac{\cos^{-1}(1 - x^2)}{x}$$

$$\text{let } \cos^{-1}(1 - x^2) = \theta$$

$$1 - x^2 = \cos \theta$$

$$x^2 = 1 - \cos \theta$$

$$x = \sqrt{1 - \cos \theta}$$

when $x \rightarrow 0^+$ then $\theta \rightarrow 0$

$$= \frac{\pi}{2\sqrt{2}} \lim_{\theta \rightarrow 0^+} \frac{\theta}{\sqrt{1 - \cos \theta}}$$

$$= \frac{\pi}{2\sqrt{2}} \lim_{\theta \rightarrow 0^+} \frac{\theta}{\sqrt{2 - \sin^2 \theta/2}} = \frac{\pi}{4} \lim_{\theta \rightarrow 0^+} \frac{\theta}{|\sin \theta/2|}$$

$$\text{RHL}|_{x=0} = \frac{\pi}{4} \lim_{\theta \rightarrow 0^+} 2 \left(\frac{\theta/2}{\sin \theta/2} \right) = \frac{\pi}{2}$$

$$\text{LHL}|_{x=0} = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{x \rightarrow 0^-} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1 - x - [x])^2 \right) \sin^{-1}(1 - x + [x])}{\sqrt{2}(x - [x]) - (x - [x])^3}$$

$$= \lim_{x \rightarrow 0^-} \frac{\frac{\pi}{2} - \sin^{-1}(1 - (x+1)^2) \sin^{-1}(-x)}{\sqrt{2}(x+1 - (x+1)^3)}$$

$$= \lim_{x \rightarrow 0^-} \frac{\left(\frac{\pi}{2} \sin^{-1}(-x^2 - 2x) \right) \sin^{-1}(-x)}{\sqrt{2}(x+1)(-x^2 - 2x)}$$

$$= \lim_{x \rightarrow 0^-} \frac{\cos^{-1}(-x^2 - 2x) \sin^{-1}(x)}{\sqrt{2}(x+1)(x^2 + 2x)}$$

$$= \lim_{x \rightarrow 0^-} \frac{\pi - \cos^{-1}(x^2 + 2x)}{\sqrt{2}(x+1)(x+2)} \cdot \frac{\sin^{-1} x}{x}$$

$$\text{LHL}|_{x=0} = \frac{\pi}{4\sqrt{2}}$$

for $f(x)$ since $\text{LHL}|_{x=0} \neq \text{RHL}|_{x=0}$ so the function is discontinuous at $x=0$.

for $g(x) \Rightarrow$

$$\text{RHL}|_{x=0} = \lim_{x \rightarrow 0^+} g(x)$$

$$= \lim_{x \rightarrow 0^+} f(x) = \frac{\pi}{2}$$

$$\text{LHL}|_{x=0} = \lim_{x \rightarrow 0^-} 2\sqrt{2}f(x)$$

$$= 2\sqrt{2} \lim_{x \rightarrow 0^-} f(x)$$

$$= 2\sqrt{2} \cdot \frac{\pi}{4\sqrt{2}} = \frac{\pi}{2}$$

$$g(0) = f(0) = \frac{\pi}{2}$$

- 9 If the function $f(x)$ defined as $f(x) = \begin{cases} -\frac{x^2}{2} & \text{for } x \leq 0 \\ x^n \sin \frac{1}{x} & \text{for } x > 0 \end{cases}$ is continuous but not derivable at $x=0$ then find the range of n .

Sol. $f(x) = \begin{cases} -\frac{x^2}{2} & \text{for } x \leq 0 \\ x^n \sin \frac{1}{x} & \text{for } x > 0 \end{cases}$

$f(x)$ is continuous at $x=0$

$$f(0) = 0$$

$$L_1 = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(\frac{-x^2}{2} \right) = 0$$

$$L_2 = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^n \sin \frac{1}{x}$$

for continuous,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} x^n \sin \left(\frac{1}{x} \right) = 0$$

limit is defined only when

$$\therefore n > 0$$

since $f(x)$ is non-differentiable at $x=0$

$$f'(0^-) = \lim_{h \rightarrow 0^-} \frac{f(h+0) - f(0)}{2} = \lim_{h \rightarrow 0^-} \frac{-\frac{h^2}{2} - 0}{h} = \lim_{h \rightarrow 0^-} \left(\frac{-h}{2} \right)$$

$$f'(0^+) = \lim_{h \rightarrow 0^+} \frac{f(h+0) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^n \sin \frac{1}{h}}{h}$$

$$f'(0^+) \neq f'(0^-)$$

$$\Rightarrow \lim_{h \rightarrow 0^+} h^{n-1} \sin\left(\frac{1}{h}\right) \neq 0$$

only when $n - 1 \leq 0$

$$\Rightarrow n \leq 1 \quad \dots(ii)$$

from equation (i) & (ii)

$$n \in (0, 1]$$

10 ... $f(0) = 0$ and $f'(0) = 1$. For a positive integer k , show that

$$\lim_{x \rightarrow 0} \frac{1}{x} \left(f(x) + f\left(\frac{x}{2}\right) + \dots + f\left(\frac{x}{k}\right) \right) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

Sol.
$$\lim_{x \rightarrow 0} \frac{1}{x} \left[f(x) + f\left(\frac{x}{2}\right) + \dots + f\left(\frac{x}{k}\right) \right]$$

$$= \lim_{x \rightarrow 0} \frac{f(x)}{x} + \frac{f\left(\frac{x}{2}\right)}{x} + \dots + \frac{f\left(\frac{x}{k}\right)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{f(x+0) - f(0)}{x} + \lim_{x \rightarrow 0} \frac{f\left(\frac{1}{x} + 0\right) - f(0)}{\frac{x}{2}} \cdot \frac{1}{2} + \dots + \lim_{x \rightarrow 0^+} \frac{f\left(\frac{x}{k} + 0\right) - f(0)}{\frac{x}{k}} \cdot \frac{1}{k}$$

$$= f'(0) + \frac{1}{2} f'(0) + \dots + \frac{1}{k} f'(0)$$

$$= 1 + \frac{1}{2} + \dots + \frac{1}{k}$$

11 If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at $x = 1$. Find the values of a & b .

Sol.
$$f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$$

$f(x)$ is differentiable at $x = 1$, hence it is also continuous at $x = 1$

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\Rightarrow \boxed{a - b = -1} \quad \dots(i)$$

$$f'(1) = \lim_{h \rightarrow 0^-} \frac{f(h+1) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{a(h+1)^2 - b + 1}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{ah^2 + 2ah + a - b + 1}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{ah^2 + 2ah}{h} = \lim_{h \rightarrow 0^-} (ah + 2a) = 2a$$

$$f'(1^+) = \lim_{h \rightarrow 0^+} \frac{f(h+1) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{-1}{|h+1|} + 1}{h}$$

$$= \lim_{h \rightarrow 0^+} \frac{\frac{-1+1+h}{1+h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{1+h} = 1$$

$$f'(1^-) = f'(1^+)$$

$$\Rightarrow 2a = 1$$

$$a = 1/2$$

$$b = 3/2$$

12 The function $f(x) = \begin{cases} ax(x-1) + b & \text{when } x < 1 \\ x-1 & \text{when } 1 \leq x \leq 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

Find the values of the constants a, b, p, q so that

(i) $f(x)$ is continuous for all x (ii) $f'(1)$ does not exist

(iii) $f'(x)$ is continuous at $x = 3$

Sol. $f(x) = \begin{cases} ax(x-1) + b & \text{when } x < 1 \\ x-1 & \text{when } 1 \leq x \leq 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

$f(x)$ is continuous at $x = 1$

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

$$\Rightarrow \lim_{x \rightarrow 1^-} ax(x-1) + b = 0$$

$$\Rightarrow \boxed{b = 0 \& a \in \mathbb{R}}$$

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(h+1) - f(1)}{h} = \begin{cases} \lim_{h \rightarrow 0^-} \frac{a(h+1)(h+1-1) + b}{h} \\ \lim_{h \rightarrow 0^+} \frac{h+1-1}{h} \end{cases}$$

$$= \begin{cases} \lim_{h \rightarrow 0^-} \frac{a(h+1)h + 0}{h} \\ \lim_{h \rightarrow 0^+} \frac{h}{h} \end{cases} = \begin{cases} \lim_{h \rightarrow 0^-} a(h+1) \\ 1 \end{cases}$$

$$= \begin{cases} a \\ 1 \end{cases}$$

$$\therefore f'(1) = \text{DNE} \Rightarrow a \neq 1$$

$$\therefore a \in \mathbb{R} - \{1\} \text{ \& } b = 0$$

$f(x)$ is cont. at $x = 3$

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

$$\Rightarrow \lim_{x \rightarrow 3} (px^2 + 9x + 2) = 2$$

$$\Rightarrow 9p + 3q + 2 = 2$$

$$\Rightarrow 9p + 3q = 0 \quad \dots(i)$$

$\therefore f'(x)$ is cont. at $x = 3$, hence $f(x)$ is diff. at $x = 3$

$$f'(3) = \lim_{h \rightarrow 0} \frac{f(h+3) - f(3)}{h} = \begin{cases} \lim_{h \rightarrow 0^-} \frac{3+h-1-2}{h} \\ \lim_{h \rightarrow 0^+} \frac{p(h+3)^2 + q(h+3) + 2 - 2}{h} \end{cases}$$

$$= \begin{cases} \lim_{h \rightarrow 0^-} \frac{h}{h} \\ \lim_{h \rightarrow 0^+} \frac{ph^2 + 6ph + qh + 9p + 3q}{h} \end{cases} = \begin{cases} 1 \\ \lim_{h \rightarrow 0^+} \frac{ph^2 + 6ph + qh}{h} \end{cases}$$

[from equation (i) $9p + 3q = 0$]

$$= \begin{cases} 1 \\ \lim_{h \rightarrow 0^+} (ph + 6p + q) \end{cases} = \begin{cases} 1 \\ 6p + q \end{cases}$$

$$\therefore f'(3^+) = f'(3^-) \Rightarrow 6p + q = 0 \quad \dots(ii)$$

solving equation (i) & (ii) $p = 1/3, q = -1$

$$a \in \mathbb{R} - \{1\}, b = 0, p = 1/3, q = -1$$

13 Discuss the continuity on $0 \leq x \leq 1$ & differentiability at $x = 0$ for the function.

$$f(x) = x \cdot \sin \frac{1}{x} \cdot \sin \frac{1}{x \cdot \sin \frac{1}{x}} \text{ where } x \neq 0, x \neq 1/r\pi \text{ \& } f(0) = f(1/r\pi) = 0,$$

$$r = 1, 2, 3, \dots$$

Sol. $f(x) = x \cdot \sin \frac{1}{x} \cdot \sin \frac{1}{x \cdot \sin \frac{1}{x}} \text{ } x \neq 0, 1/r\pi$

$$f(0) = 0 = f\left(\frac{1}{r\pi}\right), r = 1, 2, 3, \dots$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h+0) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h \sin\left(\frac{1}{h}\right) \cdot \sin\left(\frac{1}{h \sin\left(\frac{1}{h}\right)}\right) - 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h \sin\left(\frac{1}{h}\right) \cdot \sin\left(\frac{1}{h \sin\left(\frac{1}{h}\right)}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) \cdot \sin\left(\frac{1}{h \sin\left(\frac{1}{h}\right)}\right)$$

$$= \lim_{h \rightarrow 0} \underbrace{\sin\left(\frac{1}{h}\right)}_{-1 \leq \leq 1} \underbrace{\sin\left(\frac{1}{h \sin(1/h)}\right)}_{-1 \leq \leq 1}$$

$$= \text{DNE}$$

so $f(x)$ is not differentiable at $x = 0$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) \cdot \sin\left(\frac{1}{x \sin(1/x)}\right)$$

$$= \lim_{x \rightarrow 0} \underbrace{x}_{\rightarrow 0} \cdot \underbrace{\sin(1/x)}_{-1 \leq \leq 1} \cdot \underbrace{\sin\left(\frac{1}{n \sin(1/x)}\right)}_{-1 \leq \leq 1}$$

$$= 0$$

$$= f(0)$$

$$\lim_{x \rightarrow \frac{1}{r\pi}} f(x) = \lim_{x \rightarrow \frac{1}{r\pi}} x \sin\left(\frac{1}{x}\right) \cdot \sin\left(\frac{1}{x \sin\left(\frac{1}{x}\right)}\right)$$

$$= \lim_{x \rightarrow \frac{1}{r\pi}} x \sin\left(\frac{1}{x}\right) \cdot \sin\left(\frac{\frac{1}{\sin\left(\frac{1}{x}\right)}}{\left(\frac{1}{x}\right)}\right)$$

$$= \lim_{x \rightarrow \frac{1}{r\pi}} \underbrace{x \sin\left(\frac{1}{x}\right)}_{\rightarrow 0} \cdot \underbrace{\sin\left(\frac{1}{x \sin\left(\frac{1}{x}\right)}\right)}_{-1 \leq \leq 1}$$

$$= 0$$

$$= f\left(\frac{1}{r\pi}\right)$$

Hence function is continuous $\forall x \in [0, 1]$

14 $f(x) = \begin{cases} 1-x & , (0 \leq x \leq 1) \\ x+2 & , (1 < x < 2) \\ 4-x & , (2 \leq x \leq 4) \end{cases}$ Discuss the continuity & differentiability of $y = f[f(x)]$ for $0 \leq x \leq 4$.

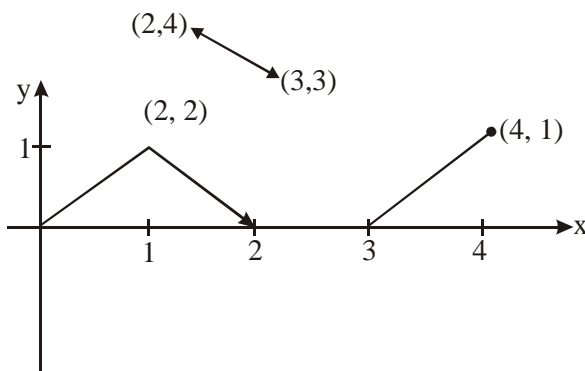
Sol. $f(x) = \begin{cases} 1-x & , (0 \leq x \leq 1) \\ x+2 & , (1 < x < 2) \\ 4-x & , (2 \leq x \leq 4) \end{cases}$

$$f(f(x)) = \begin{cases} 1-f(x) & ; 0 \leq f(x) \leq 1 \\ f(x)+2 & ; 1 < f(x) < 2 \\ 4-f(x) & ; 2 \leq f(x) \leq 4 \end{cases}$$

$$= \begin{cases} 1-1-x & ; 0 \leq x \leq 1 \cap 1 \leq 1-x \leq 1 \Rightarrow 0 \leq x \leq 1 \\ 1-x-2 & ; 1 < x < 2 \cap 0 \leq x+2 \leq 1 \Rightarrow -2 \leq x \leq -1 \\ 1-4+x & ; 2 \leq x \leq 4 \cap 0 \leq 4-x \leq 1 \Rightarrow 3 \leq x \leq 4 \\ 1-x+2 & ; 0 \leq x \leq 1 \cap 1 < 1-x < 2 \Rightarrow -1 < x < 0 \\ x+2+2 & ; 1 < x < 2 \cap 1 < x+2 < 2 \Rightarrow -1 < x < 0 \\ 4-x+2 & ; 2 \leq x \leq 4 \cap 1 < 4-x < 2 \Rightarrow 2 < x < 3 \\ 4-1+x & ; 0 \leq x \leq 1 \cap 2 \leq 1-x \leq 4 \Rightarrow -3 \leq x \leq -1 \\ 4-x-2 & ; 1 < x < 2 \cap 2 \leq 4-x \leq 4 \Rightarrow 0 \leq x \leq 2 \\ 4-4+x & ; 2 \leq x \leq 4 \cap 2 \leq 4-x \leq 4 \Rightarrow 0 \leq x \leq 2 \end{cases}$$

$$= \begin{cases} x & ; 0 \leq x \leq 1 \\ x-3 & ; 3 \leq x \leq 4 \\ -x+6 & ; 2 < x < 3 \\ -x+2 & ; 1 < x < 2 \end{cases}$$

$$f(f(x)) = \begin{cases} x & ; 0 \leq x \leq 1 \\ -x-2 & ; 1 < x < 2 \\ x & ; x = 2 \\ -x+6 & ; 2 < x < 3 \\ x-3 & ; 3 \leq x \leq 4 \end{cases}$$



$\therefore f(x)$ is continuous at $x = 1$ & discunt.

at $x = 2, 3$ & non diff. at $x = 1, 2, 3$

15 Let f be a function that is differentiable every where and that has the following properties:

- (i) $f(x+h) = f(x) \cdot f(h)$ (ii) $f(x) > 0$ for all real x . (iii) $f'(0) = -1$

Use the definition of derivative to find $f'(x)$ in terms of $f(x)$.

Sol. $f(x+h) = f(x) \cdot f(h)$ $\left. \begin{array}{l} x=0 \\ h=0 \end{array} \right\} f(0)(f(0)-1) = 0 \Rightarrow f(0) = 1$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x) \cdot f(h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} f(x)$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} f(x)$$

$$= f'(0) f(x)$$

$$\Rightarrow f'(x) = -f(x)$$

$$\therefore f'(x) = -f(x)$$

16 Discuss the continuity & the derivability of 'f' where $f(x) = \text{degree of } (u^x + u^2 + 2u - 3) \text{ at } x = \sqrt{2}$.

Sol. $f(x) = \text{degree of } (u^x + u^2 + 2u - 3) \text{ at } x = \sqrt{2}$

$$= \begin{cases} 2 & ; \quad x \leq \sqrt{2} \\ x^2 & ; \quad x > \sqrt{2} \end{cases}$$

$$f'(\sqrt{2}) = \lim_{h \rightarrow 0} \frac{f(h + \sqrt{2}) - f(\sqrt{2})}{h}$$

$$= \begin{cases} \lim_{h \rightarrow 0^+} \frac{f(h + \sqrt{2}) - f(\sqrt{2})}{h} \\ \lim_{h \rightarrow 0^-} \frac{f(h + \sqrt{2}) - f(\sqrt{2})}{h} \end{cases}$$

$$= \begin{cases} \lim_{h \rightarrow 0^+} \frac{(h + \sqrt{2})^2 - 2}{h} \\ \lim_{h \rightarrow 0^-} \frac{2 - 2}{h} \end{cases}$$

$$= \begin{cases} \lim_{h \rightarrow 0^+} \frac{2 + 2\sqrt{2}h + h^2 - 2}{h} \\ 0 \end{cases}$$

$$= \begin{cases} \text{Lim}_{h \rightarrow 0^+} \frac{h^2 + 2\sqrt{2}h}{h} \\ 0 \end{cases}$$

$$= \begin{cases} \text{Lim}_{h \rightarrow 0^+} (h + 2\sqrt{2}) \\ 0 \end{cases}$$

$$= \begin{cases} 2\sqrt{2} \\ 0 \end{cases}$$

$$\therefore f'(\sqrt{2}^-) \neq f'(\sqrt{2}^+)$$

Hence $f(x)$ is non differentiable at $x = \sqrt{2}$

$$\begin{aligned} \text{Lim}_{x \rightarrow \sqrt{2}} f(x) &= \text{Lim}_{x \rightarrow \sqrt{2}} x^2 \\ &= 2 \\ &= f(\sqrt{2}) \end{aligned}$$

$$\Rightarrow f(\sqrt{2}) = \text{Lim}_{x \rightarrow \sqrt{2}} f(x)$$

Hence $f(x)$ is continuous at $x = \sqrt{2}$

17 Let $f(x)$ be a function defined on $(-a, a)$ with $a > 0$. Assume that $f(x)$ is continuous at $x = 0$ and

$\text{Lim}_{x \rightarrow 0} \frac{f(x) - f(kx)}{x} = \alpha$, where $k \in (0, 1)$ then compute $f'(0^+)$ and $f'(0^-)$, and comment upon the differentiability of f at $x = 0$.

Sol. $\therefore \text{Lim}_{x \rightarrow 0} \frac{f(x) - f(kx)}{x} = \alpha$

$$\Rightarrow \text{Lim}_{x \rightarrow 0} \frac{f(x) - f(0) + f(0) - f(kx)}{x} = \alpha$$

$$\Rightarrow \text{Lim}_{x \rightarrow 0} \frac{f(x) - f(0) - f(kx) + f(0)}{x} = \alpha$$

$$\Rightarrow \text{Lim}_{x \rightarrow 0} \left(\frac{f(x) - f(0)}{x} - \frac{f(kx) - f(0)}{x} \right) = \alpha$$

$$\Rightarrow \left(\text{Lim}_{x \rightarrow 0} \frac{f(x) - f(0)}{x} \right) - \left(\text{Lim}_{x \rightarrow 0} \frac{f(kx) - f(0)}{kx} \right) k = \alpha$$

$$\Rightarrow \begin{cases} \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} - \lim_{x \rightarrow 0^-} \frac{f(kx) - f(0)}{kx} \cdot k = \alpha \\ \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} - \lim_{x \rightarrow 0^+} \frac{f(kx) - f(0)}{kx} \cdot k = \alpha \end{cases}$$

$$\Rightarrow \begin{cases} f'(0^-) - kf'(0^-) = \alpha \\ f'(0^+) - kf'(0^+) = \alpha \end{cases}$$

$$\Rightarrow \begin{cases} (1-k)f'(0^-) = \alpha \\ (1-k)f'(0^+) = \alpha \end{cases}$$

$$\Rightarrow \begin{cases} f'(0^-) = \frac{\alpha}{1-k} \\ f'(0^+) = \frac{\alpha}{1-k} \end{cases}$$

$$\therefore f'(0) = f'(0^-) = f'(0^+) = \frac{\alpha}{1-k}$$

- 18 A derivable function $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the condition $f(x) - f(y) \geq \ln(x/y) + x - y$ for every $x, y \in \mathbb{R}^+$. If g denotes the derivative of f then compute the value of the sum $\sum_{n=1}^{100} g\left(\frac{1}{n}\right)$.

Sol. $f(x) - f(y) \geq \ln(x/y) + x - y$

$$\Rightarrow f(x) - f(y) \geq \ln x - \ln y + x - y$$

$$\Rightarrow \frac{f(x) - f(y)}{x - y} \geq \frac{\ln x - \ln y}{x - y} + 1 \quad [\text{for } x \neq y]$$

$$\Rightarrow \lim_{x \rightarrow y} \frac{f(x) - f(y)}{x - y} \geq \lim_{x \rightarrow y} \frac{\ln x - \ln y}{x - y} + 1$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(y+h) - f(y)}{h} \geq \lim_{h \rightarrow 0} \frac{\ln\left(\frac{y+h}{y}\right)}{h} + 1$$

$$\Rightarrow f'(y) \geq \lim_{h \rightarrow 0} \ln\left(1 + \frac{h}{y}\right)^{1/h} + 1$$

- 19 If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ prove that $2y = xy' + \ln y'$. where ' denotes the derivative.

[Sol. $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$

$$y' = x + \frac{1}{2} \left[\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} \right] + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2} \left[\frac{2x^2+1}{\sqrt{x^2+1}} \right] + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2\sqrt{x^2+1}} [2(x^2+1)]$$

$$y' = x + \sqrt{x^2+1}$$

$$\text{Also } 2y = x^2 + x\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1})$$

$$= x(x + \sqrt{x^2+1}) + \ln(x + \sqrt{x^2+1}) = xy' + \ln y' \quad \text{Hence proved]}$$

20 If $y = \sec 4x$ and $x = \tan^{-1}(t)$, prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$.

[Sol. $y = \frac{1}{\cos 4x} = \frac{1+\tan^2 2x}{1-\tan^2 2x}$ (1)

using $\tan x = t$ (given)

$$\tan 2x = \frac{2t}{1-t^2}$$

substituting in (1)

$$y = \frac{1 + \frac{4t^2}{(1-t^2)^2}}{1 - \frac{4t^2}{(1-t^2)^2}} = \frac{(1+t^2)^2}{(1-t^2)^2 - 4t^2} = \frac{(1+t^2)^2}{1-6t^2+t^4}$$

$$\frac{dy}{dt} = \frac{(1-6t^2+t^4) \cdot 2(1+t^2) \cdot 2t - (1+t^2)(4t^3-12t)}{(1-6t^2+t^4)^2}$$

$$= \frac{4t(1+t^2)[(1-6t^2+t^4) - (1+t^2)(t^2-3)]}{(1-(t^2+t^4)^2)} = \frac{4t(1+t^2)(1-t^2)}{(1-6t^2+t^4)^2} = \frac{4t(1-t^4)}{(1-6t^2+t^4)^2}]$$

21 If $x = \frac{1+\ln t}{t^2}$ and $y = \frac{3+2\ln t}{t}$. Show that $y \frac{dy}{dx} = 2x \left(\frac{dy}{dx} \right)^2 + 1$.

[Sol. $\frac{dx}{dt} = \frac{t-(1+\ln t)2t}{t^4} = \frac{t(1-2-\ln t)}{t^4} = -\frac{(1+2\ln t)}{t^3}$

$$\frac{dy}{dt} = \frac{t\left(\frac{2}{t}\right) - (3+2\ln t)}{t^2} = -\frac{(1+2\ln t)}{t^2}$$

$$\frac{dy}{dx} = \frac{1+2\ln t}{t^2} \cdot \frac{t^3}{1+2\ln t} = t$$

Now L.H.S. = $\frac{3+2\ln t}{t} \cdot t = 3+2\ln t$

$$\text{R.H.S.} = \frac{2(1 + \ln t)}{t^2} \cdot t^2 + 1 = 3 + 2 \ln 2$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}]$$

22 If $y = 1 + \frac{x_1}{x-x_1} + \frac{x_2 \cdot x}{(x-x_1)(x-x_2)} + \frac{x_3 \cdot x^2}{(x-x_1)(x-x_2)(x-x_3)} + \dots$ upto $(n+1)$ terms then prove that

$$\frac{dy}{dx} = \frac{y}{x} \left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \frac{x_3}{x_3-x} + \dots + \frac{x_n}{x_n-x} \right]$$

[Sol. adding term by term

$$y = \frac{x^n}{(x-x_1)(x-x_2)(x-x_3) \dots (x-x_n)}$$

$$y = \frac{x}{(x-x_1)} \cdot \frac{x}{(x-x_2)} \cdot \frac{x}{(x-x_3)} \dots \frac{x}{(x-x_n)}$$

$$\ln y = \ln \frac{x}{(x-x_1)} + \ln \frac{x}{(x-x_2)} + \ln \frac{x}{(x-x_3)} + \dots + \ln \frac{x}{(x-x_n)}$$

$$\text{now } D\left(\frac{x}{x-x_n}\right) = \frac{x-x_n}{x} \left(\frac{(x-x_n)-x}{(x-x_n)^2} \right) = \frac{1}{x} \left(\frac{x_n}{x_n-x} \right)$$

$$\text{Hence } \frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \dots + \frac{x_n}{x_n-x} \right]$$

$$\frac{dy}{dx} = \frac{y}{x} \left[\frac{x_1}{x_1-x} + \frac{x_2}{x_2-x} + \dots + \frac{x_n}{x_n-x} \right]$$

23 Suppose $f(x) = \tan(\sin^{-1}(2x))$

(a) Find the domain and range of f .

(b) Express $f(x)$ as an algebraic function of x .

(c) Find $f'(1/4)$. [Ans. (a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-\infty, \infty)$; (b) $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; (c) $\frac{16\sqrt{3}}{9}$]

[Sol. $f(x) = \tan(\sin^{-1}(2x))$

(a) for f to be well defined

$$-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2} \quad [\because \text{for } x = \pm \frac{1}{2}, \tan \frac{\pi}{2} \text{ is not defined}]$$

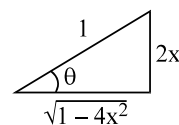
Hence domain is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\sin^{-1}2x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ hence for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ can take all real values.

Hence range of f is $x \in \mathbb{R}$

(b) $f(x) = \tan \theta$ where $\sin^{-1}(2x) = \theta \Rightarrow \sin \theta = 2x$

$$f(x) = \frac{2x}{\sqrt{1-4x^2}}$$



(c) $f'(x) = \frac{\sec^2(\sin^{-1}(2x))}{\sqrt{1-4x^2}} \cdot 2$

$$f'\left(\frac{1}{4}\right) = \frac{2\sec^2\left(\sin^{-1}\left(\frac{1}{2}\right)\right)}{\sqrt{1-\frac{1}{4}}} = \frac{2 \times 2 \cdot \frac{4}{3}}{\sqrt{\frac{3}{4}}} = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$$

24 If $x = \tan \frac{y}{2} - \ln \left[\frac{\left(1 + \tan \frac{y}{2}\right)^2}{\tan \frac{y}{2}} \right]$. Show that $\frac{dy}{dx} = \frac{1}{2} \sin y (1 + \sin y + \cos y)$.

Sol Put $\tan \frac{y}{2} = t \quad \therefore \quad \sin y = \frac{2t}{1+t^2}, \cos y = \frac{1-t^2}{1+t^2}$

$$\therefore 1 + \sin y + \cos y = \frac{2+2t}{1+t^2}$$

and $y = 2 \tan^{-1} t \quad \dots(1)$

$$\therefore \frac{dy}{dt} = \frac{2}{1+t^2} \quad \dots(2)$$

Now $x = t - 2 \log(1+t) + \log t$

$$\therefore \frac{dx}{dt} = 1 - \frac{2}{1+t} + \frac{1}{t} = \frac{t^2+1}{t(t+1)}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} + \frac{dx}{dt} = \frac{2}{1+t^2} \cdot \frac{t^2+t}{1+t^2}, \text{ by (2) \& (3)}$$

$$\begin{aligned} \text{or } \frac{dy}{dx} &= \frac{2t}{1+t^2} \cdot \frac{1}{2} \frac{2t+2}{1+t^2} \\ &= \frac{1}{2} \sin y (1 + \sin y + \cos y), \text{ by (1)} \end{aligned}$$

25 If $y = \arccos \sqrt{\frac{\cos 3x}{\cos^3 x}}$ then show that $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}, \sin x > 0$.

Sol We have,

$$y = \cos^{-1} \sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\therefore \cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\Rightarrow \cos y = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$$

$$\Rightarrow \cos y = \sqrt{4 - 3\sec^2 x}$$

$$\Rightarrow \cos^2 y = 4 - 3(1 + \tan^2 x)$$

$$\Rightarrow 1 - \cos^2 y = 3 \tan^2 x$$

$$\Rightarrow \sin^2 y = 3 \tan^2 x$$

$$\Rightarrow \sin y = \sqrt{3} \tan x$$

Differentiating both side with respect to x, we get, $\cos y \frac{dy}{dx} = \sqrt{3} \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{3}}{\cos y \cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{3}}{\cos^2 x} \sqrt{\frac{\cos^3 x}{\cos 3x}} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

Hence Proved $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}$, $\sin x > 0$.

26 $a_1 \sin x + a_2 \sin 2x + \dots + a_n \sin nx \leq |\sin x|$ for $x \in \mathbb{R}$,

then $|a_1 + 2a_2 + 3a_3 + \dots + na_n| \leq 1$

[Sol. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \dots + a_n \sin nx$

$$f'(x) = a_1 \cos x + 2a_2 \cos 2x + \dots + na_n \cos nx$$

$$f'(0) = a_1 + 2a_2 + \dots + na_n$$

Hence TPT $|f'(0)| \leq 1$

Given $|f(x)| \leq |\sin x|$ for $x \in \mathbb{R}$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h)}{h} \quad (\text{as } f(0) = 0)$$

$$|f'(0)| = \lim_{h \rightarrow 0} \left| \frac{f(h)}{h} \right| \leq \lim_{h \rightarrow 0} \left| \frac{\sin h}{h} \right| = 1 \quad [\text{as } |f(x)| \leq |\sin x|]$$

Hence $|f'(0)| \leq 1$]

27 Show that the substitution $z = \ln\left(\tan \frac{x}{2}\right)$ changes the equation $\frac{d^2 y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$ to $(d^2 y/dz^2) + 4y = 0$.

Sol Since $x = \ln \tan\left(\frac{x}{2}\right)$

$$\therefore \frac{dz}{dx} = \operatorname{cosec} x \quad \text{or} \quad \frac{dx}{dz} = \sin x \quad \dots(1)$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \operatorname{cosec} x \cdot \frac{dy}{dz} \quad [\text{From (1)}] \quad \dots(2)$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\operatorname{cosec} x \frac{dy}{dz} \right)$$

$$\begin{aligned}
&= \operatorname{cosec} x \frac{d}{dx} \left(\frac{dy}{dz} \right) + \frac{dy}{dz} (-\operatorname{cosec} x \cot x) \\
&= \operatorname{cosec} x \cdot \frac{d}{dz} \left(\frac{dy}{dz} \right) \cdot \frac{dz}{dx} - \operatorname{cosec} x \cot x \frac{dy}{dz} \\
&= \operatorname{cosec}^2 x \frac{d^2 y}{dz^2} - \operatorname{cosec} x \cot x \frac{dy}{dz} \quad [\text{From (1)}] \quad \dots(3)
\end{aligned}$$

But given $\frac{d^2 y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$

$$\operatorname{cosec}^2 x \frac{d^2 y}{dz^2} - \operatorname{cosec} x \cot x \frac{dy}{dz} + \cot x \operatorname{cosec} x \frac{dy}{dz} + 4y \operatorname{cosec}^2 x = 0 \quad [\text{From (2) and (3)}]$$

$$\Rightarrow \operatorname{cosec}^2 x \frac{d^2 y}{dz^2} + 4y \operatorname{cosec}^2 x = 0 \quad \text{or} \quad \frac{d^2 y}{dz^2} + 4y = 0$$

28 Let $f(x) = \begin{cases} xe^x & x \leq 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then prove that

(a) f is continuous and differentiable for all x . (b) f' is continuous and differentiable for all x .

[Sol. $f'(x) = \begin{cases} xe^x + e^x = e^x(x+1), & x < 0 \\ 1 + 2x - 3x^2 & x > 0 \end{cases}$

$$\lim_{x \rightarrow 0^-} f'(x) = 1; \quad \lim_{x \rightarrow 0^+} f'(x) = 1$$

hence $f(x)$ is continuous hence f is continuous and differentiable at $x = 0$

$$\text{Again } f''(x) = \begin{cases} e^x + (x+1)e^x = e^x(x+2), & x < 0 \\ 2 - 6x & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f''(x) = \lim_{x \rightarrow 0^-} f''(x) = 2 \quad \Rightarrow \quad f'(x) \text{ is also continuous and differentiable]}$$

29 Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x) = 0$ and that $f(x) = f(0) + kx$ where k denotes the sum

of all the co-factors of the elements in $f(0)$.

[Hint: $f'(x) = \begin{vmatrix} 1 & 1 & 1 \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ 1 & 1 & 1 \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ 1 & 1 & 1 \end{vmatrix}$

$$f''(x) = 0 \text{ (obviously - two identical rows)}$$

$$f'(x) = k \Rightarrow f(x) = kx + x, \quad f(0) = c$$

$$\Rightarrow f(x) = f(0) + kx. \text{ Note that } f'(x) = k$$

$$\begin{aligned}
\Rightarrow f'(0) = k &= \begin{vmatrix} 1 & 1 & 1 \\ \ell & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ \ell & m & n \\ 1 & 1 & 1 \end{vmatrix} \\
&= (c_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23}) + (c_{31} + c_{32} + c_{33}) \\
&= \text{sum of co-factors of elements } f(0)]
\end{aligned}$$

30 If $Y = sX$ and $Z = tX$, where all the letters denotes the functions of x and suffixes denotes the differentiation

w.r.t. x then prove that
$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

Sol Since $Y = sX$ and $Z = tX$... (1)

$\therefore Y_1 = sX_1 + Xs_1$ and $Z_1 = tX_1 + Xt_1$... (2)

$\Rightarrow Y_2 = sX_2 + Xs_2 + 2s_1X_1$ and $Z_2 = tX_2 + Xt_2 + 2t_1X_1$... (3)

L.H.S =
$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$\begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + Xs_1 & tX_1 + Xt_1 \\ X_2 & sX_2 + Xs_2 + 2s_1X_1 & tX_2 + Xt_2 + 2t_1X_1 \end{vmatrix}$$
 [From (1),(2) and (3)]

Applying $C_2 \rightarrow C_2 - sC_1$ and $C_3 \rightarrow C_3 - tC_1$

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

Expand w.r.t. first row, then

$$= X \begin{vmatrix} Xs_1 & Xt_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X^3 \begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - 2X_1R_1 = X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} = R.H.S.$

28 A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = \lim_{n \rightarrow \infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c \cdot e^{nx}}$ where f is continuous on \mathbb{R} . Find the value of a , b and c .

Sol. $f(x) = \lim_{n \rightarrow \infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c \cdot e^{nx}}$

$$= \begin{cases} \lim_{n \rightarrow \infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c \cdot e^{nx}} & ; x < 0 \\ \lim_{n \rightarrow \infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c \cdot e^{nx}} & ; x = 0 \\ \lim_{n \rightarrow \infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c \cdot e^{nx}} & ; x > 0 \end{cases}$$

$$= \begin{cases} \frac{ax^2 + bx + c + 0}{1 + c \cdot 0} & ; \quad x < 0 \left(\lim_{n \rightarrow \infty} e^{nx} = 0 \right) \\ \frac{c+1}{c+1} & ; \quad x = 0 \\ \lim_{n \rightarrow \infty} \frac{\frac{ax^2}{e^{nx}} + \frac{bx}{e^{nx}} + \frac{c}{e^{nx}} + 1}{\frac{1+c}{e^{nx}}} & ; \quad x > 0 \end{cases}$$

$$\left(\lim_{h \rightarrow \infty} e^{hx} = \infty \right)$$

$$= \begin{cases} ax^2 + bx + c & ; \quad x < 0 \\ 1 & ; \quad x = 0 \\ \frac{1}{c} & ; \quad x > 0 \end{cases}$$

since $f(x)$ is continuous function $\forall x \in \mathbb{R}$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \left(\frac{1}{c} \right) = \lim_{x \rightarrow 0^-} (ax^2 + bx + c) = 1 \quad \Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{c} = 1 \quad \& \quad \lim_{x \rightarrow 0^-} (ax^2 + bx + c) = 1$$

$$\Rightarrow \frac{1}{c} = 1 \quad \Rightarrow a - 0 + 3 \cdot 0 + c = 1$$

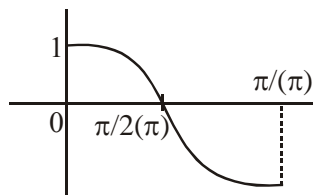
$$\therefore c = 1 \quad \Rightarrow c = 1$$

$$\therefore c = 1, a, b \in \mathbb{R}$$

- 29 Discuss the continuity of f in $[0, 2]$ where $f(x) = \begin{cases} 4x - 5 & [x] \text{ for } x > 1 \\ \cos \pi x & \text{for } x \leq 1 \end{cases}$; where $[x]$ is the greatest integer not greater than x .

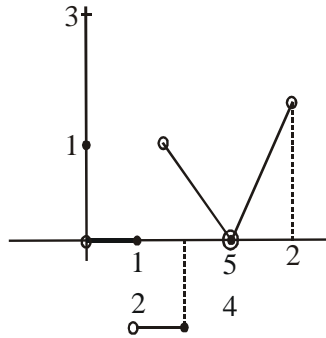
Sol. $f(x) = \cos \pi x$

$$[\cos \pi x] = \begin{cases} 1 & ; \quad x = 0 \\ 0 & ; \quad x < x \leq \frac{1}{2} \\ -1 & ; \quad \frac{1}{2} < x \leq 1 \end{cases}$$



$$|4x-5| [x] = \begin{cases} |4x-5|; & 1 < x < 2 \\ 6 & ; \quad x=2 \end{cases} = \begin{cases} (4x-5) & ; \quad 1 < x < \frac{5}{4} \\ 4x-5 & ; \quad \frac{5}{4} \leq x < 2 \\ 6 & ; \quad x=2 \end{cases}$$

$$f(x) = \begin{cases} 1 & ; \quad x=0 \\ 0 & ; \quad 0 < x \leq \frac{1}{2} \\ -1 & ; \quad \frac{1}{2} < x \leq 1 \\ -(4x-5) & ; \quad 1 < x < \frac{5}{4} \\ 4x-5 & ; \quad \frac{5}{4} \leq x < 2 \\ 6 & ; \quad x=2 \end{cases}$$



function dis at $0, \frac{1}{2}, 1, 2$

30 If $f(x) = x + \{-x\} + [x]$, where $[x]$ is the integral part & $\{x\}$ is the fractional part of x . Discuss the continuity of f in $[-2, 2]$.

Sol. $f(x) = x + \{-x\} + [x]$

$$\because \{x\} = x - [x]$$

$$\{-x\} = -x - [-x]$$

$$f(x) = x + (-x - [-x]) + [x]$$

$$f(x) = [x] - [-x] \begin{cases} x - (-x) = 2x; & x \in I \\ [x] - (-[x] - 1) = 1 - 2[x]; & x \notin I \end{cases}$$

$$f(x) = \begin{cases} 2x & ; \quad x \in I \\ 1 - 2[x] & ; \quad x \notin I \end{cases}$$

$$f(x) = \begin{cases} -4 & ; & x = -2 \\ 5 & ; & -2 < x < -1 \\ -2 & ; & x = -1 \\ 3 & ; & -1 < x < 0 \\ 0 & ; & x = 0 \\ 1 & ; & 0 < x < 1 \\ 2 & ; & x = 1 \\ -1 & ; & 1 < x < 2 \\ 4 & ; & x = 2 \end{cases}$$

so the function is discontinuous at all integers in $[-2, 2]$.

- 31 Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax - b & \text{for } x \leq 1 \\ 3x & \text{for } 1 < x < 2 \\ bx^2 - a & \text{for } x \geq 2 \end{cases}$ is continuous at $x = 1$ but discontinuous at $x = 2$.

Sol. conti at $x = 1$

$$a - b = 3 \quad \dots(1)$$

dis at $x = 2$

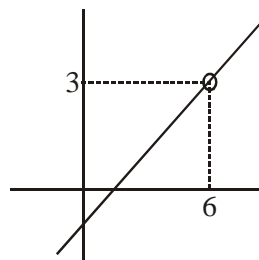
$$6 \neq 4b - a$$

$$6 \neq 4b - 3 - b$$

$$6 \neq 3b - 3$$

$$\boxed{b \neq 3}$$

$$\boxed{a \neq 6}$$



$$(a, b) \neq (6, 3)$$

$$(x, y) \neq (6, 3) \quad \text{Ans}$$

- 32 $f(x) = \frac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for $x > 0$
 $= \frac{\ln(1+x+x^2) + \ln(1-x+x^2)}{\sec x - \cos x}$ for $x < 0$, if f is continuous at $x = 0$, find 'a'

now if $g(x) = \ln\left(2 - \frac{x}{a}\right) \cdot \cot(x - a)$ for $x \neq a$, $a \neq 0$, $a > 0$. If g is continuous at $x = a$ then show that $g(e^{-1}) = -e$.

Sol. Since the function is conti at $x = 0$ then

$$\text{V.F.}|_{x=0} = \text{RHL}|_{x=0} = \text{LHL}|_{x=0}$$

since the function is conti then

$$\text{RHL}|_{x=0} = \lim_{x \rightarrow 0^+} f(x)$$

$$f(0) = \text{LHL}|_{x=0} = \text{RHL}|_{x=0}$$

$$= \lim_{x \rightarrow 0^+} \frac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$$

$$- \ln a = 1$$

$$= \lim_{x \rightarrow 0^+} \frac{a^{\tan x} (a^{\sin x - \tan x} - 1)}{-1(\sin x - \tan x)}$$

$$\boxed{a = \frac{1}{e}}$$

since $g(x)$ conti at $x = a$

$$g(a) = \lim_{x \rightarrow a} g(x)$$

$$= \lim_{x \rightarrow a} \ln \left(2 - \frac{x}{a} \right) \cot(x - a)$$

$$= \lim_{x \rightarrow a} \frac{\ln \left(2 - \frac{x}{a} \right)}{\tan(x - a)}$$

$$\boxed{\text{RHL}|_{x=0} = -\ln a}$$

$$\text{LHL}|_{x=0} = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{x \rightarrow 0^-} \frac{\ln(1+x+x^2) + \ln(1-x+x^2)}{\sec x - \cos x}$$

$$= \lim_{x \rightarrow 0^-} \frac{\ln((1+x+x^2)(1-x+x^2)) \cdot \cos x}{1 - \cos 2x}$$

put $x = a + h$

$$= \lim_{h \rightarrow 0} \frac{\ln \left(1 - \frac{h}{a} \right)}{\left(-\frac{h}{a} \right)} \cdot \frac{h}{\tan \left(-\frac{1}{a} \right)}$$

put $x = 0 - h$

$$= \lim_{h \rightarrow 0} \frac{\ln(1+h^2+h^4) \cosh}{\sin^2 h}$$

$$= \lim_{h \rightarrow 0} (h^2 + h^4) \frac{\cosh}{\sin^2 h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{h}{\sinh} \right)^2 (1+h^2) \cosh$$

$$g(a) = -\frac{1}{a}$$

$$\boxed{g\left(\frac{1}{e}\right) = -e}$$

33 Find the value of $\lim_{x \rightarrow 0^+} x^{(x^x-1)}$.

[Ans. 1]

[Sol. $l = \lim_{x \rightarrow 0^+} x^{(x^x-1)}$ (0^0 form)]

$$\ln l = \lim_{x \rightarrow 0} (x^x - 1) \cdot \ln x = \lim_{x \rightarrow 0} \frac{(e^{x \ln x} - 1)}{x \ln x} \lim_{x \rightarrow 0} x \ln x \cdot \ln x$$

$$= \lim_{x \rightarrow 0} x (\ln x)^2 \quad (\text{as } x \rightarrow 0 \text{ } x \ln x \rightarrow 0)$$

$$= \lim_{x \rightarrow 0} \frac{(\ln x)^2}{1/x} = \lim_{x \rightarrow 0} -\frac{2 \ln x}{x} \cdot x^2 \quad (\text{use Lopital's rule})$$

$$= \lim_{x \rightarrow 0} -2 \ln x \cdot x = 0 \quad \Rightarrow \quad l = e^0 = 1$$

34 $\dots \dots (x) = \sum_{r=1}^n \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right) ; r, n \in \mathbb{N}$

$$g(x) = \lim_{n \rightarrow \infty} \frac{\ell n \left(f(x) + \tan \frac{x}{2^n} \right) - \left(f(x) + \tan \frac{x}{2^n} \right)^n \cdot \left[\sin \left(\tan \frac{x}{2} \right) \right]}{1 + \left(f(x) + \tan \frac{x}{2^n} \right)^n}$$

= k for $x = \frac{\pi}{4}$ and the domain of $g(x)$ is $(0, \pi/2)$.

where $[]$ denotes the greatest integer function.

Find the value of k, if possible, so that $g(x)$ is continuous at $x = \pi/4$. Also state the points of discontinuity of $g(x)$ in $(0, \pi/4)$, if any.

Sol. $\tan \frac{x}{2} \sec x = \frac{\sin x / 2}{\cos \frac{x}{2} \cdot \cos x} = \frac{\sin \left(x - \frac{x}{2} \right)}{\cos \frac{x}{2} \cdot \cos x} = \frac{\sin x \cos \frac{x}{2} - \cos x \sin \frac{x}{2}}{\cos \frac{x}{2} \cdot \cos x} = \tan x - \tan \frac{x}{2}$

$$\tan \frac{x}{2} \sec x = \tan x - \tan \frac{x}{2}$$

$$\tan \frac{x}{2^2} \cdot \sec \frac{x}{2} = \tan \frac{x}{2} - \tan \frac{x}{2^2}$$

$$\tan \frac{x}{2^3} \cdot \sec \frac{x}{2^2} = \tan \frac{x}{2^2} - \tan \frac{x}{2^3}$$

•
•
•

$$\tan \frac{x}{2^n} \cdot \sec \frac{x}{2^{n-1}} = \tan \frac{x}{2^{n-1}} - \tan \frac{x}{2^n}$$

$$f(x) = \tan x - \tan \left(\frac{x}{2^n} \right)$$

$$f(x) + \tan \left(\frac{x}{2^n} \right) = \tan x \quad \dots(1)$$

using (1)

$$g(x) = \begin{cases} \lim_{n \rightarrow \infty} \frac{\ell n(\tan x) - (\tan x)^n \left[\sin \left(\tan \frac{x}{2} \right) \right]}{1 + (\tan x)^n} & ; x \neq \frac{\pi}{4} \\ k & ; x = \frac{\pi}{4} \end{cases}$$

$$g(x) = \begin{cases} \lim_{h \rightarrow \infty} \frac{\ln(\tan x)}{1 + (\tan x)^h} & ; x \neq \frac{\pi}{4} \\ k & ; x = \frac{\pi}{4} \end{cases}$$

$$k = 0$$

$$\lim_{n \rightarrow \infty} x^n = \begin{cases} 0 & ; x < 1 \\ 1 & ; x = 1 \\ \infty & ; x > 1 \end{cases}$$

$$\lim_{n \rightarrow \infty} (\tan x)^n = \begin{cases} 0 & ; x < \frac{\pi}{4} \\ 1 & ; x = \frac{\pi}{4} \\ \infty & ; x > \frac{\pi}{4} \end{cases}$$

35 Let f be continuous on the interval $[0, 1]$ to \mathbb{R} such that $f(0) = f(1)$. Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$

such that $f(c) = f\left(c + \frac{1}{2}\right)$

Sol. Consider a conti function

$$g(x) = f\left(x + \frac{1}{2}\right) - f(x); g \text{ is conti } \forall x \in \left[0, \frac{1}{2}\right]$$

Now

$$g(0) = f\left(\frac{1}{2}\right) - f(0) \Rightarrow g(0) = f\left(\frac{1}{2}\right) - f(1)$$

$$g\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right) \Rightarrow g\left(\frac{1}{2}\right) = f(1) - f\left(\frac{1}{2}\right)$$

since g is continuous and $g(0)$ and $g\left(\frac{1}{2}\right)$ are of opposite sign hence the equation $g(x) = 0$ must have at least

one root in $\left[0, \frac{1}{2}\right]$.

\therefore for some $c \in \left[0, \frac{1}{2}\right]; g(c) = 0$

$$\Rightarrow f\left(c + \frac{1}{2}\right) = f(c)$$

36 Consider the function $g(x) = \begin{cases} \frac{1 - a^x + xa^x \ln a}{a^x x^2} & ; x < 0 \\ \frac{2^x a^x - x \ln 2 - x \ln a - 1}{x^2} & ; x > 0 \end{cases}$

where $a > 0$, find the value of 'a' & 'g(0)' so that the function $g(x)$ is continuous at $x = 0$.

Sol. $\text{LHL}|_{x=0} = \lim_{x \rightarrow 0^-} g(x)$

$$\boxed{\text{RHL}|_{x=0} = \frac{(\ln 2a)^2}{2}}$$

$$= \lim_{x \rightarrow 0^-} \left(\frac{1 - a^x + xa^x \ln a}{a^x x^2} \right)$$

since the function is conti

put $x = a - h$

$$g(0) = \text{LHL}|_{x=0} = \text{RHL}|_{x=0}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1 - a^{-h} - ha^{-h} \ln a}{a^{-h} h^2} \right)$$

$$\frac{(\ln(2a))^2}{2} = \frac{(\ln a)^2}{2}$$

$$= \lim_{x \rightarrow 0} \left(\frac{a^h - 1 - h \ln a}{2h} \right); \frac{0}{0} \text{ form}$$

$$(\ln 2a + \ln a)(\ln 2a - \ln a) = 0$$

$$= \lim_{h \rightarrow 0} \left(\frac{a^h \ln a - 0 - \ln a}{2h} \right); \frac{0}{0} \text{ Ans}$$

$$\ln(2a^2) \cdot \ln 2 = 0$$

$$= \lim_{h \rightarrow 0} \left(\frac{a^h (\ln a)^2}{2} \right)$$

$$\ln 2a^2 = 0$$

$$\boxed{\text{LHL}|_{x=0} = \frac{(\ln a)^2}{2}}$$

$$2a^2 = 1, a = \pm \frac{1}{\sqrt{2}} ; a > 0$$

$$\text{RHL}|_{x=0} = \lim_{x \rightarrow 0^+} g(x)$$

$$\boxed{a = \frac{1}{\sqrt{2}}}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{2^x a^x - x \ln 2 - x \ln a - 1}{x^2} \right)$$

$$\therefore g(0) = \frac{(\ln 2a)^2}{2}$$

put $x = 0 + h$

$$= \frac{1}{2} \left(\ln 2 \cdot \frac{1}{\sqrt{2}} \right)^2$$

$$= \lim_{h \rightarrow 0} \left(\frac{(2a)^h - h \ln 2 - h \ln a - 1}{h^2} \right); \frac{0}{0} \text{ form}$$

$$= \frac{1}{2} (\ln \sqrt{2})^2$$

$$= \lim_{h \rightarrow 0} \frac{(2a)^h \ln 2a - \ln 2a}{2h}; \frac{0}{0} \text{ form} = \frac{1}{2} \left(\frac{1}{4} (\ln 2)^2 \right)$$

$$= \lim_{h \rightarrow 0} \frac{(2a)^h (\ln 2a)^2}{2} = \frac{1}{8} (\ln 2)^2$$

37 A function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation $f(x+y) = f(x) \cdot f(y)$ for all x, y in \mathbb{R} and $f(x) \neq 0$ for any x in \mathbb{R} . Let the function be differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in \mathbb{R} . Hence determine $f(x)$.

Sol Given that $f(x+y) = f(x) \cdot f(y)$ for all $x \in \mathbb{R}$... (1)

Putting $x = y = 0$ in (1), we get

$$f(0)\{f(0) - 1\} = 0 \quad \Rightarrow \quad f(0) = 0 \text{ or } f(0) = 1$$

If $f(0) = 0$, then $f(x) = f(x+0) = f(x) \cdot f(0) = 0$ for all $x \in \mathbb{R}$

Which is not true (given $f(x) \neq 0$)

So, $f(0) = 1$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h - 0} && (\because f(0) = 1) \\ &= f(x) f'(0) = 2f(x) && (\because f'(0) = 2) \end{aligned}$$

$$\Rightarrow \frac{f'(x)}{f(x)} = 2$$

Integrating both sides w.r.t. x and taking limit 0 to x

$$\int_0^x \frac{f'(x)}{f(x)} dx = \int_0^x 2 dx$$

$$\Rightarrow \ln f(x) - \ln f(0) = 2x \quad \Rightarrow \quad \ln f(x) - \ln 1 = 2x$$

$$\Rightarrow \ln f(x) - 0 = 2x \quad \therefore \quad f(x) = e^{2x}.$$

38 Let f be a function such that $f(x+f(y)) = f(f(x)) + f(y) \quad \forall x, y \in \mathbb{R}$ and $f(h) = h$ for $0 < h < \varepsilon$ where $\varepsilon > 0$, then determine $f'(x)$ and $f(x)$.

Sol Given $f(x+f(y)) = f(f(x)) + f(y)$ (1)

Putting $x = y = 0$ in (1), then

$$f(0 + f(0)) = f(f(0)) + f(0) \quad \Rightarrow \quad f(f(0)) = f(f(0)) + f(0)$$

$$\therefore f(0) = 0 \quad \dots(2)$$

$$\text{Now } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (\text{for } 0 < h < \varepsilon)$$

$$= \lim_{h \rightarrow 0} \frac{f(h+x) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(f(h))}{h} \quad (\text{form (1)})$$

$$= \lim_{h \rightarrow 0} \frac{f(h)}{h} \quad (\because f(h) = h)$$

$$= \lim_{h \rightarrow 0} \frac{h}{h} = 1 \quad (\because f(h) = h)$$

Integrating both sides with limites 0 to x then $f(x) = x$

$$\therefore f'(x) = 1.$$

- 39 Let $f(x) = \begin{cases} -2 & , \quad -3 \leq x \leq 0 \\ x-2 & , \quad 0 < x \leq 3 \end{cases}$, where $g(x) = f(|x|) + |f(x)|$. Test the differentiability of $g(x)$ in the interval $(-3, 3)$.

Sol From the given function

$$f(|x|) = \begin{cases} -x-2 & \text{for } -3 \leq x \leq 0 \\ x-2 & \text{for } 0 < x \leq 3 \end{cases} \quad \text{and} \quad |f(x)| = \begin{cases} 2 & \text{for } -3 \leq x \leq 0 \\ -x+2 & \text{for } 0 < x \leq 2 \\ x-2 & \text{for } 2 < x \leq 3 \end{cases}$$

$$\therefore g(x) = f(|x|) + |f(x)|$$

$$= \begin{cases} -x & \text{for } -3 \leq x \leq 0 \\ 0 & \text{for } 0 < x \leq 2 \\ 2x-4 & \text{for } 2 < x \leq 3 \end{cases}$$

Check the differentiability

At

$$x = 0 : \quad \text{Lg}'(0) = \lim_{h \rightarrow 0} \frac{g(0-h) - g(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-(0-h) - 0}{-h} = -1$$

$$\text{Rg}'(0) = \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(0-0)}{h} = 0$$

$$\therefore Lg'(0) \neq Rg'(0)$$

$\therefore g(x)$ is not differentiable at $x = 0$

Check at

$$\begin{aligned} x = 2 : \quad Lg'(2) &= \lim_{h \rightarrow 0} \frac{g(2-h) - g(2)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{0 - 0}{-h} = 0 \end{aligned}$$

$$\begin{aligned} \text{and} \quad Rg'(2) &= \lim_{h \rightarrow 0} \frac{g(2+h) - g(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(2+h) - 4 - 0}{h} = 2 \end{aligned}$$

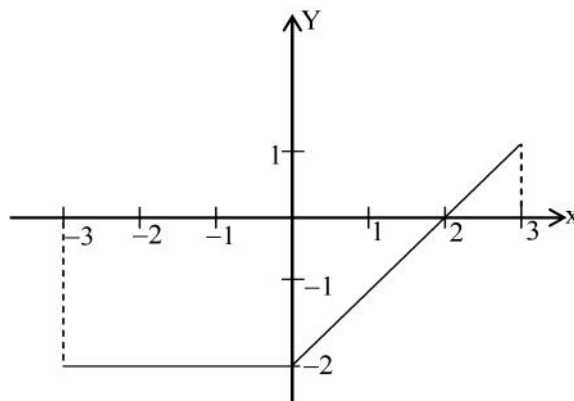
$$\therefore Lg'(2) \neq Rg'(2)$$

Hence $g(x)$ is not differentiable at $x = 2$.

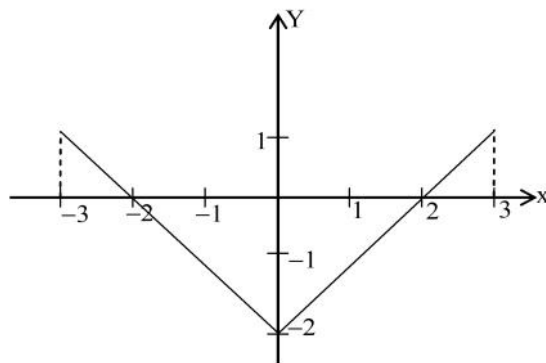
Graphical method :

$$\therefore f(x) = \begin{cases} -2 & ; -3 \leq x \leq 0 \\ x-2 & ; 0 < x \leq 3 \end{cases}$$

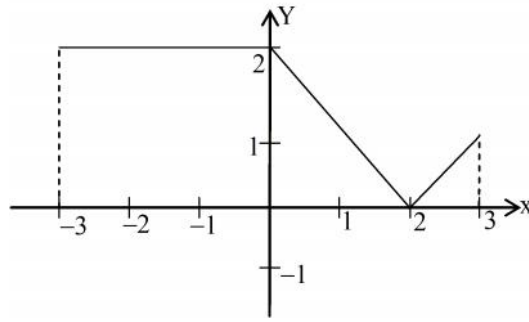
Graph of $f(x)$:



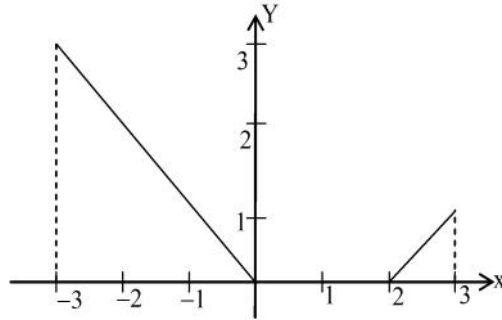
Graph of $f(|x|)$:



Graph of $|f(x)|$:



Graph of $g(x) = |f(x)| + f(|x|)$:



It is clear from the graph that $g(x)$ is not differentiable at $x = 0$ and 2 .

40 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ is a real valued function $\forall x, y \in \mathbb{R}$ such that $|f(x) - f(y)| \leq |x - y|^3$.

Prove that $h(x) = \int f(x) dx$ is continuous function of $x \forall x \in \mathbb{R}$.

Sol Since $|f(x) - f(y)| \leq |x - y|^3 \quad x \neq y$

$$\therefore \left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|^2$$

Taking lim as $y \rightarrow x$, we get

$$\lim_{y \rightarrow x} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{y \rightarrow x} |x - y|^2$$

$$\Rightarrow \left| \lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y} \right| \leq \left| \lim_{y \rightarrow x} (x - y)^2 \right|$$

$$\Rightarrow |f'(x)| \leq 0 \quad \Rightarrow \quad |f'(x)| = 0 \quad (\because |f'(x)| \geq 0)$$

$$\therefore f'(x) = 0 \quad \Rightarrow \quad f(x) = c \text{ (constant)}$$

$$\therefore h(x) = \int f(x) dx = \int c dx = cx + d \quad \text{where } d \text{ is constant of integration.}$$

$$\therefore h(x) \text{ is a linear function of } x \text{ which is continuous for all } x \in \mathbb{R}.$$

41 Let $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$ for all real x and y . If $f'(0)$ exists and equals -1 and $f(0) = 1$,

then find $f'(2)$.

Sol Since $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ (1)

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f\left(\frac{2x+2h}{2}\right) - f\left(\frac{2x+0}{2}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(2x)+f(2h)}{2} - \frac{f(2x)+f(0)}{2}}{h} \quad [\text{from (1)}] \\ &= \lim_{h \rightarrow 0} \frac{f(2h) - f(0)}{2h - 0} \\ &= f'(0) \\ &= -1 \quad \forall x \in \mathbb{R} \quad (\text{given}) \end{aligned}$$

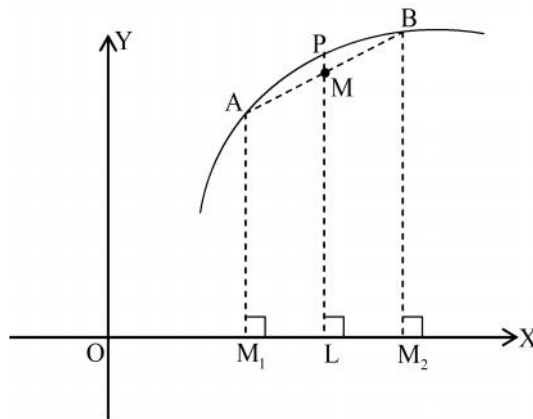
Integrating, we get $f(x) = -x + c$

Putting $x = 0$, then $f(0) = 0 + c = 1$ (given)

$\therefore c = 1$ then $f(x) = 1 - x$ $\therefore f(2) = 1 - 2 = -1$

Graphical method :

Suppose $A(x, f(x))$ and $B(y, f(y))$ be any two points on the curve $y = f(x)$.



If M is the mid-point of AB then co-ordinates of M are $\left(\frac{x+y}{2}, \frac{f(x)+f(y)}{2}\right)$

According to the graph, co-ordinates of P are $\left(\frac{x+y}{2}, f\left(\frac{x+y}{2}\right)\right)$ and $PL > ML$

$$\Rightarrow f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2}$$

But given $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ which is possible when $P \rightarrow M$

i.e. P lies on AB. Hence $y = f(x)$ must be a linear function.

$$\text{Let } f(x) = ax + b \quad \Rightarrow \quad f(0) = 0 + b = 1 \quad (\text{given})$$

$$\text{and } f'(x) = a \quad \Rightarrow \quad f'(0) = a = -1 \quad (\text{given})$$

$$\therefore f(x) = -x + 1 \quad \therefore f(2) = -2 + 1 = -1.$$

42 Let $f\left(\frac{x+y}{n}\right) = \frac{f(x)+f(y)}{n} \quad \forall x, y \in \mathbb{R}; n \neq 0, 2$ and if $f'(0) = k$ (A finite quantity) then prove that $f(x) = kx \quad \forall x \in \mathbb{R}$.

$$\text{Sol Given } f\left(\frac{x+y}{n}\right) = \frac{f(x)+f(y)}{n} \quad \dots(1)$$

Putting $x = y = 0$, we get $(n-2)f(0) = 0$

$$\therefore f(0) = 0 \quad (\because n-2 \neq 0)$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f\left(\frac{nx+nh}{n}\right) - f\left(\frac{nx+0}{n}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(nx) + f(nh)}{n} - \frac{f(nx) + f(0)}{n}}{h} \quad [\text{from (1)}] \\ &= \lim_{h \rightarrow 0} \frac{f(nh) - f(0)}{nh - 0} \end{aligned}$$

$$\Rightarrow f'(x) = k$$

On integrating we get $f(x) = kx + c$

$$\text{Putting } x = 0, \text{ then } f(0) = 0 + c = 0 \quad (\because f(0) = 0)$$

$$\therefore c = 0 \text{ then } f(x) = kx.$$

43 If $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$ then determine $y = f(x)$.

$$\text{Sol } \therefore f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3} \quad \dots(1)$$

Differentiating both sides w.r.t. x treating y as constant,

$$\text{then } f'\left(\frac{x+y}{3}\right) \left(\frac{1}{3}\right) = \frac{2+f'(x)+0}{3}$$

Now replacing x by 0 and y by $3x$, then

$$f'(x) = f'(0) = c \quad (\text{say})$$

$$\text{At } x = 2, \quad f'(2) = c = 2 \quad (\text{given})$$

$$\therefore f'(x) = 2$$

On integrating we get $f(x) = 2x + d$

Putting $x = 0$, then $f(0) = 0 + d = 2$ [from (1)]

$$\therefore f(x) = 2x + 2$$

Hence $y = 2x + 2$.

44 If $f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \quad \forall x, y \in \mathbb{R}$ and $f'(0) = 1$; prove that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Sol $\therefore f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3}$

Differentiating both sides w.r.t. x treating y as constant

$$f'\left(\frac{x+2y}{3}\right) \cdot \frac{1}{3} = \frac{f'(x)+0}{3}$$

and replacing x by 0 and y by $\frac{3x}{2}$

then $f'(x) = f'(0) = 1$ (given)

On integrating, we get

$f(x) = x + d$, d is constant of integration which is linear function in x and hence it is always continuous function for all x .

45 If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in \mathbb{R}$ and $xy \neq 1$ and $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 2$, find $f(\sqrt{3})$ and $f'(-2)$.

Sol Given $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Putting $x = 0, y = 0$, we get $f(0) = 0$... (1)

And putting $y = -x$, we get $f(x) + f(-x) = f(0) = 0$

$\therefore f(x) = -f(-x)$... (2)

Now $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) + f(-x)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{f\left(\frac{h}{1+x(x+h)}\right)}{\frac{h}{1+x(x+h)}} \cdot \frac{1}{1+x(x+h)}$$

$$= 2 \cdot \frac{1}{1+x^2} \quad \left(\because \lim_{x \rightarrow 0} \frac{f(x)}{x} = 2 \right)$$

$$= \frac{2}{1+x^2}$$

$$\therefore f(x) = 2 \tan^{-1} x + c \quad \text{or} \quad f(0) = 2 \tan^{-1} 0 + c = 0$$

$$\Rightarrow 0 = 0 + c \quad \therefore c = 0$$

then $f(x) = 2 \tan^{-1} x$

$$\therefore f(\sqrt{3}) = 2 \tan^{-1}(\sqrt{3}) = \frac{2\pi}{3} \quad \text{and} \quad f'(-2) = \frac{2}{1+(-2)^2} = \frac{2}{5}$$

46 Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in \mathbb{R}$. If $f(x)$ is differentiable and $f'(0) = \sin \phi$ then prove that $f(x) > 0 \quad \forall x \in \mathbb{R}$.

Sol Given $f(x+y) = f(x) + f(y) + 2xy - 1 \quad \forall x, y \in \mathbb{R}$... (1)

Putting $x = y = 0$ in (1), we get

$$f(0) = 1 \quad \dots (2)$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + 2xh - 1 - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h) + 2xh - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} + \lim_{h \rightarrow 0} \left(\frac{2xh}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} + \lim_{h \rightarrow 0} (2x) \\ &= f'(0) + 2x \\ &= \sin \phi + 2x \quad (\because f(0) = \sin \phi) \end{aligned}$$

Integrating both sides w.r.t. x and taking limit 0 to x , then

$$\int_0^x f'(x) dx = \int_0^x (\sin \phi + 2x) dx$$

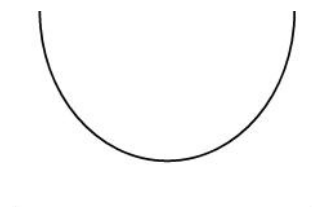
$$\Rightarrow f(x) - f(0) = x \sin \phi + x^2$$

$$\Rightarrow f(x) = x^2 + x \sin \phi + 1 \quad (\because f(0) = 1)$$

Here coefficient of x^2 is $1 > 0$ and Discriminant

$$D = \sin^2 \phi - 4 < 0.$$

Hence it is clear from graph $f(x) > 0 \quad \forall x \in \mathbb{R}$.



47 Let f be a one-one function such that $f(x)f(y) + 2 = f(x) + f(y) + f(xy) \quad \forall x, y \in \mathbb{R} \setminus \{0\}$ and $f(0) = 1, f'(1) = 2$ then prove that $3 \int f(x) dx - x(f(x) + 2)$ is constant.

Sol We have $f(x)f(y) + 2 = f(x) + f(y) + f(xy) \quad \dots(1)$

Putting $x = 1$ and $y = 1$, we get

$$(f(1))^2 + 2 = 3f(1)$$

$$\therefore f(1) = 1, 2 \quad \Rightarrow \quad f(1) = 2 \quad \dots(2)$$

$$f(1) \neq 1 \quad (\because f(0) = 1 \text{ and } f \text{ is one-one function})$$

In (1), replacing y by $\frac{1}{x}$

$$\therefore f(x)f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1)$$

$$\Rightarrow f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \quad (\because f(1) = 2)$$

$$\therefore f(x) = 1 \pm x^n \quad (x \in \mathbb{N})$$

$$\Rightarrow f'(x) = \pm nx^{n-1} \quad \Rightarrow \quad f'(1) = \pm n = 2$$

Taking positive sign $\Rightarrow n = 2$ then $f(x) = 1 + x^2$

Now, $3 \int f(x) dx - x(f(x) + 2)$

$$= 3 \int (1 + x^2) dx - x(1 + x^2 + 2)$$

$$= 3 \left(x + \frac{x^3}{3} \right) + c - 3x - x^3$$

$$= c = \text{constant.}$$

48 If $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y) \quad \forall x, y \in \mathbb{R}^+$, and $f'(1) = e$, determine $f(x)$.

Sol Given $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y) \quad \dots(1)$

Putting $x = y = 1$ in (1) we get $f(1) = 0 \quad \dots(2)$

$$\text{Now, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x \cdot 1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{x+h} \cdot \left\{ e^{-x}f(x) + e^{-1-\frac{h}{x}}f\left(1 + \frac{h}{x}\right) \right\} - e^x (e^{-x}f(x) + e^{-1}f(1))}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{e^h f(x) + e^{x+h-1} \frac{h}{x} f\left(1 + \frac{h}{x}\right) - f(x) - e^{x-1} f(1)}{h} \\
&= f(x) \lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) + e^{(x-1)} \lim_{h \rightarrow 0} \frac{e^{\frac{h}{x}} f\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}} \quad (\because f(1) = 0) \\
&= f(x) \cdot 1 + e^{x-1} \cdot \frac{f'(1)}{x} \\
&= f(x) + \frac{e^{x-1} \cdot e}{x} \quad (\because f'(1) = e)
\end{aligned}$$

$$f'(x) = f(x) + \frac{e^x}{x} \quad \Rightarrow \quad e^{-x} f'(x) - e^{-x} f(x) = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx} (e^{-x} f(x)) = \frac{1}{x}$$

On integrating we have $e^{-x} f(x) = \ln x + c$ at $x = 1, c = 0$

$$\therefore f(x) = e^x \ln x.$$

49 Let $f: \mathbb{R} \rightarrow \mathbb{R}$, such that $f'(0) = 1$

and $f(x+y) = f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy \quad \forall x, y \in \mathbb{R}$ then determine $f(x)$.

Sol Given $f(x+y) = f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy \quad \dots(1)$

Putting $x = y = 0$, we get $f(0) = 0 \quad \dots(2)$

$$\begin{aligned}
\text{Now, } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + e^{x+h}(x+h) - xe^x - he^h + 2xh - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(h) + xe^x(e^h - 1) + he^{x+h} - he^h + 2xh}{h} \\
&= \lim_{h \rightarrow 0} \left\{ \frac{f(h)}{h} + xe^x \frac{(e^h - 1)}{h} + e^{x+h} - e^h + 2x \right\} \\
&= f'(0) + xe^x \cdot 1 + e^x - 1 + 2x \\
&= 1 + xe^x + e^x + 2x - 1 \\
&= xe^x + e^x + 2x
\end{aligned}$$

Integrating both sides w.r.t. x with limit 0 to x

$$\therefore f(x) - f(0) = xe^x - e^x + e^x + x^2$$

$$f(x) - 0 = xe^x + x^2$$

Hence $f(x) = x^2 + xe^x$

50 Let $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}_+$ and $f(x)$ be differentiable in $(0, \infty)$ then determine $f(x)$.

Sol Given $f(xy) = xf(y) + yf(x)$

Differentiating both sides w.r.t. x treating y as constant,

$$f'(xy) \cdot y = f(y) + yf'(x)$$

Putting $y = x$ and $x = 1$, then $f'(x) \cdot x = f(x) + xf'(1)$

$$\Rightarrow \frac{xf'(x) - f(x)}{x^2} = \frac{f'(1)}{x} \quad \Rightarrow \quad \frac{d}{dx} \left(\frac{f(x)}{x} \right) = \frac{f'(1)}{x}$$

Integrating both sides w.r.t. x taking limit 1 to x ,

$$\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1) \{ \ln x - \ln 1 \}$$

$$\Rightarrow \frac{f(x)}{x} - 0 = f'(1) \ln x \quad (\because f(1) = 0)$$

Hence, $f(x) = f'(1)(x \ln x)$.

51 Let $f(xy) = f(x)f(y) \quad \forall x, y \in \mathbb{R}$ and f is differentiable at $x = 1$ such that $f'(1) = 1$ also $f(1) \neq 0$ then show that f is differentiable for all $x \neq 0$. Hence, determine $f(x)$.

Sol Given $f(xy) = f(x)f(y)$

Putting $x = y = 1$ then we get $f(1) = 1$.

Differentiating both sides w.r.t. x treating y as constant,

$$f'(xy) \cdot y = f'(x)f(y)$$

Replacing y by x and x by 1, then

$$f'(x) \cdot x = f'(1)f(x)$$

$$\Rightarrow f'(x) = \frac{f(x)f'(1)}{x} = \frac{f(x)}{x} \quad (\because f'(1) = 1)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating both sides w.r.t. x and taking limit 1 to x , then

$$\int_1^x \frac{f'(x)}{f(x)} dx = \int_1^x \frac{1}{x} dx$$

$$\Rightarrow \ln f(x) - \ln f(1) = \ln x - \ln 1 \quad (\because f(1) = 1)$$

$$\Rightarrow \ln f(x) - 0 = \ln x - 0 \quad \therefore f(x) = x.$$

52 If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all $x, y \in \mathbb{R}^+$, $f(1) = 0$ and $f'(1) = 1$, then find $f(e)$ and $f'(2)$.

Sol Given $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$... (1)

Replacing x by y and y by x in (1), then

$$2f(y) = f(xy) + f\left(\frac{y}{x}\right) \quad \dots(2)$$

Subtract (2) from (1), we get

$$2\{f(x) - f(y)\} = f\left(\frac{x}{y}\right) - f\left(\frac{y}{x}\right) \quad \dots(3)$$

Putting $x = 1$ in (1) then $2f(1) = f(y) + f\left(\frac{1}{y}\right) = 0 \quad (\because f(1) = 0)$

$$\therefore f(y) = -f\left(\frac{1}{y}\right) \quad \therefore f\left(\frac{y}{x}\right) = -f\left(\frac{x}{y}\right) \quad \dots(4)$$

Now from (3) and (4), we get

$$2\{f(x) - f(y)\} = 2f\left(\frac{x}{y}\right)$$

or $f(x) - f(y) = f\left(\frac{x}{y}\right) \quad \dots(5)$

Now,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} \quad [\text{From (5)}]$$

$$= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x} \cdot x} = \frac{1}{x} f'(1) = \frac{1}{x} \quad \{\because f'(1) = 1\}$$

$$\therefore f'(x) = \frac{1}{x} \quad \Rightarrow \quad f'(2) = \frac{1}{2}$$

and $f(x) = \ln x + \ln c$ for $x = 1$, and $f(1) = \ln 1 + \ln c$

$$\Rightarrow 0 = 0 + \ln c \quad \therefore \ln c = 0$$

then $f(x) = \ln x \quad \therefore f(e) = \ln e = 1 .$

..... $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$. If $|p(x)| \leq |e^{x-1} - 1|$ for all $x \geq 0$, prove that $|a_1 + 2a_2 + \dots + na_n| \leq 1$.

Sol Given $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

$$\therefore p'(x) = 0 + a_1 + 2a_2x + \dots + na_nx^{n-1}$$

$$\Rightarrow p'(1) = a_1 + 2a_2 + \dots + na_n \quad \dots(1)$$

Now, $|p(1)| \leq |e^{1-1} - 1|$

$$= |e^0 - 1| = |1 - 1| = 0$$

$$\Rightarrow |p(1)| \leq 0 \quad \Rightarrow \quad p(1) = 0 \quad (\because |p(1)| \geq 0)$$

As $|p(x)| \leq |e^{x-1} - 1|$

we get $|p(1+h)| \leq |e^h - 1| \quad \forall h > -1, h \neq 0$

$$\Rightarrow |p(1+h) - p(1)| \leq |e^h - 1| \quad (\because p(1) = 0)$$

$$\Rightarrow \left| \frac{p(1+h) - p(1)}{h} \right| \leq \left| \frac{e^h - 1}{h} \right|$$

Taking limit as $h \rightarrow 0$, then

$$\Rightarrow \lim_{h \rightarrow 0} \left| \frac{p(1+h) - p(1)}{h} \right| \leq \lim_{h \rightarrow 0} \left| \frac{e^h - 1}{h} \right|$$

$$\Rightarrow \left| \lim_{h \rightarrow 0} \frac{p(1+h) - p(1)}{h} \right| \leq \left| \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \right|$$

$$\Rightarrow |p'(1)| \leq 1$$

$$\Rightarrow |a_1 + 2a_2 + \dots + na_n| \leq 1 \quad [\text{from (1)}]$$

54 Let $f\left(\frac{xy}{2}\right) = \frac{f(x)f(y)}{2}$ for all real x and y . If $f(1) = f'(1)$, show that $f(x) + f(1-x) =$ constant, for all non-zero real x .

Sol Given $f\left(\frac{xy}{2}\right) = \frac{f(x)f(y)}{2}$

Replacing x by $2x$ and y by 1 , we get

$$2f(x) = f(2x)f(1) \quad \dots(1)$$

and,

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{x\left(1+\frac{y}{x}\right)}{2}\right) = \frac{f(x)f\left(1+\frac{y}{x}\right)}{2}, x \neq 0 \quad \dots(2)$$

now,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{f(2x)f\left(1 + \frac{h}{x}\right) - f(x)}{2}}{h} \quad [\text{from (2) }]$$

$$= \lim_{h \rightarrow 0} \frac{f(2x)f\left(1 + \frac{h}{x}\right) - 2f(x)}{2h}$$

$$= \lim_{h \rightarrow 0} \frac{f(2x)f\left(1 + \frac{h}{x}\right) - f(2x)f(1)}{2h} \quad [\text{from (1) }]$$

$$= \frac{f(2x)}{2} \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right) - f(1)}{x \cdot \frac{h}{x}}$$

$$= \frac{f(2x)}{2x} \cdot f'(1)$$

$$= \frac{2f(x)}{f(1) \cdot 2x} \cdot f'(1) = \frac{f(x)}{x} \quad (\because f'(1) = f(1))$$

$$= \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating both sides w.r.t. x , we get

$$\ln f(x) = \ln x + \ln c$$

$$\Rightarrow f(x) = cx \quad (c \text{ is constant } > 0)$$

$$\therefore f(x) + f(1-x) = cx + c(1-x) = cx + c - cx = c = \text{constant.}$$

55 Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \max\{f(t) : 0 \leq t \leq x\}, 0 \leq x \leq 1 = 3 - x, 1 < x \leq 2$.

Discuss the continuity and differentiability of the function $g(x)$ in the interval $(0, 2)$.

Sol Given $f(x) = x^3 - x^2 + x + 1$

$$\therefore f'(x) = 3x^2 - 2x + 1$$

$$= 3 \left\{ x^2 - \frac{2x}{3} + \frac{1}{3} \right\}$$

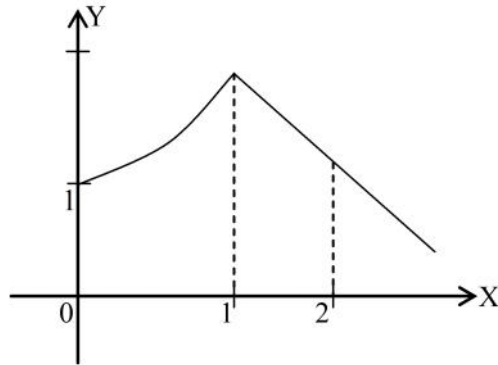
$$= 3 \left\{ \left(x - \frac{1}{3} \right)^2 + \frac{2}{9} \right\} > 0$$

$\therefore f(x)$ is strictly increasing in $(0, 2)$

\therefore maximum value of $f(t)$ in $0 \leq t \leq x$ is $f(x)$

$$\begin{aligned} \therefore g(x) &= \begin{cases} f(x) & , 0 \leq x \leq 1 \\ 3-x & , 1 < x \leq 2 \end{cases} \\ &= \begin{cases} x^3 - x^2 + x + 1 & , 0 \leq x \leq 1 \\ 3-x & , 1 < x \leq 2 \end{cases} \end{aligned}$$

Graph of $g(x)$:



Clearly, $g(x)$ is continuous for all $x \in (0, 2)$ and differentiable at all points in this interval except $x = 1$.

- 56 Let $f(x) = x^3 - 9x^2 + 15x + 6$, and $g(x) = \begin{cases} \min f(t) : 0 \leq t \leq x & , 0 \leq x \leq 6 \\ x - 18 & , x > 6 \end{cases}$, then draw the graph of $g(x)$ and discuss the continuity and differentiability of $g(x)$.

Sol $\therefore f(x) = x^3 - 9x^2 + 15x + 6$,

$$\therefore f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-1)(x-5)$$

If $f'(x) > 0$ then $x \in (-\infty, 1) \cup (5, \infty)$

and if $f'(x) < 0$ then $x \in (1, 5)$



Hence $f(x)$ is increasing in

$(-\infty, 1) \cup (5, \infty)$ and decreasing in $(1, 5)$.

$$\text{Now, } f(x) = 6 \quad \Rightarrow \quad x^3 - 9x^2 + 15x + 6 = 6$$

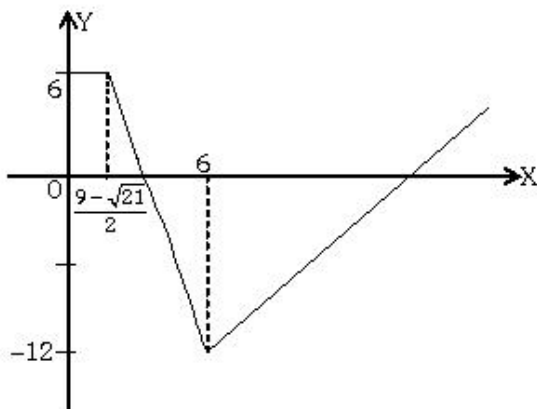
$$\Rightarrow x^3 + 9x^2 + 15x = 0 \quad \Rightarrow \quad x(x^2 - 9x + 15) = 0$$

$$\Rightarrow x = 0, \frac{9 \pm \sqrt{21}}{2}$$

$$\Rightarrow x = 0, \frac{9 - \sqrt{21}}{2} \quad \left(x \neq \frac{9 + \sqrt{21}}{2}, \because \frac{9 - \sqrt{21}}{2} > 6 \right)$$

$$\therefore g(x) = \begin{cases} 6 & , 0 \leq x < \frac{9-\sqrt{21}}{2} \\ x^3 - 9x^2 + 15x + 6 & , \frac{9-\sqrt{21}}{2} \leq x \leq 6 \\ x - 18 & , x > 6 \end{cases}$$

Graph of $g(x)$:



Clearly $g(x)$ is continuous in $[0, \infty)$ and differentiable at all points in this interval

other than $\frac{9-\sqrt{21}}{2}$ and 6.

57 Let $f(x) = \begin{cases} b \sin^{-1}\left(\frac{x+c}{2}\right) & , -\frac{1}{2} < x < 0 \\ \frac{1}{2} & , x = 0 \\ \frac{e^{ax/2} - 1}{x} & , 0 < x < \frac{1}{2} \end{cases}$, If $f(x)$ is differentiable at $x = 0$. Find the

value of a also prove that $64b^2 = 4 - c^2$.

Sol $Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{e^{\frac{ah}{2}} - 1}{h} - \frac{1}{2}}{h}$

$$= \lim_{h \rightarrow 0} \frac{\frac{a}{2} \cdot \left(\frac{e^{\frac{ah}{2}} - 1}{\frac{ah}{2}} \right) - \frac{1}{2}}{h}$$

at $h \rightarrow 0$ numerator must be = 0, then $\frac{a}{2} \cdot 1 - \frac{1}{2} = 0$

$$\therefore a = 1$$

$$\Rightarrow Rf'(0) = \lim_{h \rightarrow 0} \frac{\frac{e^{\frac{h}{2}} - 1}{h} - \frac{1}{2}}{h} = \lim_{h \rightarrow 0} \frac{2\left(e^{\frac{h}{2}} - 1\right) - h}{2h^2} = P \text{ (say)} \quad \dots(1)$$

$$\therefore P = \lim_{h \rightarrow 0} \frac{2\left(e^{\frac{h}{2}} - 1\right) - h}{2h^2}$$

$$\text{Replacing } h \text{ by } -h \text{ then } P = \lim_{h \rightarrow 0} \frac{2\left(e^{-\frac{h}{2}} - 1\right) + h}{2h^2} \quad \dots(2)$$

$$\begin{aligned} \text{Adding (1) and (2) then } 2P &= \lim_{h \rightarrow 0} \frac{e^{\frac{h}{2}} + e^{-\frac{h}{2}} - 2}{h^2} = \lim_{h \rightarrow 0} \frac{e^h - 2e^{\frac{h}{2}} + 1}{h^2 e^{\frac{h}{2}}} \\ &= \lim_{h \rightarrow 0} \left(\frac{e^{\frac{h}{2}} - 1}{\frac{h}{2}} \right)^2 \cdot \frac{1}{4e^{\frac{h}{2}}} = \frac{1}{4} \end{aligned}$$

$$\therefore P = \frac{1}{8} \quad \Rightarrow \quad Rf'(0) = \frac{1}{8} \quad \dots(3)$$

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{b \sin^{-1}\left(\frac{-h+c}{2}\right) - \frac{1}{2}}{-h}$$

Now, at $h \rightarrow 0$ numerator must be 0

$$\therefore b \sin^{-1}\left(\frac{c}{2}\right) - \frac{1}{2} = 0$$

then,

$$\begin{aligned} Lf'(0) &= b \lim_{h \rightarrow 0} \frac{\sin^{-1}\left(\frac{c-h}{2}\right) - \sin^{-1}\left(\frac{c}{2}\right)}{-h} \\ &= b \lim_{h \rightarrow 0} \frac{\sin^{-1}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)} - \frac{c}{2}\sqrt{\left(1-\left(\frac{c-h}{2}\right)^2\right)}\right\}}{-h} \\ &= b \lim_{h \rightarrow 0} \frac{\sin^{-1}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)} - \frac{c}{2}\sqrt{\left(1-\left(\frac{c-h}{2}\right)^2\right)}\right\}}{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)} - \frac{c}{2}\sqrt{\left(1-\left(\frac{c-h}{2}\right)^2\right)}} \end{aligned}$$

$$\begin{aligned}
& \frac{\left\{ \left(\frac{c-h}{2} \right) \sqrt{1-\frac{c^2}{4}} - \frac{c}{2} \sqrt{1-\left(\frac{c-h}{2} \right)^2} \right\}}{-h} \\
= & -b \lim_{h \rightarrow 0} \frac{\left\{ \left(\frac{c-h}{2} \right) \sqrt{1-\frac{c^2}{4}} - \frac{c}{2} \sqrt{1-\left(\frac{c-h}{2} \right)^2} \right\} \left\{ \left(\frac{c-h}{2} \right) \sqrt{1-\frac{c^2}{4}} + \frac{c}{2} \sqrt{1-\left(\frac{c-h}{2} \right)^2} \right\}}{h \left\{ \left(\frac{c-h}{2} \right) \sqrt{1-\frac{c^2}{4}} - \frac{c}{2} \sqrt{1-\left(\frac{c-h}{2} \right)^2} \right\}} \\
= & -b \lim_{h \rightarrow 0} \frac{\left(\frac{c-h}{2} \right)^2 \left(1-\frac{c^2}{4} \right) - \frac{c^2}{4} \left(1-\left(\frac{c-h}{2} \right)^2 \right)}{h \left\{ \left(\frac{c-h}{2} \right) \sqrt{1-\frac{c^2}{4}} + \frac{c}{2} \sqrt{1-\left(\frac{c-h}{2} \right)^2} \right\}} \\
= & -b \lim_{h \rightarrow 0} \frac{(2c-h)(-h)}{4h \left\{ \left(\frac{c-h}{2} \right) \sqrt{1-\frac{c^2}{4}} + \frac{c}{2} \sqrt{1-\left(\frac{c-h}{2} \right)^2} \right\}} \\
= & \frac{2bc}{4 \left\{ c \sqrt{1-\frac{c^2}{4}} \right\}} = \frac{b}{2 \sqrt{1-\frac{c^2}{4}}} \quad \dots(5)
\end{aligned}$$

From (3) and (5),

$$\frac{1}{8} = \frac{b}{2 \sqrt{1-\frac{c^2}{4}}}$$

$$\Rightarrow 64b^2 = 4 - c^2$$

58 Let $\alpha \in \mathbb{R}$. Prove that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at $x = \alpha$ if and only if there is a function $g: \mathbb{R} \rightarrow \mathbb{R}$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $\alpha \in \mathbb{R}$.

Sol Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x = \alpha \in \mathbb{R}$, then

$$\lim_{x \rightarrow \alpha} \frac{f(x) - f(\alpha)}{(x - \alpha)} = f'(\alpha) \text{ exists and finite.}$$

$$\text{i.e. } Lf'(\alpha) = Rf'(\alpha) = f'(\alpha)$$

$$\Rightarrow \lim_{x \rightarrow \alpha^-} \frac{f(x) - f(\alpha)}{(x - \alpha)} = \lim_{x \rightarrow \alpha^+} \frac{f(x) - f(\alpha)}{(x - \alpha)} = f'(\alpha)$$

$$\lim_{x \rightarrow \alpha^-} g(x) = \lim_{x \rightarrow \alpha^+} g(x) = f'(\alpha) \quad \left\{ \because f(x) - f(\alpha) = g(x)(x - \alpha) \right\} \quad \dots(1)$$

$$\begin{aligned}\text{Again } f'(\alpha) &= \lim_{x \rightarrow \alpha} \frac{f(x) - f(\alpha)}{(x - \alpha)} \\ &= \lim_{x \rightarrow \alpha} g(x) = g(\alpha)\end{aligned}$$

From (1) and (2), we get $\lim_{x \rightarrow \alpha^-} g(x) = \lim_{x \rightarrow \alpha^+} g(x) = g(\alpha)$

$$\text{L.H.L} = \text{R.H.L} = \text{V.F.}$$

\Rightarrow $g(x)$ is continuous function at $x = \alpha \in \mathbb{R}$.

59 Let $g(x) = 0$ if $-e \leq x < 1$

$$= \left\{ 1 + \frac{1}{3} \sin(\ln x^{2\pi}) \right\} \text{ if } 1 \leq x \leq e.$$

where $\{ \}$ denotes the fractional part function and

$$\begin{aligned}f(x) &= x g(x) \text{ for } g(x) = 1 + \frac{1}{3} \sin(\ln x^{2\pi}) \\ &= x(g(x) + 1) \text{ otherwise}\end{aligned}$$

Discuss the continuity and differentiability of $f(x)$ over its domain.

Sol Given $g(x) = \left\{ 1 + \frac{1}{3} \sin(\ln x^{2\pi}) \right\}$ for $1 \leq x \leq e$
 $= 0$ for $-e \leq x < 1$

$$\begin{aligned}\text{i.e., } g(x) &= 1 + \frac{1}{3} \sin(\ln x^{2\pi}) - \left[1 + \frac{1}{3} \sin(\ln x^{2\pi}) \right] \\ &= \frac{1}{3} \sin(\ln x^{2\pi}) - \left[\frac{1}{3} \sin(\ln x^{2\pi}) \right], 1 \leq x \leq e \\ &= 0, -e \leq x < 1\end{aligned}$$

where $[.]$ denotes the greatest integer function.

consider: $1 \leq x \leq e$

$$\Rightarrow (1)^{2\pi} \leq x^{2\pi} \leq e^{2\pi} \quad \Rightarrow \ln(1) \leq \ln(x^{2\pi}) \leq \ln(e^{2\pi})$$

$$\Rightarrow 0 \leq \ln(x^{2\pi}) \leq 2\pi$$

Case I: If $0 \leq \ln(x^{2\pi}) \leq \pi$ i.e., $1 \leq x \leq \sqrt{e}$ then $0 \leq \sin(\ln(x^{2\pi})) \leq 1$

$$\Rightarrow 0 \leq \frac{1}{3} \sin(\ln(x^{2\pi})) \leq \frac{1}{3} \quad \therefore \left[\frac{1}{3} \sin(\ln(x^{2\pi})) \right] = 0$$

$$\therefore g(x) = \frac{1}{3} \sin(\ln x^{2\pi}) \quad \text{for } 1 \leq x \leq \sqrt{e}$$

Case II: If $\pi < \ln(x^{2\pi}) < 2\pi$ i.e., $\sqrt{e} < x < e$ then $-1 \leq \sin(\ln(x^{2\pi})) < 0$

$$\Rightarrow -\frac{1}{3} \leq \frac{1}{3} \sin(\ln(x^{2\pi})) < 0 \quad \therefore \left[\frac{1}{3} \sin(\ln(x^{2\pi})) \right] = -1$$

$$\therefore g(x) = 1 + \frac{1}{3} \sin(\ln(x^{2\pi})) \quad \text{for } \sqrt{e} < x < e$$

$$\text{Case III : } \quad \text{If } \ln(x^{2\pi}) = 2\pi \quad \Rightarrow \quad x = e \quad \Rightarrow \quad g(x) = \{1\} = 0$$

Combining all cases, we get

$$f(x) = x \left(1 + \frac{1}{3} \sin(\ln(x^{2\pi})) \right) \quad \text{for } \sqrt{e} < x < e$$

$$= x \left(1 + \frac{1}{3} \sin(\ln(x^{2\pi})) \right) \quad \text{for } 1 \leq x \leq \sqrt{e}$$

$$= x(1+0) \quad \text{for } -e \leq x < 1$$

$$= x(1+0) \quad \text{for } x = e$$

$$\Rightarrow \quad f(x) = x \left(1 + \frac{1}{3} \sin(\ln(x^{2\pi})) \right) \quad \text{for } 1 \leq x \leq e$$

$$= x$$

\therefore f is differentiable in $(-e, 1)$ and $(1, e)$

Check the differentiability of $f(x)$ at $x = 1$.

$$Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{(1-h) - 1}{-h} = 1$$

$$\text{and } Rf'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h) \cdot \left(1 + \frac{1}{3} \sin(\ln(1+h)^{2\pi}) \right) - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h + \frac{(1+h)}{3} \sin(\ln(1+h)^{2\pi})}{h}$$

$$= \lim_{h \rightarrow 0} \left(1 + \frac{(1+h) \sin\{\ln(1+h)^{2\pi}\}}{3h} \right)$$

$$= 1 + \lim_{h \rightarrow 0} \frac{(1+h)}{3} \lim_{h \rightarrow 0} \frac{\sin(\ln(1+h)^{2\pi})}{h}$$

$$= 1 + \lim_{h \rightarrow 0} \frac{(1+h)}{3} \lim_{h \rightarrow 0} \frac{\sin\{2\pi \ln(1+h)\}}{2\pi \ln(1+h)} \cdot \frac{2\pi \ln(1+h)}{h}$$

$$= 1 + \left(\frac{1+0}{3} \right) \cdot 1 \cdot 2\pi \cdot 1$$

$$= 1 + \frac{2\pi}{3}$$

Thus f is not differentiable at $x = 1$.

Hence f is continuous and differentiable for all $x \in$ domain of except not differentiable at $x = 1$.

60 Suppose that f and g are non-constant differentiable real valued functions on \mathbb{R} .

If for every $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y) - g(x)g(y)$, $g(x+y) = g(x)f(y) + f(x)g(y)$ and

$f'(0) = 0$ then prove that $\{f(x)\}^2 + \{g(x)\}^2 = 1 \quad \forall x \in \mathbb{R}$.

Sol We have $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x+0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\{f(x)f(h) - g(x)g(h)\} - \{f(x)f(0) - g(x)g(0)\}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x)(f(h) - f(0))}{(h-0)} - \lim_{h \rightarrow 0} \frac{g(x)(g(h) - g(0))}{(h-0)}$$

$$= f(x)f'(0) - g(x)g'(0)$$

$$= 0 - g(x)g'(0) \quad (\because f'(0) = 0)$$

$$\therefore f'(x) = -g(x)g'(0) \quad \dots(1)$$

and $g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x+0)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\{g(x)f(h) + f(x)g(h)\} - \{g(x)f(0) + f(x)g(0)\}}{h}$$

$$= g(x) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h-0} + f(x) \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h-0}$$

$$= g(x)f'(0) + f(x)g'(0)$$

$$= 0 + f(x)g'(0) \quad (\because f'(0) = 0)$$

$$= f(x)g'(0) \quad \dots(2)$$

Multiplying (1) by $f(x)$ and (2) by $g(x)$ and adding we get

$$f(x)f'(x) + g(x)g'(x) = 0$$

or $2f(x)f'(x) + 2g(x)g'(x) = 0$ on integrating we get

$$\{f(x)\}^2 + \{g(x)\}^2 = c \quad \dots(3)$$

Putting $x = 0, y = 0$ in the given equation then

$$f(0) = \{f(0)\}^2 - \{g(0)\}^2 \quad \text{and} \quad g(0) = 2f(0)g(0)$$

$$\text{or} \quad g(0)\{2f(0) - 1\} = 0 \quad \text{or} \quad g(0) = 0 \quad \text{or} \quad f(0) = \frac{1}{2}$$

If $g(0) = 0$, then $f(0) = (f(0))^2 - 0$ or $f(0) = 1$

$$\text{and for } f(0) = \frac{1}{2}, \frac{1}{2} = \left(\frac{1}{2}\right)^2 - (g(0))^2$$

$$\Rightarrow (g(0))^2 = -\frac{1}{4} \quad (\text{Impossible})$$

Hence $f(0) = 1$ and $g(0) = 0$ from (3), $\{f(0)\}^2 + \{g(0)\}^2 = c$

$$\Rightarrow 1 + 0 = c \quad \therefore c = 1$$

Hence $\{f(x)\}^2 + \{g(x)\}^2 = 1$.

61 Let $f(x)$ be a real valued function not identically zero such that

$$f(x + y^n) = f(x) + \{f(y)\}^n; \forall x, y \in \mathbb{R} \text{ (where } n \text{ is odd natural number } > 1) \text{ and } f'(0) \geq 0.$$

Find out the values of $f'(10)$ and $f(5)$.

Sol Given that $f(x + y^n) = f(x) + (f(y))^n$

$$\text{Putting } x = y = 0 \quad \Rightarrow \quad f(0) = 0$$

$$\begin{aligned} \therefore f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lambda \text{ (say)} \quad \dots(1) \end{aligned}$$

Also,

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(0 + \left(h^{1/n}\right)^n\right) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(0) + \{f(h^{1/n})\}^n - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \left\{ \frac{f(h^{1/n})}{h^{1/n}} \right\}^n \\ &= \lambda^n \quad [\text{from (1)}] \end{aligned}$$

From (1) and (2), $\lambda = \lambda^n$

$$\therefore \lambda = -1, 0, 1 \quad (\because n \text{ is odd and } \lambda \in \mathbb{R})$$

$$\therefore f'(0) \geq 0 \quad (\because \lambda \neq -1)$$

$$\therefore f'(0) = 0, 1$$

$$\text{Again } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{f(x + (h^{1/n})^n) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x) + (f(h^{1/n}))^n - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \left(\frac{f(h^{1/n})}{h^{1/n}} \right)^n = \lambda^n
\end{aligned}$$

For $\lambda = 0, f'(x) = 0$

On intergrating we get $f(x) = c$

At $x = 0, f(0) = c = 0$ ($\because f(0) = 0$)

$\therefore f(x) = 0$

which is impossible as $f(x)$ is not identically zero, i.e., $f(x) \neq 0$

and for $\lambda = 1 \quad f'(x) = 1$

On intergrating w.r.t. x and taking limit 0 to x ,

$$\text{then } \int_0^x f'(x) dx = \int_0^x 1 dx$$

$$\Rightarrow f(x) - f(0) = x \quad \Rightarrow f(x) - (0) = x \quad (\because f(0) = 0)$$

Hence $f(x) = x$ and $f'(x) = 1 \quad \therefore f'(10) = 1$ and $f(5) = 5$.

62 Let $a_1 > a_2 > a_3 \dots \dots \dots a_n > 1; p_1 > p_2 > p_3 \dots \dots \dots > p_n > 0$; such that $p_1 + p_2 + p_3 + \dots + p_n = 1$

Also $F(x) = (p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x)^{1/x}$. Compute

(a) $\lim_{x \rightarrow 0^+} F(x)$ (b) $\lim_{x \rightarrow 0} F(x)$ (c) $\lim_{x \rightarrow -\infty} F(x)$ [Ans. (a) $a_1^{p_1} \cdot a_2^{p_2} \dots a_n^{p_n}$; (b) a_1 ; (c) a_n]

[Sol.

$$(1) \quad \lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow 0^+} (p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x)^{1/x} \quad (1^\infty \text{ form})$$

$$= e^l \text{ where } l = \lim_{x \rightarrow 0} \frac{p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x - 1}{x} \quad \left(\frac{0}{0} \right)$$

using L'Hospital's Rule

$$\begin{aligned}
l &= \lim_{x \rightarrow 0} (p_1 \ln a_1 a_1^x + p_2 \ln a_2 a_2^x + \dots + p_n \ln a_n a_n^x) \\
&= p_1 \ln a_1 + p_2 \ln a_2 + \dots + p_n \ln a_n \\
&= \ln (a_1^{p_1} \cdot a_2^{p_2} \dots a_n^{p_n})
\end{aligned}$$

$$\therefore L_1 = e^l = a_1^{p_1} \cdot a_2^{p_2} \dots a_n^{p_n} \quad \text{Ans.}$$

$$(2) \quad \lim_{x \rightarrow \infty} F(x) = L_2 = \lim_{x \rightarrow \infty} (p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x)^{1/x} \quad (\infty^0 \text{ form}) \quad [\text{only when } a_1 a_2 \text{ etc. } > 1]$$

$$\therefore \ln L_2 = \lim_{x \rightarrow \infty} \frac{\ln (p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x)}{x}$$

using L'Hospital's Rule

$$L_2 = \lim_{x \rightarrow \infty} \frac{(p_1 \ln a_1 a_1^x + p_2 \ln a_2 a_2^x + \dots + p_n \ln a_n a_n^x)}{p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x} \quad \dots(1)$$

dividing by a_1^x and taking limit, we get

$$\lim_{x \rightarrow \infty} \left(\frac{a_2}{a_1} \right)^x, \left(\frac{a_3}{a_2} \right)^x, \text{ etc all vanishes as } x \rightarrow \infty$$

$$= \frac{p_1 \ln a_1}{p_1} = \ln a_1$$

hence $\ln L_2 = \ln a_1 \Rightarrow L_2 = a_1$ **Ans.**

(3) $\lim_{x \rightarrow -\infty} F(x) = L_3$ (say)

$$\therefore \ln L_3 = \lim_{x \rightarrow -\infty} \frac{(p_1 \ln a_1 a_1^x + p_2 \ln a_2 a_2^x + \dots + p_n \ln a_n a_n^x)}{p_1 a_1^x + p_2 a_2^x + \dots + p_n a_n^x}$$

dividing by $(a_n)^x$ and taking $\lim_{x \rightarrow -\infty} \left(\frac{a_1}{a_n} \right)^x, \left(\frac{a_2}{a_n} \right)^x$ etc vanishes

$$\therefore \ln L_3 = \frac{p_n \ln a_n}{p_n} \Rightarrow L_3 = a_n$$

63 Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a differentiable function with $f(1) = 3$ and satisfying :

$$\int_1^{xy} f(t) dt = y \int_1^x f(t) dt + x \int_1^y f(t) dt ; \forall x, y \in \mathbb{R}^+$$

then find $f(x)$.

Sol We have $\int_1^{xy} f(t) dt = y \int_1^x f(t) dt + x \int_1^y f(t) dt$

Differentiating both sides w.r.t. x treating y as constant; we get

$$f(xy) \cdot y = yf(x) + \int_1^y f(t) dt$$

Putting $x = 1$, we get $yf(y) = yf(1) + \int_1^y f(t) dt$

$$\Rightarrow yf(y) = 3y + \int_1^y f(t) dt \quad (\because f(1) = 3)$$

Again differentiating both sides w.r.t. y , we get

$$yf'(y) + f(y) \cdot 1 = 3 + f(y)$$

$$\Rightarrow f'(y) = \frac{3}{y}$$

Integrating both sides w.r.t. y with limit 1 to x then

$$yf'(1) = 3 \ln x - 3 \ln 1$$

$$f(x) - f(1) = 3 \ln x - 3 \ln 1$$

$$\Rightarrow f(x) - 3 = 3 \ln x - 0 \quad (\because f(1) = 3)$$

$$\begin{aligned}\Rightarrow f(x) &= 3 + 3 \ln x \\ &= 3 \ln e + 3 \ln x = 3 \ln(ex)\end{aligned}$$

Hence $f(x) = 3 \ln(ex)$.

64 Let $f(x^m y^n) = mf(x) + nf(y) \quad \forall x, y \in \mathbb{R}^+$ and $\forall m, n \in \mathbb{R}$. If $f'(x)$ exists and has the value

$$\frac{e}{x}, \text{ then find } \lim_{x \rightarrow 0} \frac{f(1+x)}{x}.$$

Sol $\therefore f(x^m y^n) = mf(x) + nf(y) \quad \dots(1)$

Putting $x = y = m = n = 1$, then $f(1) = f(1) + f(1)$

$$\Rightarrow f(1) = 0$$

$$\begin{aligned}\therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x \cdot 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left\{\left(x^{1/m}\right)^m \left\{\left(1 + \frac{h}{x}\right)^{1/n}\right\}^n\right\} - f\left\{\left(x^{1/m}\right)^m \left\{(1)^{1/n}\right\}^n\right\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{mf\left(x^{1/m}\right) + nf\left\{\left(1 + \frac{h}{x}\right)^{1/n}\right\} - mf\left(x^{1/m}\right) - nf(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{nf\left\{\left(1 + \frac{h}{x}\right)^{1/n}\right\}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{x\left(\frac{h}{x}\right)} \quad \left(\text{Putting } y = 1 \text{ in (1) then } f(x^m) = mf(x)\right)\end{aligned}$$

$$\Rightarrow \frac{e}{x} = \frac{1}{x} \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{\left(\frac{h}{x}\right)} \quad \Rightarrow \quad \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{\left(\frac{h}{x}\right)} = e$$

Hence $\lim_{h \rightarrow 0} \frac{f(1+x)}{x} = e$

65 Let f be a continuous and differentiable function in (x_1, x_2) . If $f(x).f'(x) \geq x\sqrt{1 - (f(x))^4}$

and $\lim_{x \rightarrow x_1^+} (f(x))^2 = 1$ and $\lim_{x \rightarrow x_2^-} (f(x))^2 = \frac{1}{2}$ for $x \in (x_1, x_2)$, then prove that $x_1^2 - x_2^2 \geq \frac{\pi}{3}$

(assume that $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$ holds everywhere).

Sol Given $f(x) \cdot f'(x) \geq x\sqrt{1-(f(x))^4}$

$$\Rightarrow \frac{f(x)f'(x)}{\sqrt{1-(f(x))^4}} - x \geq 0 \quad \text{or} \quad \frac{2f(x)f'(x)}{\sqrt{1-(f(x))^4}} - 2x \geq 0$$

$$\text{or} \quad \frac{d}{dx} \left\{ \sin^{-1}(f(x))^2 - x^2 \right\} \geq 0$$

$\Rightarrow F(x) = \sin^{-1}(f(x))^2 - x^2$ is a non decreasing function.

$$\Rightarrow \lim_{x \rightarrow x_1^+} F(x) \leq \lim_{x \rightarrow x_2^-} F(x)$$

$$\Rightarrow \lim_{x \rightarrow x_1^+} \left\{ \sin^{-1}(f(x))^2 - x^2 \right\} \leq \lim_{x \rightarrow x_2^-} \left\{ \sin^{-1}(f(x))^2 - x^2 \right\}$$

$$\Rightarrow \frac{\pi}{2} - x_1^2 \leq \frac{\pi}{6} - x_2^2 \quad \Rightarrow \quad x_1^2 - x_2^2 \geq \frac{\pi}{3}$$

66 Are there any non-constant differentiable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(f(f(x))) = f(x) \geq 0 \quad \forall x \in \mathbb{R}?$$

Sol Given $f(f(f(x))) = f(x)$ (1)

Applying f to both sides of the equation (1), then

$$f(f(f(f(x)))) = f\{f(x)\} \quad \dots(2)$$

If $g(x) = f(f(x)) \quad \forall x \in \mathbb{R}$ then equation (2) can be written as $g(g(x)) = g(x)$; g is also a differentiable function on \mathbb{R} and $g(x) \geq 0 \quad \forall x \in \mathbb{R}$.

Then the range $T = g(\mathbb{R})$ of g is an interval in $[0, \infty)$. Let a be the infimum of T .

Since $g(t) = t$ for all $t \in T$ and g is continuous.

$$\Rightarrow g(a) = a$$

Assume T has more than one element. Choose $\delta > 0$ such that $(a, a + \delta) \subseteq T$.

Then $x \in (a - \delta, a)$

$$\Rightarrow g(x) \geq g(a) = a \quad \therefore \quad \frac{g(x) - g(a)}{x - a} \leq 0$$

$$\therefore Lg'(a) = \lim_{x \rightarrow a^-} \frac{g(x) - g(a)}{x - a} \leq 0$$

$$= \lim_{h \rightarrow 0} \frac{g(a-h) - g(a)}{-h} \leq 0 \quad \dots(3)$$

For $x \in (a, a + \delta)$ we have $\frac{g(x) - g(a)}{x - a} = 1$

Hence $Rg'(a) = \lim_{x \rightarrow a^+} \frac{g(x) - g(a)}{x - a} = 1 \dots(4)$

As g is differentiable at a , therefore (3) and (4) are contradictory. This concludes that T is a single point i.e., g is a constant function,

$$g(x) = c \quad \forall x \in \mathbb{R}, \quad (c \text{ is constant})$$

from (1), $f(c) = f(x) \quad \forall x \in \mathbb{R}$

This shows that f is a constant function. Thus there is no non-constant differentiable function satisfying (1).

67 Let $f(x) = x^3 - 3x^2 + 6 \quad \forall x \in \mathbb{R}$ and

$$g(x) = \begin{cases} \max\{f(t) : x+1 \leq t \leq x+2, -3 \leq x < 0\} \\ 1-x, & \text{for } x \geq 0 \end{cases}$$

Test continuity of $g(x)$ for $x \in [-3, 1]$.

Sol Since $f(x) = x^3 - 3x^2 + 6$

$$\begin{aligned} \Rightarrow f'(x) &= 3x^2 - 6x \\ &= 3x(x - 2) \end{aligned}$$

for maximum and minima $f'(x) = 0$

$$\therefore x = 0, 2$$

$$f''(x) = 6x - 6$$

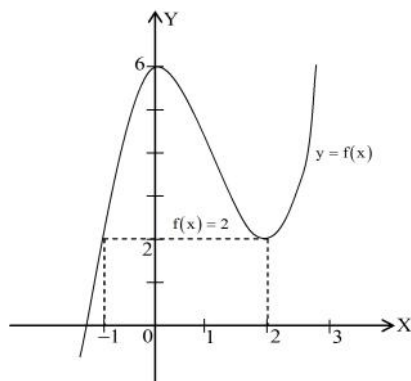
$$f''(0) = -6 < 0 \quad (\text{local maxima at } x = 0)$$

$$f''(2) = 6 > 0 \quad (\text{local minima at } x = 2)$$

Cut off x-axis $x^3 - 3x^2 + 6 = 0$ has maximum 2 positive and 1 negative real roots.

Cut off y-axis. $F(0) = 6$.

Now graph of $f(x)$ is :



Clearly $f(x)$ is increasing in $(-\infty, 0) \cup (2, \infty)$ and decreasing in $(0, 2)$

$$\Rightarrow x+2 < 0 \quad \Rightarrow \quad x < -2 \quad \Rightarrow \quad -3 \leq x < -2$$

$$\Rightarrow -2 \leq x+1 < -1 \quad \text{and} \quad -1 \leq x+2 < 0$$

in both cases $f(x)$ increases (maximum) of $g(x) = f(x+2)$

$$\therefore g(x) = f(x+2); -3 \leq x < -2 \quad \dots(1)$$

$$\text{and if } x+1 < 0 \text{ and } 0 \leq x+2 < 2 \quad \Rightarrow \quad -2 \leq x < -1$$

then $g(x) = f(0)$

$$\text{Now for } x+1 \geq 0 \text{ and } x+2 < 2 \quad \Rightarrow \quad -1 \leq x < 0, g(x) = f(x+1)$$

$$\text{Hence } g(x) = \begin{cases} f(x+2) & ; -3 \leq x < -2 \\ f(0) & ; -2 \leq x < -1 \\ f(x+1) & ; -1 \leq x < -0 \\ 1-x & ; x \geq 0 \end{cases}$$

Hence $g(x)$ is continuous in the interval $[-3, 1]$.

68 $f: [0, 1] \rightarrow \mathbb{R}$ is defined as $f(x) = \begin{cases} x^3(1-x) \sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$, then prove that

(a) f is differentiable in $[0, 1]$ (b) f is bounded in $[0, 1]$ (c) f' is bounded in $[0, 1]$

Sol. $f(x) = \begin{cases} x^3(1-x) \sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$

$$f'(0^+) = \lim_{h \rightarrow 0} \frac{h^3(1-h) \sin \frac{1}{h^2} - 0}{h} = 0$$

$$f'(1^-) = \lim_{h \rightarrow 0} \frac{(1-h)^3(+h) \sin \frac{1}{(1-h)^2} - 0}{-h} = \lim_{h \rightarrow 0} -(1-h)^3 \sin \frac{1}{(1-h)^2} = -\sin 1$$

Hence f is derivable in $[0, 1]$, obviously f is continuous in $[0, 1]$ hence f is bounded

$$\text{hence } f'(x) = \begin{cases} (x^3 - x^4) \cos\left(\frac{1}{x^2}\right) \left(-\frac{2}{x^3}\right) + \sin \frac{1}{x^2} (3x^2 - 4x^3) & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\lim_{x \rightarrow 1^-} = (0) + \sin 1(3-4), \quad \text{hence } f' \text{ is also bounded.}$$

- 69 Discuss the continuity of f in $[0,2]$ where $f(x) = \begin{cases} 4x - 5[x] & \text{for } x > 1 \\ \cos \pi x & \text{for } x \leq 1 \end{cases}$; where $[x]$ is the greatest integer not greater than x .

Sol. $f(x) = \begin{cases} 4x - 5[x] & \text{for } 1 < x \leq 2 \\ \cos \pi x & \text{for } 0 \leq x \leq 1 \end{cases} = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x \leq 1 \\ (5-4x) & \text{if } 1 < x < \frac{5}{4} \\ (4x-5) & \text{if } \frac{5}{4} \leq x < 2 \\ 6 & \text{if } x = 2 \end{cases}$

Clearly $f(x)$ is discont. for $x = 0, 1/2, 1$ & 2 .

- 70 If $f(x) = x + \{-x\} + [x]$, where $[x]$ is the integral part & $\{x\}$ is the fractional part of x . Discuss the continuity of f in $[-2, 2]$.

Sol. $f(x) = x + \{-x\} + [x]$

if $n < x < n+1$, then $f(x) = 2n+1$

{as for nonintegral values $\{-x\} = 1-x + [x]$ and $[x] = n$ }

if $x = n$, then $f(x) = 2n$

Hence $f(x) = \begin{cases} 2n & \text{if } x = n \\ 2n+1 & \text{if } n < x < n+1 \\ 2n+2 & \text{if } x = n+1 \end{cases}$