CONTINUITY & DIFFERENTIABILITY

EXERCISE 1(A)

1. (d)

L.H.L. at $x = 3$, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+\lambda) = \lim_{h\to 0}(3-h+\lambda) = 3+\lambda$ …..(i)

R.H.L. at $x = 3$, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(3x-5) = \lim_{h\to 0}\{3(3+h)-5\} = 4$ …..(ii)

Value of function $f(3) = 4$ …..(iii)

For continuity at $x = 3$

Limit of function = value of function $3 + \lambda = 4 \Rightarrow \lambda = 1$.

2. (c)

If function is continuous at $x = 0$, then by the definition of continuity $f(0) = \lim_{x\to 0} f(x)$

Since f(0) = k. Hence, $f(0) = k = \lim_{x\to 0}(x)\left(\sin\frac{1}{x}\right)$

$\Rightarrow k = 0$ (a finite quantity lies between –1 to 1)

$\Rightarrow k = 0$.

3. (c)

Since f(x) is continuous at $x = 1$,

$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ …..(i)

Now $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 2(1-h)+1 = 3$ *i.e.*, $\lim_{x\to 1^-} f(x) = 3$

Similarly, $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 5(1+h)-2$ *i.e.*, $\lim_{x\to 1^+} f(x) = 3$

So according to equation (i), we have $k = 3$.

4. (d)

We have $\lim_{x\to 0} f(x) = \lim_{x\to 0}\sin\frac{1}{x}$ = An oscillating number which oscillates between –1 and 1.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Consequently f(x) cannot be continuous at x = 0 for any value of *k*.

5. (c)

LHL = $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} m(1-h)^2 = m$

RHL = $\lim_{x\to 1^+} f(x) = \lim_{h\to 0} 2(1+h) = 2$ and $f(1) = m$

Function is continuous at x = 1, $\therefore$ LHL = RHL = f(1)

Therefore m = 2.

6. (a)

$$\lim_{x\to 0}(\cos x)^{1/x} = k \Rightarrow \lim_{x\to 0}\frac{1}{x}\log(\cos x) = \log k$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x}\lim_{x\to 0}\log\cos x = \log k$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x}\times 0 = \log_e k \Rightarrow k = 1\ .$$

7. (b)

Since f is continuous at $x=\frac{\pi}{4}$; $\therefore f\left(\frac{\pi}{4}\right)=\underset{h\to 0}{f}\left(\frac{\pi}{4}+h\right)=\underset{h\to 0}{f}\left(\frac{\pi}{4}-h\right)$

$\Rightarrow \frac{\pi}{4}+b=\frac{\pi}{4}+a^2 \Rightarrow b=a^2$

Also as f is continuous at $x=\frac{\pi}{2}$;

$\therefore f\left(\frac{\pi}{2}\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$\Rightarrow \ 2b+a=b \ \Rightarrow a=-b$.

Hence (–1, 1) & (0, 0) satisfy the above relations.

8. (c)

$\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h) = \lim_{h\to 0}\left[2+\sin\frac{\pi}{2}(1-h)\right] = 3$

Similarly, $\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h) = \lim_{h\to 0} a(1+h)+b = a+b$

$\because$ f(x) is continuous at x = 1 so $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

$\Rightarrow \ a+b=3$ …….(i)

Again, $\lim_{x\to 2^-} f(x)=\lim_{h\to 0} f(2-h)=\lim_{h\to 0} a(2-h)+b=2a+b$

and $\lim_{x\to 2^+} f(x)=\lim_{h\to 0} f(2+h) = \lim_{h\to 0} \tan\frac{\pi}{8}(2+h) = 1$

f(x) is continuous in $(-\infty, 6)$, so it is continuous at $x=2$ also, so

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \ 2a+b=1$ ……(ii)

Solving (i) and (ii) a = - 2, b = 5.

9. (a)

$\lim_{x\to \frac{\pi}{2}^-} f(x)=\frac{\pi}{2}, \lim_{x\to \frac{\pi}{2}^+} f(x)=-\frac{\pi}{2}$

Since $\lim_{x\to \frac{\pi}{2}^-} f(x)\neq \lim_{x\to \frac{\pi}{2}^+} f(x)$,

$\therefore$ Function is discontinuous at $x=\frac{\pi}{2}$

10. (b)

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(\frac{2\sin^2 3x}{(3x)^2}\right)3=6$ and

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{9+\sqrt{x}}-3}=\lim_{x\to 0^+}\left(\sqrt{9+\sqrt{x}}+3\right)=6$

Hence $a=6$.

11. (c)

The function $f\left(x\right)=\dfrac{1}{x^2+x-6}$ is discontinuous at 2 points.

The function $f\left(x\right)=\dfrac{1}{x^2+x-6}$ & $g\left(x\right)=\dfrac{1}{x-1}\Rightarrow g\left(f\left(x\right)\right)=\dfrac{1}{x^2+x-7}$

g(f(x)) is discontinuous at 4 points.

Hence, the composite $f\left(g\left(x\right)\right)$ is discontinuous at three points $x=\dfrac{2}{3},1\ \&\ \dfrac{3}{2}$

12. (b)

$$\lim_{x\to 0}\frac{\ln b\ln(a+x)-\ln a\ln(b-x)}{x}=\lim_{x\to 0}\frac{\ln b\left(\ln(a+x)-\ln a\right)-\ln a\left(\ln a\ln(b-x)-\ln b\right)}{x}$$

$$=\ln b\lim_{x\to 0}\frac{\left(\ln(a+x)-\ln a\right)}{x}+\ln a\lim_{x\to 0}\frac{\left(\ln(b-x)-\ln b\right)}{x}$$

$$=\frac{\ln b}{a}\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{a}\right)}{\frac{x}{a}}+\frac{\ln a}{b}\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{b}\right)}{\frac{x}{b}}$$

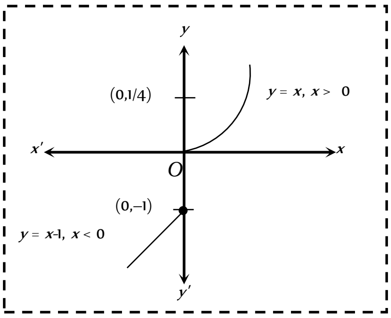
$$=\frac{\ln b}{a}+\frac{\ln a}{b}=\frac{\ln\left(b^b a^a\right)}{ab}$$

13. (b)

$$f(2)=2,\ f(2^+)=\lim_{x\to 2^+}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2^+}\frac{(x-3)}{(x+2)}=-\frac{1}{4}$$

14. (c)

Clearly from curve drawn of the given function $f(x)$, it is discontinuous at $x=0$.



15. (b)

$$f(x)=\begin{cases}(1+|\tan x|)^{\frac{a}{3|\tan x|}}, & -\frac{\pi}{6}<x<0\\ b, & x=0\\ e^{\frac{\tan 6x}{\tan 3x}}, & 0<x<\frac{\pi}{6}\end{cases}$$

For f(x) to be continuous at x = 0

$$\Rightarrow \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

$$\Rightarrow \lim_{x\to 0}(1+|\tan x|)^{\frac{a}{3|\tan x|}} = e^{\lim_{x\to 0^-}\left((1+|\tan x|-1)\frac{a}{3|\tan x|}\right)} = e^{a/3}$$

Now, $\lim_{x\to 0^+} e^{\frac{\tan 6x}{\tan 3x}} = \lim_{x\to 0^+} e^{\left(\frac{\tan 6x}{6x}.6x\right)/\left(\frac{\tan 3x}{3x}.3x\right)} = e^2$

$\therefore\ e^{a/3} = b = e^2 \Rightarrow a = 6$ and $b = e^2$.

16. (d)

Let $f(x) = \ln\frac{x}{4}$

$$\lim_{x\to 4} xf(x) = \lim_{x\to 4} x\ln\frac{x}{4} = 0$$

17. (a)

Note that $[x+2] = 0$ if $0 \le x+2 < 1$

i.e. $[x+2] = 0$ if $-2 \le x < -1$.

Thus domain of f is $R-[-2,-1)$

We have $\sin\left(\frac{\pi}{[x+2]}\right)$ is continuous at all points of $R-[-2,-1)$ and $[x]$ is continuous on $R-I$, where I denotes the set of integers.

Thus the points where f can possibly be discontinuous are……, $-3,-2,-1,0\,1,2,\ldots\ldots\ldots$. But for $-1 \le x < 0, [x+1] = 0$ and $\sin\left(\frac{\pi}{[x+2]}\right)$ is defined.

Therefore f(x) = 0 for $-1 \le x < 0$.

Also $f(x)$ is not defined on $-2 \le x < -1$.

Hence set of points of discontinuities of f(x) is $I-\{-1\}$.

18. (b)

$$f(x) = \lim_{x\to 0}\left(\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\right) = f(0) \qquad ,\left(\frac{0}{0}\text{form}\right)$$

Applying L-Hospital's rule, $f(0) = \lim_{x\to 0}\frac{\left(2-\frac{1}{\sqrt{1-x^2}}\right)}{\left(2+\frac{1}{1+x^2}\right)} = \frac{2-1}{2+1} = \frac{1}{3}$

19. (d)

For continuity at all $x \in R$, we must have

$$f\left(-\frac{\pi}{2}\right) = \lim_{x\to(-\pi/2)^-}(4\sin x) = \lim_{x\to(-\pi/2)^+}(a\sin x - b)$$

$\Rightarrow\ 4 = -a-b$ …..(i)

and $f\left(\frac{\pi}{2}\right) = \lim_{x\to(\pi/2)^-}(a\sin x - b) = a - b = \lim_{x\to(\pi/2)^+}(\cos x) = 0$

$\Rightarrow\ 0 = a - b$ ….(ii)

From (i) and (ii), $a = -2$ and $b = -2$.

20. (a)

$$f(5) = \lim_{x\to 5} f(x) = \lim_{x\to 5}\frac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x\to 5}\frac{(x-5)^2}{(x-2)(x-5)} = \frac{5-5}{5-2} = 0\ .$$

21. (c)

For continuity at 0, we must have $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}(x+1)^{\cot x} = \lim_{x\to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{x\cot x} = \lim_{x\to 0}\left\{(1+x)^{\frac{1}{x}}\right\}^{\lim_{x\to 0}\left(\frac{x}{\tan x}\right)} = e.$$

22. (a)

Conceptual question

23. (c)

f(x) is continuous at $x = \frac{\pi}{3}$, then $\lim_{x\to\pi/3} f(x) = f(0)$ or

$$\lambda = \lim_{x\to\pi/3}\frac{1 - \sin\frac{3x}{2}}{\pi - 3x}\ ,\left(\frac{0}{0}\text{ form}\right)$$

Applying L-Hospital's rule, $\lambda = \lim_{x\to\pi/3}\frac{-\frac{3}{2}\cos\frac{3x}{2}}{-3} = 0$

24. (d)

If f(x) is continuous at x = 0 then,

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{2 - \sqrt{x+4}}{\sin 2x}\ ,\left(\frac{0}{0}\text{ form}\right)$$

Using L–Hospital's rule, $f(0) = \lim_{x\to 0}\frac{\left(-\frac{1}{2\sqrt{x+4}}\right)}{2\cos 2x} = -\frac{1}{8}$.

25. (d)

$x^2 + 2 = 3x \Rightarrow x = 1, 2$

F(x) will be continuous only at x = 1 & 2.

26. (b)

$$f(x) = \left[x^2 + e^{\frac{1}{2-x}}\right]^{-1} \text{ and } f(2) = k$$

If f(x) is continuous from right at x = 2 then $\lim_{x\to 2^+} f(x) = f(2) = k$

$$\Rightarrow \lim_{x\to 2^+}\left[x^2 + e^{\frac{1}{2-x}}\right]^{-1} = k \Rightarrow k = \lim_{h\to 0} f(2+h) \Rightarrow k = \lim_{h\to 0}\left[(2+h)^2 + e^{\frac{1}{2-(2+h)}}\right]^{-1}$$

$$\Rightarrow k=\lim_{h\to 0}\left[4+h^2+4h+e^{-1/h}\right]^{-1} \Rightarrow k=[4+0+0+e^{-\infty}]^{-1} \Rightarrow k=\frac{1}{4}.$$

27. (c)

$$\lim_{x\to\pi} f(x)=\lim_{x\to\pi}\frac{2\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to\pi}\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}=\lim_{x\to\pi}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$\therefore$ At $x=\pi,\ f(\pi)=-\tan\frac{\pi}{4}=-1$.

28. (c)

$$\text{L.H.L.}=\lim_{x\to 0^{-1}}\frac{\sqrt{4+kx}-\sqrt{4-kx}}{x}=\lim_{x\to 0^{-1}}\frac{2kx}{x}\times\frac{1}{\sqrt{4+kx}+\sqrt{4-kx}}=\frac{k}{2}$$

$$\text{R.H.L.}=\lim_{x\to 0^+}\frac{2x^2+3x}{\sin x}=\lim_{x\to 0^+}\frac{x}{\sin x}(2x+3)=3$$

Since it is continuous, hence L.H.L = R.H.L $\Rightarrow k=6$.

29. (c)

$|x|$ is continuous at $x=0$ and $\frac{|x|}{x}$ is discontinuous at $x=0$

$\therefore f(x)=|x|+\frac{|x|}{x}$ is discontinuous at $x=0$.

30. (b)

$$\lim_{x\to 0^+}\frac{x(e^x-1)}{|\tan x|}=\lim_{x\to 0^+}\frac{x(e^x-1)}{\tan x}=0$$

$$\lim_{x\to 0^-}\frac{x(e^x-1)}{|\tan x|}=-\lim_{x\to 0^-}\frac{x(e^x-1)}{\tan x}=0$$

So f(x) is continuous at x = 0.

Now $\text{L.H.D.}=\lim_{x\to 0^-}\frac{\frac{x(e^x-1)}{|\tan x|}-0}{x-0}=-\lim_{x\to 0^-}\frac{x}{\tan x}\times\frac{e^x-1}{x}=-1$

$$\text{R.H.D.}=\lim_{x\to 0^+}\frac{\frac{x(e^x-1)}{|\tan x|}-0}{x-0}=\lim_{x\to 0^-}\frac{x}{\tan x}\times\frac{e^x-1}{x}=1$$

$\text{L.H.D.}\neq\text{R.H.D.}$

F(x) is continuous but not differentiable at $x = 0$

31. (a)

We have, $f(x)=\dfrac{x}{1+|x|}=\begin{cases}\dfrac{x}{1+x} & , \quad x>0 \\ 0 & , \quad x=0 \; ; \\ \dfrac{x}{1-x} & , \quad x<0\end{cases}$

$$\text{L.H.D.}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{\frac{-h}{1+h}-0}{-h}=1$$

$$\text{R.H.D.}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\frac{\lim_{h\to 0}\frac{h}{1+h}-0}{h}=\lim_{h\to 0}\frac{1}{1+h}=1$$

So, f(x) is differentiable at x = 0; Also f(x) is differentiable at all other points.

Hence, f(x) is everywhere differentiable.

32. (b)

Let $f(x)=|x-1|+|x-3|=\begin{cases}-(x-1)-(x-3) & , \quad x<1 \\ (x-1)-(x-3) & , \quad 1\le x<3 \\ (x-1)+(x-3) & , \quad x\ge 3\end{cases}=\begin{cases}-2x+4 & , \quad x<1 \\ 2 & , \quad 1\le x<3 \\ 2x-4 & , \quad x\ge 3\end{cases}$

Since, $f(x)=2$ for $1\le x<3$. Therefore $f'(x)=0$ for all $x\in(1,3)$.

Hence, $f'(x)=0$ at $x=2$.

33. (a)

We have, $\lim_{x\to 0}f(x)=\lim_{x\to 0}\dfrac{\sin x^2}{x}=\lim_{x\to 0}\left(\dfrac{\sin x^2}{x^2}\right)x=1\times 0=0=f(0)$

So, f(x) is continuous at x = 0,

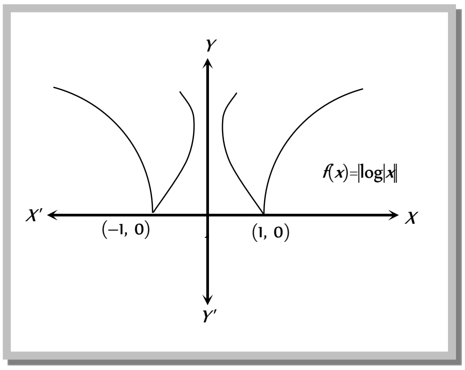
$f(x)$ is also derivable at

x = 0, because $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\dfrac{\sin x^2}{x^2}=1$

exists finitely.

34. (b)

It is evident from the graph of $f(x)=|\log|x||$ than $f(x)$ is everywhere continuous but not differentiable at $x=\pm 1$.



35. (a)

$f(x)=[x]\sin(\pi x)$

If x is just less than k, [x] = k – 1. $\therefore$ $f(x)=(k-1)\sin(\pi x)$, when $x<k \;\; \forall\; k\in I$

Now L.H.D. at $x=k$,

$$=\lim_{x\to k}\frac{(k-1)\sin(\pi x)-k\sin(\pi k)}{x-k}=\lim_{x\to k}\frac{(k-1)\sin(\pi x)}{(x-k)} \quad [\text{as } \sin(\pi k)=0 \quad k\in I]$$

$= \lim_{h\to 0} \frac{(k-1)\sin(\pi(k-h))}{-h}$ [Let $x=(k-h)$]

$= \lim_{h\to 0} \frac{(k-1)(-1)^{k-1}\sin h\pi}{-h}$

$= \lim_{h\to 0} (k-1)(-1)^{k-1} \frac{\sin h\pi}{h\pi} \times (-\pi)$

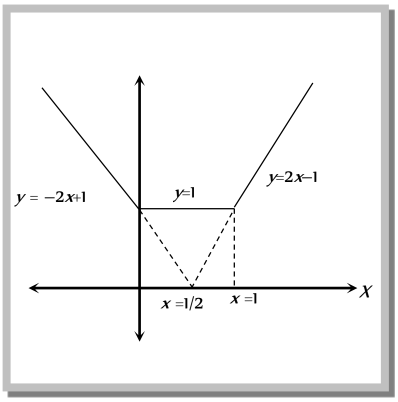
$= (k-1)(-1)^k \pi = (-1)^k (k-1)\pi$.

36. (a)

We have, $f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x<0 \\ 1, & 0 \le x < 1 \\ 2x-1, & x \ge 1 \end{cases}$

Since, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1, \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 1$ and $f(1) = 2\times 1 - 1 = 1$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$. So, f(x) is continuous at *x* = 1.



Now, $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{1-1}{-h} = 0$ and

$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{2(1+h)-1-1}{h} = 2$.

$\therefore$ (LHD at x = 1) $\neq$ (RHD atx = 1).

So, *f*(*x*) is not differentiable at *x* = 1.

Alternately

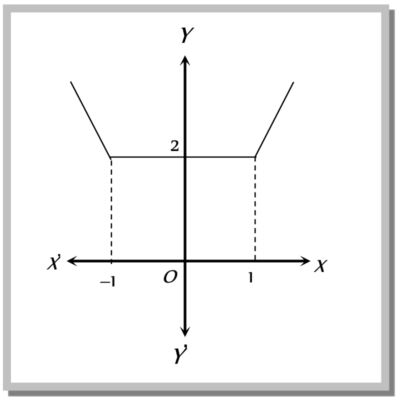
By graph, it is clear that the function is not differentiable at x = 0, 1 as there it has sharp edges.

37. (c)

Here $f(x) = |x-1| + |x+1| \Rightarrow f(x) = \begin{cases} 2x, & \text{when } x > 1 \\ 2, & \text{when } -1 \le x \le 1 \\ -2x, & \text{when } x < -1 \end{cases}$

Alternate

The graph of the function is shown alongside,



From the graph it is clear that the function is continuous at all real *x*, also differentiable at all real *x* except at $x = \pm 1$; Since sharp edges at $x = -1$ and $x = 1$.

At $x = 1$ we see that the slope from the right *i.e.*, R.H.D. = 2, while slope from the left *i.e.*, L.H.D.= 0

Similarly, at $x = -1$ it is clear that R.H.D. = 0 while L.H.D. = – 2

Here, $f'(x) = \begin{cases} -2, & x < -1 \\ 0, & -1 < x < 1 \\ 2, & x > 1 \end{cases}$ (No equality on –1 and +1)

Now, at $x = 1$, $f'(1^+) = 2$ while $f'(1^-) = 0$ and

at $x = -1$, $f'(-1^+) = 0$ while $f'(-1^-) = -2$

Thus, f(x) is not differentiable at $x = \pm 1$.

38. (d)

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^{-1}} (ax^2 + bx + 2) = a - b + 2$ and

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (bx^2 + ax + 4) = b - a + 4$

For continuity $a - b + 2 = b - a + 4 \Rightarrow a - b = 1 \ldots (i)$

Now $f'(x) = \begin{cases} 2ax + b & , \quad x < -1 \\ 2bx + a & , \quad x > -1 \end{cases} \Rightarrow \text{R.H.D.} = -2a + b \;\&\; \text{L.H.D.} = -2b + a$

For differentiability $-2a + b = -2b + a \Rightarrow a = b \ldots (ii)$

From (i) & (ii) no value of (a, b) is possible.

39. (b)

$h(x) = e^{(f(x))^3 + (g(x))^3 + x} \Rightarrow h'(x) = e^{(f(x))^3 + (g(x))^3 + x}\left(3(f(x))^2 f'(x) + 3(g(x))^2 g'(x) + 1\right)$

$\Rightarrow h'(x) = h(x)\left(3(f(x))^2 \dfrac{g(x)}{f(x)} - 3(g(x))^2 \dfrac{f(x)}{g(x)} + 1\right)$

$\Rightarrow h'(x) = h(x) \Rightarrow h(x) = e^{x+c}$

Now $h(5) = e^6 \Rightarrow h(x) = e^{x+1}$

Hence $h(10) = e^{11}$

40. (c)

$[2+h] = 2, [2-h] = 1, [1+h] = 1, [1-h] = 0$

At *x* = 2, we will check $\text{RHL} = \text{LHL} = f(2)$

$\text{RHL} = \lim\limits_{h\to 0} |4 + 2h - 3|[2+h] = 2, f(2) = 1.2 = 2$

$\text{LHL} = \lim\limits_{h\to 0} |4 - 2h - 3|[2-h] = 1, R \neq L$, $\therefore$ not continuous

At $x = 1, \text{RHL} = \lim |2 + 2h - 3|[1+h] = 1.1 = 1,$

$f(1) = |-1|[1] = 1$

$\text{LHL} = \lim\limits_{h\to 0} \sin\dfrac{\pi}{2}(1-h) = 1$

continuous at x = 1

$\text{R.H.D.} = \lim\limits_{h\to 0} \dfrac{|2 + 2h - 3|[1+h] - 1}{h} = \lim\limits_{h\to 0} \dfrac{|-1|.1 - 1}{h} = \lim\limits_{h\to 0} \dfrac{1-1}{h} = 0$

$\text{L.H.D.} = \lim\limits_{h\to 0} \dfrac{|2 - 2h - 3|[1-h] - 1}{-h} = \lim\limits_{h\to 0} \dfrac{1.0 - 1}{-h} = \lim\limits_{h\to 0} \dfrac{1}{h} = \infty$

Since R.H.D. $\neq$ L.H.D. $\therefore$ not differentiable. at x = 1.

41. (b)

Clearly, $f(x)$ is differentiable for all non-zero values of *x*,

For $x \neq 0$, we have $f'(x) = \dfrac{xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$

Now, (L.H.D. at *x* = 0)

$= \lim_{x \to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim_{h \to 0} \dfrac{f(0-h)-f(0)}{-h}$

$= \lim_{h \to 0} \dfrac{\sqrt{1-e^{-h^2}}}{-h} = \lim_{h \to 0} -\dfrac{\sqrt{1-e^{-h^2}}}{h}$

$= -\lim_{h \to 0} \sqrt{\dfrac{e^{h2}-1}{h^2}} \times \dfrac{1}{\sqrt{e^{h^2}}} = -1$

and, (RHD at *x* = 0) = $\lim_{x \to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim_{h \to 0} \dfrac{\sqrt{1-e^{-h^2}}-0}{h}$

$= \lim_{h \to 0} \sqrt{\dfrac{e^{h^2}-1}{h^2}} \times \dfrac{1}{\sqrt{e^{h^2}}} = 1$.

So, $f(x)$ is not differentiable at $x = 0$,

Hence, the points of differentiability of $f(x)$ are $(-\infty, 0) \cup (0, \infty)$

42. (a)

We have, $f(x) = \begin{cases} e^{\sin x}, -\dfrac{\pi}{2} \leq x < 0 \\ e^{-\sin x}, 0 \leq x \leq \dfrac{\pi}{2} \end{cases}$

Clearly, $f(x)$ is continuous and differentiable for all non-zero *x*.

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} e^{\sin x} = 1$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} e^{-\sin x} = 1$

Also, $f(0) = e^0 = 1$

So, f(x) is continuous for all *x*.

(LHD at x = 0) = $\left(\dfrac{d}{dx}(e^x)\right)_{x=0} = (e^x)_{x=0} = e^0 = 1$

(RHD at $x = 0$) = $\left(\dfrac{d}{dx}(e^{-x})\right)_{x=0} = (-e^{-x})_{x=0} = -1$

So, f(x) is not differentiable at x = 0.

43. (b)

We have, $f(x) = \sqrt{1-\sqrt{1-x^2}}$. The domain of definition of $f(x)$ is [–1, 1].

For $x \neq 0, x \neq 1, \quad x \neq -1$ we have $f'(x) = \dfrac{1}{\sqrt{1-\sqrt{1-x^2}}} \times \dfrac{x}{\sqrt{1-x^2}}$

Since $f(x)$ is not defined on the right side of $x = 1$ and on the left side of $x = -1$.

Also, $f'(x) \to \infty$ when $x \to -1^+$ or $x \to 1^-$.

So, we check the differentiability at $x = 0$.

Now, (LHD at $x=0$) = $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-h^2}}-0}{-h} = -\lim_{h\to 0}\frac{\sqrt{1-\{1-(1/2)h^2+(3/8)h^4+....\}}}{h}$

$= -\lim_{h\to 0}\sqrt{\frac{1}{2}-\frac{3}{8}h^2+.....} = -\frac{1}{\sqrt{2}}$

Similarly, (RHD at $x = 0$) = $\frac{1}{\sqrt{2}}$

Hence, $f(x)$ is not differentiable at $x = 0$.

44. (d) Since $f(x)$ is differentiable at $x=c$, therefore it is continuous at $x=c$.

Hence, $\lim_{x\to c} f(x) = f(c)$.

45. (c)

$(x^2-3x+2)=(x-1)(x-2)>0$ When $x<1$ or >2,

And $(x^2-3x+2)=(x-1)(x-2)<0$ when $1\le x\le 2$

Also $\cos|x| = \cos x$

$\therefore\ f(x) = -(x^2-4)(x^2-3x+2)+\cos x,\ \ 1\le x\le 2$

and $f(x)=(x^2-4)(x^2-3x+2)+\cos x,\ \ x<1 \text{ or } x>2$

Evidently $f(x)$ is not differentiable at x = 1.

46. (b)

$f(0)=0$ and $f(x)=x^2 e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}$

R.H.L. = $\lim_{h\to 0}(0+h)^2 e^{-2/h} = \lim_{h\to 0}\frac{h^2}{e^{2/h}} = 0$

L.H.L. = $\lim_{h\to 0}(0-h)^2 e^{-\left(\frac{1}{h}-\frac{1}{h}\right)} = 0$

$\therefore$ f(x) is continuous at x = 0.

R.H.D. at $(x=0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2 e^{-2/h}}{h} = he^{-2/h} = 0$

L.H.D. at $(x=0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{h^2 e^{-\left(\frac{1}{h}-\frac{1}{h}\right)}}{-h} = \lim_{h\to 0}(-h) = 0$

F(x) is differentiable at x = 0

47. (d)

$\lim_{x\to 0} f(x) = x^3\sin^2\left(\frac{1}{x}\right) = 0$ as $0\le \sin^2\left(\frac{1}{x}\right)\le 1$ and $x\to 0$

Therefore $f(x)$ is continuous at $x=0$.

Also, the function $f(x)=x^3\sin^2\frac{1}{x}$ is differentiable because

$RHD = \lim_{h\to 0}\frac{h^3\sin^2\frac{1}{h}-0}{h} = \lim_{h\to 0} h^2\sin^2\frac{1}{h} = 0$, $LHD = \lim_{h\to 0}\frac{h^3\sin\left(\frac{1}{-h}\right)}{-h} = 0$.

48. (b)

49. (d)

50. (c)

$\lim_{h\to 0^-} 1+(2-h)=3$, $\lim_{h\to 0^+} 5-(2+h)=3$, $f(2)=3$

Hence, *f* is continuous at $x=2$

Now $RHD=\lim_{h\to 0}\frac{5-(2+h)-3}{h}=-1$

$LHD=\lim_{h\to 0}\frac{1+(2-h)-3}{-h}=1$

$\therefore$ f(x) is not differentiable at $x=2$.

51. (c)

$g(x)=|f(|x|)|\geq 0$. So $g(x)$ cannot be onto.

If $f(x)$ is one-one and $f(x_1)=-f(x_2)$ then $g(x_1)=g(x_2)$.

So, ‘ $f(x)$ is one-one’ does not ensure that $g(x)$ is one-one.

If f(x) is continuous for $x\in R$, $|f(|x|)|$ is also continuous for $x\in R$.

So the answer (c) is correct.

The fourth answer (d) is not correct as f(x) being differentiable does not ensure |f(x)| being differentiable.

52. (b)

Given $f(4)=6, f'(4)=1$

$\therefore \lim_{x\to 4}\frac{xf(4)-4f(x)}{x-4}=\lim_{x\to 4}\frac{xf(4)-4f(4)+4f(4)-4f(x)}{x-4}$

$=\lim_{x\to 4}\frac{(x-4)f(4)}{x-4}-4\lim_{x\to 2}\frac{f(x)-f(4)}{x-4}$

$= f(4)-2f'(4)=4$

53. (c)

$f(x+2y)=2f(x)f(y)\Rightarrow 2f'(x+2y)=2f(x)f'(y)$ {partially differentiating w.r.to y}

For x = 5 & y = 0, $f'(5)=f(5)f'(0)\Rightarrow f'(5)=6$

54. (c)

By L’hospital’s rule

$\lim_{x\to 2}\frac{g^2(x)f^2(2)-f^2(x)g^2(2)}{x^2-4}=\lim_{x\to 2}\frac{g(x)g'(x)f^2(2)-f(x)f'(x)g^2(2)}{x}$

$=\frac{(-1)\times 4\times 9-3\times(-2)\times 1}{2}=-15$

55. (b)

Given $5f(2x)+3f\left(\frac{2}{x}\right)=2x+2$ ......(i)

Replacing *x* by $\frac{1}{x}$ in (i), $5f\left(\frac{2}{x}\right)+3f(2x)=\frac{2}{x}+2$ ......(ii)

On solving equation (i) and (ii), we get, $8f(2x)=5x-\frac{3}{x}+2$,

$\Rightarrow 8f(x)=\frac{5x}{2}-\frac{6}{x}+2$

$\therefore\ 8f'(x)=\frac{5}{2}+\frac{6}{x^2}$

$\because\ y=xf(x)\ \Rightarrow\ \frac{dy}{dx}=f(x)+xf'(x)$

$=\frac{1}{8}\left(\frac{5x}{2}-\frac{6}{x}+2\right)+\frac{x}{8}\left(\frac{5}{2}+\frac{6}{x^2}\right)$

at $x=1$, $\frac{dy}{dx}=\frac{1}{8}\left(\frac{5}{2}-6+2\right)+\frac{1}{8}\left(\frac{5}{2}+6\right)=\frac{7}{8}$

56. (d)

$f(x)=\begin{cases}x^3-1, & x\ge 1\\ 1-x^3, & x<1\end{cases}$ and $f'(x)=\begin{cases}3x^2, & x\ge 1\\ -3x^2, & x<1\end{cases}$

$f'(1^+)=3,\ f'(1^-)=-3$

57. (b)

$f(x)=\sin 2x.\cos 2x.\cos 3x+\log_2 2^{x+3}$,

$\Rightarrow f(x)=\frac{1}{2}\sin 4x\cos 3x+(x+3)\log_2 2$,

$\Rightarrow f(x)=\frac{1}{4}[\sin 7x+\sin x]+x+3$

Differentiate w.r.t. x,

$f'(x)=\frac{1}{4}[7\cos 7x+\cos x]+1$,

$\Rightarrow f'(\pi)=-2+1=-1$.

58. (b) In neighborhood of $x=\frac{3\pi}{4}$, $|\cos^3 x|=-\cos^3 x$ and $|\sin^3 x|=\sin^3 x$

$\therefore\ y=-\cos^3 x+\sin^3 x$

$\therefore\ \frac{dy}{dx}=3\cos^2 x\sin x+3\sin^2 x\cos x$

At $x=\frac{3\pi}{4}$, $\frac{dy}{dx}=3\cos^2\frac{3\pi}{4}\sin\frac{3\pi}{4}+3\sin^2\frac{3\pi}{4}\cos\frac{3\pi}{4}=0$.

59. (b)

$$f(x) = \log_x(\log x) = \frac{\log(\log x)}{\log x}$$

$$\Rightarrow f'(x) = \frac{\frac{1}{x} - \frac{1}{x}\log(\log x)}{(\log x)^2}$$

$$\Rightarrow f'(e) = \frac{\frac{1}{e} - 0}{1} = \frac{1}{e}$$

60. (d)

$$f(x) = |\log x| = \begin{cases} -\log x, & \text{if } 0 < x < 1 \\ \log x, & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{1}{x}, & \text{if } 0 < x < 1 \\ \frac{1}{x}, & \text{if } x > 1 \end{cases}.$$

Clearly $f'(1^-) = -1$ and $f'(1^+) = 1$,

$\therefore$ $f'(x)$ does not exist at x = 1

61. (c)

Let $y = \left[\log\left\{e^x\left(\frac{x-1}{x+1}\right)\right\}\right] = \log e^x + \log\left(\frac{x-1}{x+1}\right)$

$$\Rightarrow y = x + [\log(x-1) - \log(x+1)]$$

$$\Rightarrow \frac{dy}{dx} = 1 + \left[\frac{1}{x-1} - \frac{1}{x+1}\right] = 1 + \frac{2}{(x^2-1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+1}{x^2-1}.$$

62. (a)

$$x = \exp\left\{\tan^{-1}\left(\frac{y-x}{x}\right)\right\} \Rightarrow \log x = \tan^{-1}\left(\frac{y-x}{x}\right)$$

$$\Rightarrow \frac{y-x}{x} = \tan(\log x) \Rightarrow y = x\tan(\log x) + x$$

$$\Rightarrow \frac{dy}{dx} = \tan(\log x) + x\frac{\sec^2(\log x)}{x} + 1$$

$$\Rightarrow \frac{dy}{dx} = \tan(\log x) + \sec^2(\log x) + 1$$

At x = 1, $\frac{dy}{dx} = 2$.

63. (a)

$$y=\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)=\frac{\pi}{2}$$

$$\Rightarrow \frac{dy}{dx}=0$$

64. (d)

$$\frac{d}{dx}\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right]=\frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4}-x\right)\right]=-1 .$$

65. (b)

Let $y=\sin^2\left(\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

Put $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

$$\Rightarrow y=\sin^2\left(\cot^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)=\sin^2\left(\cot^{-1}\left(\tan\frac{\theta}{2}\right)\right)$$

$$\Rightarrow y=\sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos^2\frac{\theta}{2}=\frac{1}{2}(1+\cos\theta)=\frac{1}{2}(1+x)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}$$

66. (a)

Let $\cos\alpha=\frac{5}{13}$. Then $\sin\alpha=\frac{12}{13}$. So, $y=\cos^{-1}\{\cos\alpha.\cos x-\sin\alpha.\sin x\}$

$\therefore y=\cos^{-1}\{\cos(x+\alpha)\}=x+\alpha$ ($\because$ $x+\alpha$ is in the first or the second quadrant)

$$\therefore \frac{dy}{dx}=1 .$$

67. (c)

$$y\left(\frac{\tan^2 2x-\tan^2 x}{1-\tan^2 2x\tan^2 x}\right)\cot 3x=\left(\frac{\tan 2x-\tan x}{1+\tan 2x\tan x}\right)\left(\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}\right)\cot 3x$$

$$\Rightarrow y=\tan x\tan 3x\cot 3x=\tan x$$

$$\Rightarrow \frac{dy}{dx}=\sec^2 x$$

68. (a)

$$f(x)=\cot^{-1}\left(\frac{x^x-x^{-x}}{2}\right)$$

Put $x^x=\tan\theta$, $\therefore$ $y=f(x)=\cot^{-1}\left(\frac{\tan^2\theta-1}{2\tan\theta}\right)$

$$=\cot^{-1}(-\cot 2\theta)=\pi-\cot^{-1}(\cot 2\theta)$$

$$\Rightarrow y=\pi-2\theta=\pi-2\tan^{-1}(x^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^{2x}}.x^x(1+\log x)$$

$$\Rightarrow f'(1) = -1 .$$

69. (a)

$$y = \frac{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)}{1-x} = \frac{1-x^{16}}{1-x}$$

$$\therefore \frac{dy}{dx} = \frac{-16x^{15}(1-x)+1-x^{16}}{(1-x)^2}, \quad \therefore \text{ At } x=0, \ \frac{dy}{dx} = 1.$$

70. (c)

$$f(x) = \frac{2\sin x.\cos x.\cos 2x.\cos 4x}{2\sin x} = \frac{\sin 8x}{8\sin x}$$

$$\therefore f'(x) = \frac{1}{8}.\frac{8\cos 8x.\sin x - \cos x.\sin 8x}{\sin^2 x}$$

$$\therefore f'\left(\frac{\pi}{4}\right) = 0.$$

71. (a)

$$xe^{x+y} = y + 2\sin x \Rightarrow e^{x+y} + xe^{x+y}(1+y') = y' + 2\cos x$$

Now x = 0 gives y = 0, hence $\frac{dy}{dx} = -1$.

72. (a)

$$\sin(3x-2y) = \log(3x-2y) \Rightarrow \left(3 - 2\frac{dy}{dx}\right)\cos(3x-2y) = \left(3 - 2\frac{dy}{dx}\right)\frac{1}{3x-2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2}$$

73. (c)

$$x^4y^5 = 2(x+y)^9 \Rightarrow 4x^3y^5 + 5x^4y^4\frac{dy}{dx} = 18(x+y)^8\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow 4\frac{2(x+y)^9}{x} + 5\frac{2(x+y)^9}{y}\frac{dy}{dx} = 18(x+y)^8\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{4}{x} - \frac{9}{x+y} = \left(\frac{9}{x+y} - \frac{5}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

74. (b)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{a[\cos\theta - \theta(-\sin\theta) - \cos\theta]}{a[-\sin\theta + \theta\cos\theta + \sin\theta]} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta .$$

75. (d)

Obviously $x=\cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y=\sin^{-1}\frac{t}{\sqrt{1+t^2}}$

$\Rightarrow x=\tan^{-1}t$ and $y=\tan^{-1}t$

$\Rightarrow y=x \Rightarrow \frac{dy}{dx}=1$.

76. (c)

$x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$

Put $t=\tan\theta$ in both the equations to get

$x=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$ and $y=\frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$.

Differentiating both the equations, we get $\frac{dx}{d\theta}=-2\sin 2\theta$ and $\frac{dy}{d\theta}=2\cos 2\theta$.

Therefore $\frac{dy}{dx}=-\frac{\cos 2\theta}{\sin 2\theta}=-\frac{x}{y}$.

77. (d)

$y=\sqrt{x+1+\sqrt{x+1+\sqrt{x+1...\text{to }\infty}}} \Rightarrow y=\sqrt{x+1+y}$

$\Rightarrow y^2=x+y+1 \Rightarrow 2y\frac{dy}{dx}=1+\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2y-1)=1 \Rightarrow \frac{dy}{dx}=\frac{1}{2y-1}$

78. (b)

$y=(x+1)^{(x+1)^{(x+1)......\infty}} \Rightarrow y=(x+1)^y$

$\Rightarrow \log_e y=y\log_e(x+1)$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=\frac{y}{(x+1)}+\ln(x+1)\frac{dy}{dx}$

$\Rightarrow \left(\frac{1}{y}-\ln(x+1)\right)\frac{dy}{dx}=\frac{y}{x+1}$

$\Rightarrow (x+1)(1-\ln y)\frac{dy}{dx}=y^2$

79. (a)

$y=x^2+\frac{2}{y} \Rightarrow y^2=x^2y+2$

$\Rightarrow 2y\frac{dy}{dx}=y.2x+x^2\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{2xy}{2y-x^2}$

80. (c)

$x = e^{2y+x}$

Taking log both sides, $\log x = (2y+x)\log e = 2y+x$

$\Rightarrow 2y + x = \log x \Rightarrow 2\frac{dy}{dx} + 1 = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1-x}{2x}$

CONTINUITY & DIFFERENTIABILITY
EXERCISE 2(A)

More than one options may be correct

Q.1 If $f(x)=\begin{cases}\dfrac{x\cdot \ell n(\cos x)}{\ell n(1+x^2)} & x\neq 0\\ 0 & x=0\end{cases}$ then :

(A) f is continuous at x = 0
(B) f is continuous at x = 0 but not differentiable at x = 0
(C) f is differentiable at x = 0
(D) f is not continuous at x = 0

Sol. **[A, C]**

$$\Rightarrow f'(0^+)=\lim_{h\to 0}\frac{h\ln(\cos h)}{h\ln(1+h^2)}=\lim_{h\to 0}\frac{\ln(\cos h)^{\frac{1}{h^2}}}{\frac{\ln(1+h^2)}{h^2}}$$

$$\Rightarrow \lim_{h\to 0}\frac{1}{h^2}(\cos h-1)=-\frac{1}{2}$$

$\Rightarrow$ Paralally $f'(0^-)=-\dfrac{1}{2}$

Hence f is continuous and derivable at x = 0

Q.2 Given that the derivative f ' (a) exists. Indicate which of the following statement(s) is/are always true.

(A) $f'(x)=\lim\limits_{h\to a}\dfrac{f(h)-f(a)}{h-a}$
(B) $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a)-f(a-h)}{h}$
(C) $f'(a)=\lim\limits_{t\to 0}\dfrac{f(a+2t)-f(a)}{t}$
(D) $f'(a)=\lim\limits_{t\to 0}\dfrac{f(a+2t)-f(a+t)}{2t}$

Sol. **[A, B]**

$\Rightarrow$ (C) is false and is True only if f ' (a) = 0 limit is 2f ' (a). In (D) same logic limit is $\dfrac{1}{2}f'(a)$

Q.3 Let [x] denote the greatest integer less than or equal to x. If $f(x)=[x\sin \pi x]$, then f (x) is:

(A) continuous at x = 0
(B) continuous in (– 1, 0)
(C) differentiable at x = 1
(D) differentiable in (– 1, 1)

Sol. **[A, B, D]**

$$\Rightarrow f(x)=\begin{bmatrix}0 & 0<x<1\\ 0 & x=0 \text{ or } 1 \text{ or } -1\\ 0 & -1<x<0\end{bmatrix}$$

$\Rightarrow$ f (x) = 0 for all in [– 1, 1]

Q.4 The function, $f(x)=[|x|]-|[x]|$ where [x] denotes greatest integer function

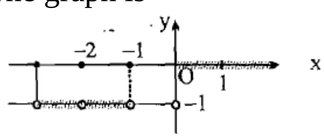
(A) is continuous for all positive integers
(B) is continuous for all non positive integers
(C) has finite number of elements in its range
(D) is such that its graph does not lie above the x – axis.

Sol. **[A, B, C, D]**

$$\Rightarrow [|x|]-|[x]| = \begin{cases} 0 & x=-1 \\ -1 & -1<x<0 \\ 0 & 0\le x\le 1 \\ 0 & 1<x\le 2 \end{cases}$$

$\Rightarrow$ range is {0, – 1}

The graph is



Q.5 Let f (x + y) = f (x) + f (y) for all $x, y \in R$. Then:

(A) f (x) must be continuous $\forall x \in R$
(B) f (x) may be continuous $\forall x \in R$
(C) f (x) must be discontinuous $\forall x \in R$
(D) f (x) may be discontinuous $\forall x \in R$

Sol. **[B, D]**

$\Rightarrow \lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x) + f(h)$

$\Rightarrow f(x) + \lim_{h\to 0} f(h)$

Hence if $h \to 0$

$\Rightarrow$ f (h) = 0

$\Rightarrow$ 'f' is continuous otherwise discontinuous

Q.6 The function $f(x) = \sqrt{1-\sqrt{1-x^2}}$

(A) has its domain $-1 \le x \le 1$.
(B) has finite one sided derivates at the point x = 0.
(C) is continuous and differentiable at x = 0.
(D) is continuous but not differentiable at x = 0.

Sol. **[A, B, D]**

$$\Rightarrow f'(0^+) = \frac{1}{\sqrt{2}}; f'(0^-) = -\frac{1}{\sqrt{2}}; f(x) = \frac{\sqrt{x^2}}{\sqrt{1+\sqrt{1-x^2}}} = \frac{|x|}{\sqrt{1+\sqrt{1-x^2}}}$$

Q.7 Consider the function *f(x)* = | x^3 + 1 | then

(A) Domain of f $x \in R$
(B) Range of f is R^+
(C)f has no inverse.
(D)f is continuous and differentiable for every $x \in R$.

Sol. **[A, C]**

Range is $R^+ \cup \{0\}$ $\Rightarrow$ B is not correct

f is not differentiable at x = – 1

$$\Rightarrow \text{ as } f(x) = \begin{cases} x^3+1 & \text{if } x \ge -1 \\ -(x^3+1) & \text{if } x < -1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 3x^2 & \text{if } x > -1 \\ -3x^2 & \text{if } x < -1 \end{cases}$$

$$\Rightarrow f'(-1^+) = 3;$$

$$\Rightarrow f'(-1^-) = -3$$

f is not differentiable at x = – 1

also since f is not bijective hence it has no inverse

Q.8 Let $f(x) = \frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}-1}.x$ then:

(A) f ’ (10) = 1

(B) $f'\left(\frac{3}{2}\right) = -1$

(C) domain of f (x) is $x \ge 1$

(D) none

Sol. **[A, B]**

$$\Rightarrow f(x) = \frac{\sqrt{\left(\sqrt{x-1}\right)^2+1-2\sqrt{x-1}}}{\sqrt{x-1}-1}\cdot x = \frac{\left|\sqrt{x-1}-1\right|}{\sqrt{x-1}-1}\cdot x = \begin{bmatrix} -x & \text{if } x \in [1,2) \\ x & \text{if } x \in (2,\infty) \end{bmatrix}$$

Q.9 f is a continuous function in [a, b]; g is a continuous function in [b, c]

A function h (x) is defined as

h (x) = f (x) for $x \in [a,b)$

= g (x) for $x \in (b,c]$

If f (b) = g (b), then

(A) h (x) has a removable discontinuity at x=b.

(B) h (x) may or may not be continuous in [a, c]

(C) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$

(D) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

Sol. **[A, C]**

Given f is continuous in [b, c] ……..(1)

g is continuous in [b, c] ………(2)

f (b) = g (b) ………(3)

$$\Rightarrow h(x) = \left.\begin{array}{ll} f(x) & \text{for } x \in [a,b) \\ = f(b) = g(b) & \text{for } x = b \\ = g(x) & \text{for } x \in (b,c] \end{array}\right\} \quad \ldots\ldots(4)$$

$\Rightarrow$ h (x) is continuous in $[a,b) \cup (b,c]$ [using (1), (2)] also $f\left(b^-\right) = f(b)$; $g(b^+) = g(b)$ ....(5) [using (1), (2)]

$\Rightarrow \therefore\ h\left(b^-\right) = f\left(b^-\right) = f(b) = g(b) = g\left(b^+\right) = h\left(b^+\right)$

$\Rightarrow$ now, verify each alternative. Of course! $g(b^-)$ and $f(b^+)$ are undefined.

$h(b^-) = f\left(b^-\right) = f(b) = g(b) = g(b^+)$

$\Rightarrow$ and $h(b^+) = g(b^+) = g(b) = f(b) = f\left(b^-\right)$

$\Rightarrow$ hence $h(b^-) = h(b^+) = f(b) = g(b)$

$\Rightarrow$ and h (b) is not defined

Q.10 The function $f(x) = \begin{cases} |x-3| & , \ x \geq 1 \\ \left(\frac{x^2}{4}\right) - \left(\frac{3x}{2}\right) + \left(\frac{13}{4}\right) & , \ x < 1 \end{cases}$ is :

(A) continuous at x = 1 (B) differentiable at x =1

(C) continuous at x = 3 (D) differentiable at x = 3

Sol. **[A, B, C]**

$$\Rightarrow f(x) = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } 1 \leq x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & \text{if } x < 1 \end{cases}$$

$$\Rightarrow f'\left(1^+\right) = \lim_{h\to 0} it \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h\to 0} it \frac{3-(1+h)-2}{h} = -1$$

$$\Rightarrow f'\left(1^-\right) = \lim_{h\to 0} it \frac{\frac{(1-h)^2}{4} - \frac{3}{2}(1-h) + \frac{13}{4} - 2}{-h}$$

$$\Rightarrow \lim_{h\to 0} it \frac{(1-h)^2 - 6(1-h) + 5}{-4h}$$

$$\Rightarrow \lim_{h\to 0} it \frac{h^2 - 2h + 6h}{-4h} = -1$$

$\Rightarrow$ f' is continuous at x = 1

Q.11 Which of the following statements are true?

(A) If $xe^{xy} = y + \sin x$, then at y I (0) = 1.

(B)If $f(x) = a_o x^{2m+} 1 + a_l x^{2m} + a_3 X^{2m-1} + ... + a_{2m}+1 = 0$ $(a_0 \neq 0)$ is a polynomial equation with rational co-efficients then the equation f''(x) = 0 must have a real root.($m \in N$).

(C) If (x – r) is a factor of the polynomial $f(x) = a_n x'' + a_{n\_1} x^{n-I} + a_{n-2} x^{n-2} + ... + a_0$ repeated m times where $1 \leq m \leq n$ then r is a root of the equation f' (x) = 0 repeated (m – 1) times.

(D) If $y = \sin^{-1}(\cos \sin^{-1} x) + \cos^{-1}(\sin \cos^{-1} x)$ then $\frac{dy}{dx}$ is independent on x.

Sol. [A, C, D]

[D] Let $\sin^{-1} x = t$

$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - t$

$\Rightarrow y = \sin^{-1}(\cos t) + \cos^{-1}\left(\sin\left(\frac{\pi}{2} - t\right)\right) = \sin^{-1}(\cos t) + \cos^{-1}(\cos t)$

$\Rightarrow y = \frac{\pi}{2}$

$\Rightarrow \frac{dy}{dx} = 0$

Q.12 Let $y = \sqrt{x + \sqrt{x + \sqrt{+..........\infty}}}$ then $\frac{dy}{dx} =$

(A) $\frac{1}{2y-1}$ (B) $\frac{x}{x+2y}$ (C) $\frac{1}{\sqrt{1+4x}}$ (D) $\frac{y}{2x+y}$

Sol. [A, C, D]

$\Rightarrow y^2 = x + y$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$

also $y = \frac{x}{y} + 1$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2x+y}$

make a quadratic in y to get explicit function

Q.13 If $\sqrt{y+x} + \sqrt{y-x} = c$ (where $c \neq 0$), then $\frac{dy}{dx}$ has the value equal to

(A) $\frac{2x}{c^2}$ (B) $\frac{x}{y+\sqrt{y^2-x^2}}$ (C) $\frac{y\sqrt{y^2-x^2}}{x}$ (D) $\frac{c^2}{2y}$

Sol. [A, B, C]

$\Rightarrow$ Square both sides, differentiate and rationalize

Q.14 If $f(x) = \cos\left[\frac{\pi}{x}\right] \cos\left(\frac{\pi}{2}(x-1)\right)$; where $[x]$ is the greatest integer function of x, then f(x) is continuous at

(A) x = 0 (B) x = 1 (C) x = 2 (D) none of these

Sol. [B, C]

$\Rightarrow$ (A) = Not defined at x = 0;

$\Rightarrow$ (B) = f (1) = cos 3; f (2) = 0 and both the limits exist

Q.15 Select the correct statements.

(A) The function f defined by f (x) = $f(x)=\begin{bmatrix} 2x^2+3 & \text{for } x\le 1 \\ 3x+2 & \text{for } x>1 \end{bmatrix}$ is neither differentiable nor continuous at x = 1

(B) The function $f(x)=x^2|x|$ is twice differentiable at x = 0.

(C) If f is continuous at x = 5 and f (5) = 2 then $\lim_{x\to 2} f(4x^2-11)$ exists.

(D) If $\lim_{x\to a}(f(x)+g(x))=2$ and $\lim_{x\to a}(f(x)-g(x))=1$ then $\lim_{x\to a} f(x)\cdot g(x)$ need not exist.

Sol. **[B, C]**

Q.16 Which of the following functions has/have removable discontinuity at x = 1.

(A) $f(x)=\frac{1}{\ell n|x|}$

(B) $f(x)=\frac{x^2-1}{x^3-1}$

(C) $f(x)=2^{-2^{\left(\frac{1}{1-x}\right)}}$

(D) $f(x)=\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

Sol. **[B, D]**

(A) $\lim_{x\to 1} f(x)$ does not exist

(B) $\lim_{x\to 1} f(x)=\frac{2}{3}$ $\therefore$ f (x) has removable discontinuity at x = 1

(C) $\lim_{x\to 1} f(x)$ does not exist

(D) $\lim_{x\to 1} f(x)=\frac{-1}{2\sqrt{2}}$ $\therefore$ f (x) has removable discontinuity at x = 1

Q.17 f (x) is an even function, x = 1 is a point of minima and x = 2 is a point of maxima for y = f (x). Further $\lim_{x\to\infty} f(x)=0$, and $\lim_{x\to\infty} f(x)=\infty$. f (x) is increasing in (1, – 2) & decreasing everywhere in $(0,1)\cup(2,\infty)$. Also f (1) = 3 & f (2) = 5 Then

(A) f (x) = 0 has no real roots

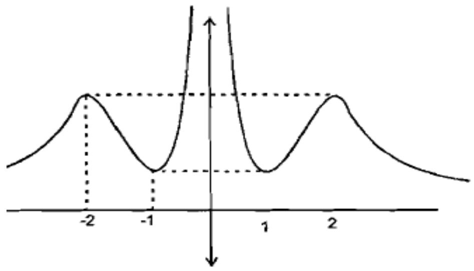
(B) y = f (x) and $y=|f(x)|$ are identical functions

(C) f ’ (x) = 0 has exactly four real roots whose sum is zero

(D) f ’ (x) = 0 has exactly four real roots whose sum is 6

Sol. **[A, B, C]**

$\lim_{x\to 0} f(x)=\infty,$ $\lim_{x\to\infty} f(x)=0$



$\Rightarrow$ f (x) is increasing in (1, 2) and decreasing in $(0,1)\cup(2,\infty)$ from the graph

Q.18

Q.19

Q.20

PASSAGE 1

A curve is represented parametrically by the equations x = f (t) = $a^{\ln(b^t)}$ and y = g (t) = $b^{-\ln(a^t)}$ a, b > 0 and $a \neq 1, b \neq 1$ where $t \in R$.

Q.21 Which of the following is not a correct expression for $\frac{dy}{dx}$?

(A) $\frac{-1}{f(t)^2}$ (B) – $(g(t))^2$ (C) $\frac{-g(t)}{f(t)}$ (D) $\frac{-f(t)}{g(t)}$

Sol. **[D]**

Q.22 The value of $\frac{d^2y}{dx^2}$ at the point where f (t) = g (t) is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

Sol. **[D]**

Q.23 The value of $\frac{f(t)}{f'(t)}\cdot\frac{f'(-t)}{f'(-t)}+\frac{f(-t)}{f'(-t)}\cdot\frac{f''(t)}{f'(t)}\forall t \in R,$ is equal to

(A) – 2 (B) 2 (C) – 4 (D) 4

Sol. **[B]**

$\Rightarrow\ x = f(t) = a^{\ln(b^t)} = a^{t\ln b}$ ....(1)

$\Rightarrow\ y = g(t) = b^{-\ln(a^t)} = \left(b^{\ln a}\right)^{-t} = \left(a^{\ln b}\right)^{-t} = a^{-t\ln b}$

$\Rightarrow\ \therefore y = g(t) = a^{\ln(b^{-1})} = f(-t)$ ....(2)

From equation (1) and (2)

$\Rightarrow$ xy = 1

(i) $\because\ y = \frac{1}{x}$

$\Rightarrow \therefore \frac{dy}{dx} = -\frac{1}{x^2} = -\frac{1}{f^2(t)}$ **(A)** is correct

$\Rightarrow$ Also xy = 1

$\Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} = -\frac{y^2}{1} = -g^2(t)$ **(B)** is correct

$\Rightarrow$ Again xy = 1 $\frac{dy}{dx} = -\frac{y}{x} = -\frac{g(t)}{f(t)}$ **(C)** is correct

(D) is incorrect

(ii) f(t) = g(t) $\Rightarrow$ f (t) = f (– t) $\Rightarrow$ t = 0
{$\because$ f (t) is one-one function}
At t = 0, x = y = 1

$\Rightarrow \therefore \frac{dy}{dx} = \frac{-1}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{2}{x^3}$

$\Rightarrow$ At x = 1, $\frac{d^2y}{dx^2} = 2$

(iii) $\therefore$ xy = 1 $\therefore$ f g = 1 $\therefore$ f g' + g f' = 0
$\Rightarrow \therefore$ f g'' + g' f' + g' f' + g f '' = 0
$\Rightarrow$ f g'' + g f '' + 2 g' f ' = 0

$\Rightarrow \frac{fg''}{f'g'} + \frac{gf''}{g'f'} = -2$ ……(3)

from equation (2)
$\Rightarrow$ g (t) = f (– t)
$\Rightarrow \therefore$ g' (t) = – f ' (– t)
and g'' (t) = f '' (– t)
substituting in equation (3)

$\Rightarrow \frac{f(t)}{f'(t)} \cdot \frac{f''(-t)}{-f'(-t)} + \frac{f(-t)}{-f'(-t)} \cdot \frac{f''(t)}{f'(t)} = -2$

$\Rightarrow \frac{f(t)}{f'(t)} \cdot \frac{f''(-t)}{f'(-t)} + \frac{f(-t)}{f'(-t)} \cdot \frac{f''(t)}{f'(t)} = 2$

$\Rightarrow$ 1

PASSAGE 2

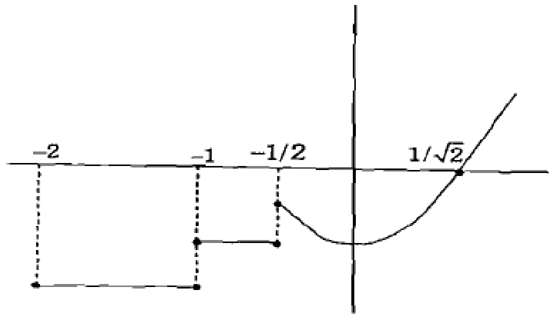
Let a function be defined as $f(x) = \begin{cases} [x], & -2 \le x \le -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \le 2 \end{cases}$, where [.] denotes greatest integer function.

Answer the following question by using the above information.

Q.24 The number of points of discontinuity of f (x) is

(A) 1 (B) 2 (C) 3 (D) 0

Sol. **[B]**

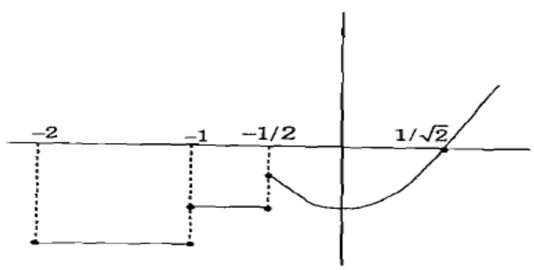


$\Rightarrow$ Two points of discontinuity $-1, -\frac{1}{2}$

Q.25 The function f (x – 1) is discontinuous at the points

(A) $-1, -\frac{1}{2}$ (B) $-\frac{1}{2}, 1$ (C) $0, \frac{1}{2}$ (D) 0, 1

Sol. **[C]**

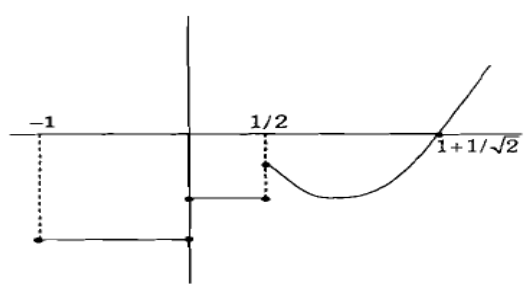


$\Rightarrow$ Discontinuous at $1, \frac{1}{2}$

Q.26 Number of points where $|f(x)|$ is not differentiable is

(A) 1 (B) 2 (C) 3 (D) 4

Sol. **[C]**



$\Rightarrow$ at $-1, -\frac{1}{2}, \frac{1}{\sqrt{2}}$ the function is not differentiable.

PASSAGE 3

Two students, A & B are asked to solve two different problem. A is asked to evaluate $\lim_{x\to 0}\frac{1-\cos(\ln(1+x))}{x^2}$ & B is asked to evaluate $\lim_{x\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n^3+1}}+\frac{\sqrt{n}}{\sqrt{n^3+1}}+......+\frac{\sqrt{n}}{\sqrt{n^3+2n}}\right), n\in N.$ A provides the following solution

$$\text{Let } h=\lim_{x\to 0}\frac{1-\cos\left(\frac{\ln(1+x)}{x}\cdot x\right)}{x^2}=\lim_{x\to 0}\frac{1-\cos x}{x^2}\left(\text{As}\lim_{x\to 0}\frac{\ln(1+x)}{x}=1\right)$$

$$l_1=\frac{1}{2}$$

B provides the following solution

$$\text{Let } l_2=\lim_{n\to\infty}\left\{\sum_{r=1}^{2n}\frac{\sqrt{n}}{\sqrt{n^3+r}}\right\}=\lim_{n\to\infty}\left\{\sum_{r=1}^{2n}\frac{1}{n}\frac{\sqrt{n}}{\sqrt{n+\frac{r}{n^2}}}\right\}$$

$$\lim_{n\to\infty}\left[\frac{1}{n}\left\{\sqrt{\frac{n}{n+\frac{1}{n^2}}}+\sqrt{\frac{n}{n+\frac{2}{n^2}}}+.......+\sqrt{\frac{n}{n+\frac{2n}{n^2}}}\right\}\right]$$

$$\lim_{n\to\infty}\left[\frac{1}{n}\left(\underbrace{1+1+.......+1}_{2n\text{ times}}\right)\right]=\lim_{n\to\infty}\frac{2n}{n}=2$$

Q.27 Identify the correct statement

(A) both of them get the correct answer

(B) both of them get the incorrect answer

(C) A gets the correct answer while B gets the incorrect answer.

(D) B gets the correct answer while A gets the incorrect answer.

Sol. **[A]**

Q.28 Who has solved the problem correctly.

(A) A (B) B (C) both of them (D) no one

Sol. **[D]**

Q.29 $f(x)=\begin{cases} 4l_1\left(\dfrac{\tan x-\sin x}{x^3}\right) & x<0 \\ k & x=0 \\ l_2\left(\dfrac{e^x-x-1}{1-\cos x}\right) & x>0 \end{cases}$ where l_1 and l_2 are correct values of the corresponding limits, if is continuous at x = 0 the K is equal to:

(A) 1 (B) 2 (C) 3 (D) no value of K

Sol. **[D]**

$$\Rightarrow l_1=\lim_{x\to 0}\frac{1-\cos(\ln(1+x))}{\ln^2(1+x)}\cdot\left(\frac{\ln(1+x)}{x}\right)^2=\frac{1}{2}$$

A & B have made the same mistake, they used the notion of limit partly in the problem, where as once the limiting notion has been used the resulting expression must be free from the variable on which the limit has been imposed

$$\Rightarrow \lim_{n\to\infty}\frac{2n\sqrt{n}}{\sqrt{n^3+1}}<l_2<\lim_{n\to\infty}\frac{2n\sqrt{2}}{\sqrt{n^3+1}}$$

Hence l_2 = 2 (sandwich theorem)

$\Rightarrow$ **Sol.1** **Hence (A)**

$\Rightarrow$ **Sol.2** **Hence (D)**

$\Rightarrow$ **Sol.3** $\lim_{x\to 0} 4\cdot\frac{1}{2}\left(\frac{\tan x-\sin x}{x^3}\right)=4\cdot\frac{1}{2}\cdot\frac{1}{2}=1$

$$\Rightarrow \lim_{x\to 0} l_2\left(\frac{e^x-x-1}{x^2}\cdot\frac{x^2}{1-\cos x}\right)=2(2\cdot 2)=8$$

$\Rightarrow$ for no value if K

Hence (D)

PASSAGE 4

Q.30

Q.31

Q.32

Matrix match type

Q.33

Q.34 **Column – I** **Column – II**

	Column – I	Column – II
(A)	$f(x)=\begin{cases} x+1 & \text{if } x<0 \\ \cos x & \text{if } x\ge 0\end{cases}$, at $x=0$ is	(P) continuous
(B)	For every $x\in R$ the function	(Q) differentiability

$g(x)=\dfrac{\sin(\pi[x-\pi])}{1+[x]^2}$ (R) discontinuous

where [x] denotes the greatest integer function is (S) non derivable

(C) $h(x)=\sqrt{\{x\}^2}$ where {x} denotes fractional part function for all $x \in I$, is

(D) $k(x)=\begin{cases} x^{\frac{1}{\ln x}} & \text{if } x \neq 1 \\ e & \text{if } x = 1 \end{cases}$ at $x=1$ is

Sol. (A) ⇒ P, S; (B) ⇒ P, Q; (C) ⇒ R, S; (D) ⇒ P, Q

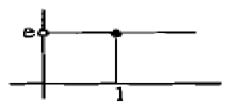
(A) $f'(0)=\lim\limits_{h\to 0}\dfrac{\cos h-0}{h}$ does not exist. Obviously f (0) = f (0^+) = 1

Hence continuous and not derivable

(B) g (x) = 0 for all x, hence continuous and derivable

(C) as $0 \le \{f(x)\} < 1$, hence $h(x)=\sqrt{\{x\}^2}=\{x\}$ which is discontinuous hence non derivable all $x \in I$

(D) $\lim\limits_{x\to 1} x^{\frac{1}{\ln x}} = \lim\limits_{x\to 1} x^{\log_x e} = e = f(1)$



⇒ Hence k (x) is constant for all x > 0 hence continuous and differentiable at x = 1.

Q.35 **Column – I** **Column – II**

(A) Number of points of discontinuity of $f(x)=\tan^2 x-\sec^2 x$ in $(0, 2\pi)$ is (p) 1

(B) Number of points at which $f(x)=\sin^{-1}x+\tan^{-1}x+\cot^{-1}x$ is non-differentiable in (– 1, 1) is (q) 2

(C) Number of points of discontinuity of $y=[\sin x], x \in [0, 2\pi)$ where [.] represents greatest integer function (r) 0

(D) Number of points where $y=\left|(x-1)^3\right|+\left|(x-2)^5\right|+|x-3|$ is non-differentiable (s) 3

Sol. (A) ⇒ q; (B) ⇒ r; (C) ⇒ q; (D) ⇒ p

(A) $\tan^2 x$ is discontinuous at $x=\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

⇒ $\sec^2 x$ is discontinuous at x = $x=\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

⇒ Number of discontinuities = 2

(B) Since $f(x)=\sin^{-1}x+\tan^{-1}x+\cot^{-1}x=\sin^{-1}x+\dfrac{\pi}{2}$

$\Rightarrow\ \therefore$ f(x) is differentiable in (– 1, 1)

$\Rightarrow$ number of points of non-differentiable = 0

(C) $y = [\sin x] = \begin{cases} 0 & , \ 0 \le x \frac{\pi}{2} \\ 1 & , \ x = \frac{\pi}{2} \\ 0 & , \ \frac{\pi}{2} < x \le \pi \\ -1 & , \pi < x < 2\pi \\ 0 & , \quad x = 2\pi \end{cases}$ 7t

$\Rightarrow\ \therefore$ Points of discontinuity are $\frac{\pi}{2}, \pi$

(D) $y = |(x-1)^3| + |(x-2)^5| + |x-3|$ is non differentiable at x = 3 only.

CONTINUITY & DIFFERENTIABILITY
EXERCISE 3

1 Let $f(x) = \begin{cases} \dfrac{\ln \cos x}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \dfrac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0), if not then indicate the nature of discontinuity.

Sol. $\text{LHL}|_{x=0} = \lim\limits_{x\to 0^-} \dfrac{e^{\sin 4x}-1}{\ell n(1+\tan 2x)}$

put x = 0 – h

$$= \lim_{x\to 0} \frac{e^{-\sin 4x}-1}{\ell n(1-\tan 2h)}$$

$$= \lim_{h\to 0} \frac{e^{-\sin 4h}-1}{-\sin 4h}\left(\frac{-\sin 4h}{4h}\right). 4h \left(\frac{1}{\dfrac{\ell n(1-\tan 2h)}{(-\tan 2h)}\left(\dfrac{-\tan 2h}{2h}\right).2h}\right)$$

$\boxed{f(0^-) = 2}$

$$\text{RHL}|_{x=0} = \lim_{x\to 0^+}\left(\frac{\ell n \cos x}{\sqrt[4]{(1+x^2)}-1}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cos x - 1}{1+\frac{1}{4}x^2-1}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{1-\cos x}{x^2}\right)(-4)$$

$\boxed{f(0^+) = -2}$

hence f(0) can not define.

and ∵ $f(0^-)$ & $f(0^+)$ are finite hence there non-removable type disconti.

2 Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots\ldots\ldots\ldots + \dfrac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim\limits_{n\to\infty} y_n(x)$

Discuss the continuity of $y_n(x)$ $(n \in N)$ and y(x) at x = 0

Sol. $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots + \dfrac{x^2}{(1+x^2)^{n-1}}$

$$y_n(x) = x^2 \frac{\left(1-\left(\frac{1}{1+x^2}\right)^n\right)}{1-\frac{1}{1+x^2}}$$

$$= x^2 \frac{\left\{1-\left(\frac{1}{1+x^2}\right)^n\right\}}{\frac{1+x^2-1}{1+x^2}}$$

$y_n(x) = (1 + x^3) \{1 - (1 + x^2)^{-n}\}$

3 Let $f(x) = \begin{cases} \frac{1-\sin \pi x}{1+\cos 2\pi x}, & x < \frac{1}{2} \\ p, & x = \frac{1}{2} \\ \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x > \frac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at x = 1/2.

Sol. $V.F.|_{x=\frac{1}{2}} = p$ ...(1)

$$LHL|_{x=\frac{1}{2}} = \lim_{x \to \frac{1}{2}^-} f(x)$$

$$= \lim_{x \to \frac{1}{2}^-} \frac{1-\sin \pi x}{1+\cos(2\pi x)}$$

put $x = \frac{1}{2} - h$

$$= \lim_{h \to 0} \frac{1-\sin\left(\frac{\pi}{2} - \pi h\right)}{1+\cos(\pi - 2\pi h)}$$

$$= \lim_{h \to 0} \left(\frac{1-\cos \pi h}{(\pi h)^2}\right) \left(\frac{1}{\frac{1-\cos(2\pi h)}{(2\pi h)^2}}\right) \left(\frac{\pi^2 h^2}{4\pi^2 h^2}\right)$$

$LHL|_{x=\frac{1}{2}} = \frac{1}{4}$ ...(2)

$\text{RHL}\big|_{x=\frac{1}{2}} = \lim_{x\to\left(\frac{1}{2}\right)^+} f(x)$

$$= \lim_{x\to\left(\frac{1}{2}\right)^+}\left(\frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}\right)$$

$$= \lim_{x\to\frac{1+}{2}}\left(\frac{\sqrt{2x-1}}{4+\sqrt{2x-1}-4}\right)\left(\sqrt{4+\sqrt{2x-1}}+2\right)$$

$\text{RHL}\big|_{x=\frac{1}{2}} = 4$

$\because\ \text{LHL}\big|_{x=\frac{1}{2}} \neq \text{RHL}\big|_{x=\frac{1}{2}}$

so the value of function cannot determine & the function is discontinuous.

4 Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate $h(g(2))$ (b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

Sol. (i) h(g(2)) =

$g(2) = \sqrt{6-4} = \sqrt{2}$

$h(x) = 2x^2 - 3x + a$

$h(\sqrt{2}) = 4 - 3\sqrt{2} + a$ **Ans**

(ii) $f(x) = \begin{cases} g(x) & ; & x \le 1 \\ h(x) & ; & x > 1 \end{cases}$

$$f(x) = \begin{cases} \sqrt{6-2x} & ; & x \le 1 \\ 2x^2 - 3x + a & ; & x > 1 \end{cases}$$

$\text{V.F.}\big|_{x=1} = 2$ ...(1)

$\text{R.H.L.}\big|_{x=1} = \lim_{x\to1^+} f(x)$

$= \lim_{x\to1^+}(2x^2 - 3x + a)$

$\text{R.H.L.}\big|_{x=1} = a - 1$ ...(2)

$\text{L.H.L.}\big|_{x=1} = \lim_{x\to1^-} f(x)$

$= \lim_{x\to1^-}\sqrt{6-2x}$

$= 2$

since function is conti

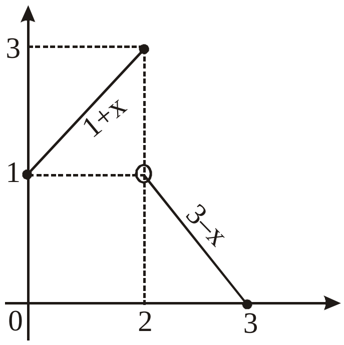
L.H.L.$|_{x=1}$ = R.H.L.$|_{x=1}$ = VF$|_{x=1}$

2 = a – 1 = 2

a – 1 = 2 ⇒ $\boxed{a = 3}$

5 Let $f(x) = \begin{cases} 1+x & , 0 \le x \le 2 \\ 3-x & , 2 < x \le 3 \end{cases}$. Determine the form of g(x) = f [f(x)] & hence find the point of discontinuity of g , if any.

Sol. $f(x) = \begin{cases} 1+x & ; \quad 0 \le x \le 2 \\ 3-x & ; \quad 2 < x \le 3 \end{cases}$

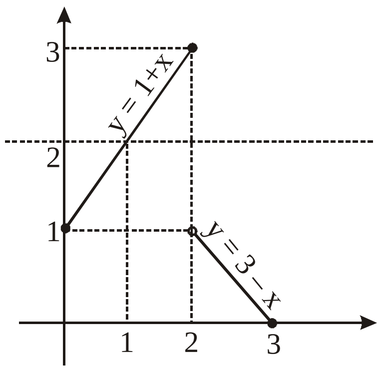


g(x) = f (f(x))

$= \begin{cases} 1+f(x) & ; \quad 0 \le f(x) \le 2 \\ 3-f(x) & ; \quad 2 < f(x) \le 3 \end{cases}$

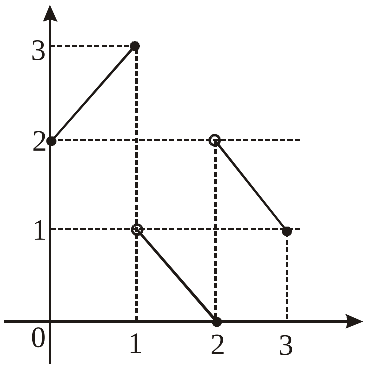
let f(x) = y

$f(y) = \begin{cases} 1+y & ; \quad 0 \le y \le 2 \\ 3-y & ; \quad 2 < y \le 3 \end{cases}$



$= \begin{cases} 1+(1+x) & ; \quad 0 \le x \le 1 \\ 1+(3-x) & ; \quad 2 < x \le 3 \\ 3-(1+x) & ; \quad 1 < x \le 2 \end{cases}$

$$= \begin{cases} 2+x & ; \quad 0 \le x \le 1 \\ 2-x & ; \quad 1 < x \le 2 \\ 4-x & ; \quad 2 \le x \le 3 \end{cases}$$



so the point of discontinuity

1, 2 **Ans**

Or

$F.V.|_x = LHL = RHL$

6 Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x) = \begin{cases} \dfrac{\left(\exp\{(x+2)\ell n4\}\right)^{\frac{[x+1]}{4}} - 16}{4^x - 16} & , x<2 \\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)} & , x>2 \end{cases}.$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

Sol. $RHL|_{x=2} = \lim\limits_{x \to 2^+} f(x)$

$$= \lim_{x \to 2^+} \frac{A(1-\cos(x-2))}{(x-2).\tan(x-2)}$$

put x = 2 + h

$$= \lim_{h \to 0} \frac{A(1-\cos h)}{h \tan h}$$

$$= \lim_{h \to 0} A\left(\frac{1-\cos h}{h^2}\right)\frac{1}{\left(\frac{\tan h}{h}\right)}$$

$$= \lim_{h \to 0} \frac{4^2.4^{\frac{-h}{2}} - 16}{4^2.4^{-h} - 16}$$

$$= \lim_{h \to 0} \frac{4^{-n/2} - 1}{4^{-h} - 1}$$

$$= \lim_{h \to 0} \left(\frac{4^{-h/2}-1}{-\frac{h}{2}}\right).\left(-\frac{h}{2}\right)\frac{1}{\left(\frac{4^{-h}-1}{-h}\right)(-h)}$$

$$= \ell n4.\ \frac{1}{2}.\frac{1}{\ell n4} = \frac{1}{2}$$

$RHL|_{x=2} = \frac{A}{2}$ since the function is contin.

$VF|_{x=2} = RHL|_{x=2} = LHL|_{x=2}$

$LHL|_{x=2} \Rightarrow \lim_{x\to 2^-} f(x)$ $V.F.|_{x=2} = \frac{A}{2} = \frac{1}{2}$

$$= \lim_{x\to 2^-} \frac{\left(e^{(x+2)\ell n4}\right) - 16^{\frac{[x+1]}{4}}}{4^x - 16}$$

$\boxed{V.F.|_{x=2} = \frac{1}{2}}$ **Ans**

$$= \lim_{x\to 2^-} \frac{4^{\frac{(x+2)([x]+1)}{4}} - 16}{4^x - 16}$$

$\boxed{A = 1}$ **Ans**

$$= \lim_{x\to 2^-} \frac{4^{\left(\frac{x+2}{2}\right)} - 16}{4^x - 16}$$

put x = 2 – h

$$= \lim_{x\to 0} \frac{4^{\frac{4-h}{2}} - 16}{4^{2-h} - 16}$$

7 The function $f(x) = \begin{cases} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \frac{\pi}{2} \\ b+2 & \text{if } x = \frac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b' , if f is continuous at x = π/2.

Sol. $V.F.|_{x=\frac{\pi}{2}} = b + 2$ ...(1)

$$LHL|_{x=\frac{\pi}{2}} = \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}$$

put $x = \frac{\pi}{2} - h$

$$LHL|_{x=\frac{\pi}{2}} = \lim_{h\to 0} \left(\frac{6}{5}\right)^{\frac{\tan(3\pi - 6h)}{\tan(5\pi/2 - 5h)}} = \lim_{h\to 0} \left(\frac{6}{5}\right)^{-\frac{\tan 6h}{\cot 5h}} = 1$$

$$RHL|_{x=\frac{\pi}{2}} = \lim_{x\to\left(\frac{\pi}{2}\right)^+} f(x)$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^+} (1 - \cos x)^{-\frac{a}{b}\tan x}$$

put $x = \frac{\pi}{2} + h$

$= \lim_{h\to 0}(1+\sin h)^{\frac{a}{b}\cot h}$; 1^{∞} form

$= e^{\lim_{h\to 0}(\sin h)\frac{a}{b}\cot h}$

$= e^{\lim_{h\to 0}\frac{a}{b}\cos h} = e^{\frac{a}{b}}$

since the function is conti so

$$LHL|_{x=\frac{\pi}{2}} = RHL|_{x=\frac{\pi}{2}} = V.F.|_{x=\frac{\pi}{2}}$$

$$1 = e^{ab} = b + 2$$

$$\boxed{a = 0,\ b = -1}$$

8 Let $f(x) = \begin{cases} \dfrac{\left(\frac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)} & ;\quad x \neq 0 \\ \dfrac{\pi}{2} & ;\quad x = 0 \end{cases}$

where $\{x\}$ is the fractional part of x. Consider another function g(x); such that

$g(x) = f(x)\ ;\ x \geq 0$

$= 2\sqrt{2}f(x)\ ;\ x < 0$

Discuss the continuity of the function f(x) & g(x)at $x = 0$.

Sol. $RHL|_{x=0} = \lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+}\frac{\left(\frac{\pi}{2} - \sin^{-1}(1-(x-[x]^2)\right)\sin^{-1}(1-x+[x])}{\sqrt{2}(x-[x]-(x-[x])^3)}$$

$$= \lim_{x\to 0^+}\frac{\left(\frac{\pi}{2} - \sin^{-1}(1-x^2)\right)\sin^{-1}(1-x)}{\sqrt{2}x(1-x^2)}$$

$$= \lim_{x\to 0^+}\frac{\cos^{-1}(1-x^2).\sin^{-1}(1-x)}{\sqrt{2}x.(1-x^2)}$$

$$= \frac{\pi}{2\sqrt{2}}\lim_{x\to 0^+}\frac{\cos^{-1}(1-x^2)}{x}$$

let $\cos^{-1}(1-x^2) = \theta$

$$1 - x^2 = \cos\theta$$
$$x^2 = 1 - \cos\theta$$
$$x = \sqrt{1-\cos\theta}$$

when $x \to 0^+$ then $\theta \to 0$

$$= \frac{\pi}{2\sqrt{2}} \lim_{\theta\to 0^+} \frac{\theta}{\sqrt{1-\cos\theta}}$$

$$= \frac{\pi}{2\sqrt{2}} \lim_{\theta\to 0^+} \frac{\theta}{\sqrt{2-\sin^2\theta/2}} = \frac{\pi}{4} \lim_{\theta\to 0^+} \frac{\theta}{|\sin\theta/2|}$$

$$\text{RHL}|_{x=0} = \frac{\pi}{4} \lim_{\theta\to 0^+} 2\left(\frac{\theta/2}{\sin\theta/2}\right) = \frac{\pi}{2}$$

$$\text{LHL}|_{x=0} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} \frac{\left(\frac{\pi}{2} - \sin^{-1}(1-x-[x])^2\right)\sin^{-1}(1-x+[x])}{\sqrt{2}(x-[x]-(x-[x])^3)}$$

$$= \lim_{x\to 0^-} \frac{\frac{\pi}{2} - \sin^{-1}(1-(x+1)^2)\sin^{-1}(-x)}{\sqrt{2}(x+1-(x+1)^3)}$$

$$= \lim_{x\to 0^-} \frac{\left(\frac{\pi}{2}\sin^{-1}(-x^2-2x)\right)\sin^{-1}(-x)}{\sqrt{2}(x+1)(-x^2-2x)}$$

$$= \lim_{x\to 0^-} \frac{\cos^{-1}(-x^2-2x)\sin^{-1}(x)}{\sqrt{2}(x+1)(x^2+2x)}$$

$$= \lim_{x\to 0^-} \frac{\pi - \cos^{-1}(x^2+2x)}{\sqrt{2}(x+1)\,(x+2)} \cdot \frac{\sin^{-1}x}{x}$$

$$\text{LHL}|_{x=0} = \frac{\pi}{4\sqrt{2}}$$

for f(x) since $\text{LHL}|_{x=0} \neq \text{RHL}|_{x=0}$ so the function is discontinuous at $x = 0$.

for $g(x) \Rightarrow$

$$\text{RHL}|_{x=0} = \lim_{x\to 0^+} g(x)$$

$$= \lim_{x\to 0^+} f(x) = \frac{\pi}{2}$$

$$\text{LHL}|_{x=0} = \lim_{x\to 0^-} 2\sqrt{2} f(x)$$

$$= 2\sqrt{2} \lim_{x\to 0^-} f(x)$$

$$= 2\sqrt{2}.\frac{\pi}{4\sqrt{2}} = \frac{\pi}{2}$$

$$g(0) = f(0) = \frac{\pi}{2}$$

9 If the function $f(x)$ defined as $f(x) = \begin{cases} -\dfrac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\dfrac{1}{x} & \text{for } x > 0 \end{cases}$ is continuous but not derivable at x = 0 then find the range of n.

Sol. $f(x) = \begin{cases} -\dfrac{x^2}{2} & \text{for } x \le 0 \\ x^n \sin\dfrac{1}{x} & \text{for } x > 0 \end{cases}$

f(x) is continuous at x = 0

f(0) = 0

$$L_1 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{-x^2}{2}\right) = 0$$

$$L_2 = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^n \sin\frac{1}{x}$$

for continuous,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+} x^n \sin\left(\frac{1}{x}\right) = 0$$

limit is defined only when

$\therefore\ n > 0$

since f(x) is non-differentiable at x = 0

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(h+0) - f(0)}{2} = \lim_{h\to 0^-} \frac{-\dfrac{h^2}{2} - 0}{h} = \lim_{h\to 0^-}\left(\frac{-h}{2}\right)$$

$$f'(0^+) = \lim_{h\to 0^+} \frac{f(h+0) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^n \sin\frac{1}{h}}{h}$$

$f'(0^+) \neq f'(0^-)$

$$\Rightarrow \lim_{h\to 0^+} h^{n-1} \sin\left(\frac{1}{h}\right) \neq 0$$

only when $n - 1 \leq 0$

$\Rightarrow n \leq 1$ .....(ii)

from equation (i) & (ii)

$n \in (0, 1]$

10 . . . $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that

$$\lim_{x\to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + ...f\left(\frac{x}{k}\right)\right) = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}$$

Sol. $$\lim_{x\to 0} \frac{1}{x}\left[f(x) + f\left(\frac{x}{2}\right) + ... + f\left(\frac{x}{k}\right)\right]$$

$$= \lim_{x\to 0} \frac{f(x)}{x} + \frac{f\left(\frac{x}{2}\right)}{x} + ... + \frac{f\left(\frac{x}{k}\right)}{x}$$

$$= \lim_{x\to 0} \frac{f(x+0) - f(0)}{x} + \lim_{x\to 0} \frac{f\left(\frac{1}{x}+0\right) - f(0)}{\frac{x}{2}} \cdot \frac{1}{2} + ... + \lim_{x\to 0^+} \frac{f\left(\frac{x}{k}+0\right) - f(0)}{\frac{x}{k}} \cdot \frac{1}{k}$$

$$= f'(0) + \frac{1}{2} f'(0) + ... + \frac{1}{k} f'(0)$$

$$= 1 + \frac{1}{2} + ... + \frac{1}{k}$$

11 If $f(x) = \begin{bmatrix} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{bmatrix}$ is derivable at x = 1. Find the values of a & b.

Sol. $$f(x) = \begin{bmatrix} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{bmatrix}$$

f(x) is differentiable at x = 1, hence it is also continuous at x = 1

$\lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \boxed{a - b = -1}$ ...(i)

$$f'(1) = \lim_{h\to 0^-} \frac{f(h+1)-f(1)}{h} = \lim_{h\to 0^-} \frac{a(h+1)^2 - b + 1}{h}$$

$$= \lim_{h\to 0^-} \frac{ah^2 + 2ah + a - b + 1}{h}$$

$$= \lim_{h\to 0^-} \frac{ah^2 + 2ah}{h} = \lim_{h\to 0^-} (ah + 2a) = 2a$$

$$f'(1^+) = \lim_{h\to 0^+} \frac{f(h+1)-f(1)}{h} = \lim_{h\to 0^+} \frac{\frac{-1}{|h+1|} + 1}{h}$$

$$= \lim_{h\to 0^+} \frac{\frac{-1+1+h}{1+h}}{h} = \lim_{h\to 0^+} \frac{1}{1+h} = 1$$

$f'(1^-) = f'(1^+)$

$\Rightarrow 2a = 1$

$a = 1/2$

$b = 3/2$

12 The function $f(x) = \begin{cases} ax(x-1)+b & \text{when } x < 1 \\ x - 1 & \text{when } 1 \le x \le 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x (ii) f ' (1) does not exist (iii) f '(x) is continuous at x = 3

Sol. $f(x) = \begin{cases} ax(x-1)+b & \text{when } x < 1 \\ x - 1 & \text{when } 1 \le x \le 3 \\ px^2 + qx + 2 & \text{when } x > 3 \end{cases}$

f(x) is continous at x = 1

$\lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \lim_{x\to 1^-} ax(x-1) + b = 0$

$\Rightarrow \boxed{b = 0 \;\&\; a \in R}$

$$f'(1) = \lim_{h\to 0}\frac{f(h+1)-f(1)}{h} = \begin{cases}\lim\limits_{h\to 0^-} & \dfrac{a(h+1)(h+1-1)+b}{h}\\ \lim\limits_{h\to 0^+} & \dfrac{h+1-1}{h}\end{cases}$$

$$= \begin{cases}\lim\limits_{h\to 0^-} & \dfrac{a(h+1)h+0}{h}\\ \lim\limits_{h\to 0^+} & \dfrac{h}{h}\end{cases} = \begin{cases}\lim\limits_{h\to 0^-} a(h+1)\\ 1\end{cases}$$

$$= \begin{cases}a\\ 1\end{cases}$$

$\because$ $f'(1)$ = DNE $\Rightarrow a \neq 1$

$\therefore$ $a \in R - \{1\}$ & $b = 0$

f(x) is cont. at x = 3

$$\lim_{x\to 3} f(x) = f(3)$$

$$\Rightarrow \lim_{x\to 3} (px^2 + 9x + 2) = 2$$

$\Rightarrow 9p + 3q + 2 = 2$

$\Rightarrow 9p + 3q = 0$ ....(i)

$\because$ $f'(x)$ is cont. at x = 3, hence f(x) is diff. at x = 3

$$f'(3) = \lim_{h\to 0}\frac{f(h+3)-f(3)}{h} = \begin{cases}\lim\limits_{h\to 0^-}\dfrac{3+h-1-2}{h}\\ \lim\limits_{h\to 0^+}\dfrac{p(h+3)^2+q(h+3)+2-2}{h}\end{cases}$$

$$= \begin{cases}\lim\limits_{h\to 0^-}\dfrac{h}{n}\\ \lim\limits_{h\to 0^+}\dfrac{ph^2+6ph+qh+9p+3q}{h}\end{cases} = \begin{cases}1\\ \lim\limits_{h\to 0^+}\dfrac{ph^2+6ph+qh}{h}\end{cases}$$

[from equation (i) 9p + 3q = 0]

$$= \begin{cases}1\\ \lim\limits_{h\to 0^+}(ph+6p+q)\end{cases} = \begin{cases}1\\ 6p+q\end{cases}$$

$\therefore$ $f'(3^+) = f'(3^-) \Rightarrow 6p + q = 0$ ...(ii)

solving equation (i) & (ii) p = 1/3, q = – 1

$a \in R - \{1\}$, b = 0, p = 1/3 , q = – 1

13 Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function.

$f(x) = x.\sin\frac{1}{x}.\sin\frac{1}{x.\sin\frac{1}{x}}$ where $x \neq 0$, $x \neq 1/r\pi$ & $f(0) = f(1/r\pi) = 0$,

$r = 1, 2, 3, \ldots\ldots\ldots$

Sol. $f(x) = x.\sin\frac{1}{x}.\sin\frac{1}{x.\sin\frac{1}{x}}\ x \neq 0, 1/r\pi$

$f(0) = 0 = f\left(\frac{1}{r\pi}\right)$, $r = 1, 2, 3....$

$$f'(0) = \lim_{h\to 0}\frac{f(h+0)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right).\sin\left(\frac{1}{h\sin\left(\frac{1}{h}\right)}\right)-0}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\left(\frac{1}{h}\right).\sin\left(\frac{1}{h\sin\left(\frac{1}{h}\right)}\right)}{h}$$

$$= \lim_{h\to 0}\sin\left(\frac{1}{h}\right).\sin\left(\frac{1}{h\sin\left(\frac{1}{h}\right)}\right)$$

$$= \lim_{h\to 0}\ \underbrace{\sin\left(\frac{1}{h}\right)}_{1-\leq\ \ \leq 1}\ \underbrace{\sin\left(\frac{1}{h\sin(1/h)}\right)}_{-1\leq\ \ \leq 1}$$

$= DNE$

so $f(x)$ is not differentiable at $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right).\sin\left(\frac{1}{x\sin(1/x)}\right)$$

$$= \lim_{x\to 0} \underbrace{x}_{\to 0} . \underbrace{\sin(1/x)}_{-1\le\ \le 1} . \underbrace{\sin\left(\frac{1}{n\sin(1/x)}\right)}_{-1\le\ \le 1}$$

$$= 0$$

$$= f(0)$$

$$\lim_{x\to\frac{1}{r\pi}} f(x) = \lim_{x\to\frac{1}{\pi r}} x\sin\left(\frac{1}{x}\right).\sin\left(\frac{1}{x\sin\left(\frac{1}{x}\right)}\right)$$

$$= \lim_{x\to\frac{1}{r\pi}} x.\sin\left(\frac{1}{x}\right).\sin\left(\frac{\frac{1}{\sin\left(\frac{1}{x}\right)}}{\left(\frac{1}{x}\right)}\right)$$

$$= \lim_{x\to\frac{1}{r\pi}} \underbrace{x.\sin\left(\frac{1}{x}\right)}_{\to 0}.\underbrace{\sin\left(\frac{1}{x\sin\left(\frac{1}{x}\right)}\right)}_{-1\le\ \ \le 1}$$

$$= 0$$

$$= f\left(\frac{1}{r\pi}\right)$$

Hence function is continuous $\forall\ x \in [0, 1]$

14 $f(x) = \begin{cases} 1-x & , \ (0\le x\le 1) \\ x+2 & , \ (1<x<2) \\ 4-x & , \ (2\le x\le 4) \end{cases}$ Discuss the continuity & differentiability of $y = f\,[f(x)]$ for $0\le x\le 4$.

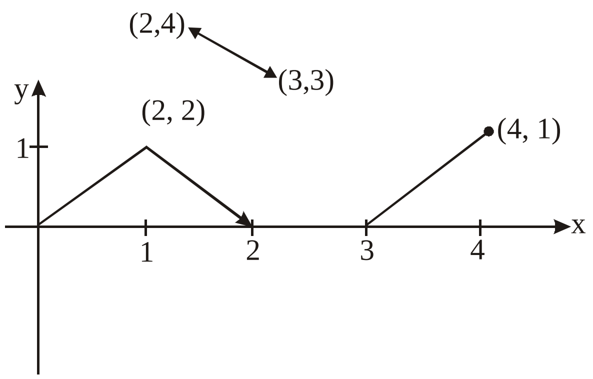
Sol. $f(x) = \begin{cases} 1-x & , \ (0\le x\le 1) \\ x+2 & , \ (1<x<2) \\ 4-x & , \ (2\le x\le 4) \end{cases}$

$$f(f(x)) = \begin{cases} 1-f(x) & ; \quad 0 \le f(x) \le 1 \\ f(x)+2 & ; \quad 1 < f(x) < 2 \\ 4-f(x) & ; \quad 2 \le f(x) \le 4 \end{cases}$$

$$= \begin{cases} 1-1-x & ; \quad 0 \le x \le 1 \cap 1 \le 1-x \le 1 \Rightarrow 0 \le x \le 1 \\ 1-x-2 & ; \quad 1 < x < 2 \cap 0 \le x+2 \le 1 \Rightarrow -2 \le x \le -1 \\ 1-4+x & ; \quad 2 \le x \le 4 \cap 0 \le 4-x \le 1 \Rightarrow 3 \le x \le 4 \\ 1-x+2 & ; \quad 0 \le x \le 1 \cap 1 < 1-x < 2 \Rightarrow -1 < X < 0 \\ x+2+2 & ; \quad 1 < x < 2 \cap 1 < x+2 < 2 \Rightarrow -1 < x < 0 \\ 4-x+2 & ; \quad 2 \le x \le 4 \cap 1 < 4x < 2 \Rightarrow 2 < x < 3 \\ 4-1+x & ; \quad 0 \le x \le 1 \cap 2 \le 1-x \le 4 \Rightarrow -3 \le x \le -1 \\ 4-x-2 & ; \quad 1 < x < 2 \cap 2 \le 4 \Rightarrow 0 \le x \le 2 \\ 4-4+x & ; \quad 2 \le x \le 4 \cap 2 \le 4-x \le 4 \Rightarrow 0 \le x \le 2 \end{cases}$$

$$= \begin{cases} x & ; \quad 0 \le x \le 1 \\ x-3 & ; \quad 3 \le x \le 4 \\ -x+6 & ; \quad 2 < x < 3 \\ -x+2 & ; \quad 1 < x < 2 \end{cases}$$

$$f(f(x)) = \begin{cases} x & ; \quad 0 \le x \le 1 \\ -x-2 & ; \quad 1 < x < 2 \\ x & ; \quad x = 2 \\ -x+6 & ; \quad 2 < x < 3 \\ x-3 & ; \quad 3 \le x \le 4 \end{cases}$$



$\therefore$ f(x) is continuous at x = 1 & discunt.

at x = 2, 3 & non diff. at x = 1, 2, 3

15 Let f be a function that is differentiable every where and that has the following properties:
(i) $f(x+h) = f(x) \cdot f(h)$ (ii) $f(x) > 0$ for all real x. (iii) $f'(0) = -1$

Use the definition of derivative to find $f'(x)$ in terms of $f(x)$.

Sol. $f(x+h) = f(x) \cdot f(h)$ $\quad \left| \begin{array}{l} x = 0 \\ h = 0 \end{array} \right.$ $\quad f(0)\,(f(0)-1) = 0 \Rightarrow f(0) = 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x).f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h} f(x)$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}\ f(x)$$

$$= f'(0)\ f(x)$$

$$\Rightarrow f'(x) = -f(x)$$

$$\therefore f'(x) = -f(x)$$

16 Discuss the continuity & the derivability of 'f' where $f(x)$ = degree of $(u^{x^2} + u^2 + 2u - 3)$ at $x = \sqrt{2}$.

Sol. $f(x)$ = degree of $(u^{x^2} + u^2 + 2u - 3)$ at $x = \sqrt{2}$

$$= \begin{cases} 2 & ; \quad x \le \sqrt{2} \\ x^2 & ; \quad x > \sqrt{2} \end{cases}$$

$$f'(\sqrt{2}) = \lim_{h\to o} \frac{f(h+\sqrt{2}) - f(\sqrt{2})}{h}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{f(h+\sqrt{2}) - f(\sqrt{2})}{h} \\ \lim\limits_{h\to o^-} \dfrac{f(h+\sqrt{2}) - f(\sqrt{2})}{h} \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{(h+\sqrt{2})^2 - 2}{h} \\ \lim\limits_{h\to o^-} \dfrac{2-2}{h} \end{cases}$$

$$= \begin{cases} \lim\limits_{h\to o^+} \dfrac{2 + 2\sqrt{2}h + h^2 - 2}{h} \\ 0 \end{cases}$$

$$= \begin{cases} \lim_{h \to o^+} \frac{h^2 + 2\sqrt{2}h}{h} \\ 0 \end{cases}$$

$$= \begin{cases} \lim_{h \to o^+} (h + 2\sqrt{2}) \\ 0 \end{cases}$$

$$= \begin{cases} 2\sqrt{2} \\ 0 \end{cases}$$

$$\therefore \; f'(\sqrt{2}^-) \neq f'(\sqrt{2}^+)$$

Hance f(x) is non differentiable at $x = \sqrt{2}$

$$\lim_{x \to \sqrt{2}} f(x) = \lim_{x \to \sqrt{2}} x^2$$

$$= 2$$

$$= f(\sqrt{2})$$

$$\Rightarrow \; f(\sqrt{2}) = \lim_{x \to \sqrt{2}} f(x)$$

Hance f(x) is confinous at $x = \sqrt{2}$

17 Let f (x) be a function defined on (–a, a) with a > 0. Assume that f (x) is continuous at x = 0 and $\lim_{x \to 0} \frac{f(x) - f(kx)}{x} = \alpha$, where k ∈ (0, 1) then compute f ' (0^+) and f ' (0^-), and comment upon the differentiability of f at x = 0.

Sol. $\because \lim_{x \to 0} \frac{f(x) - f(k\alpha)}{x} = \alpha$

$$\Rightarrow \lim_{x \to o} \frac{f(x) - f(0) + f(0) - (kx)}{x} = \alpha$$

$$\Rightarrow \lim_{x \to o} \frac{f(x) - f(0) - f(kx) + f(0)}{x} = \alpha$$

$$\Rightarrow \lim_{x \to o} \left(\frac{f(x) - f(0)}{x} - \frac{f(kx) - f(0)}{x} \right) = \alpha$$

$$\Rightarrow \left(\lim_{x \to 0} \frac{f(x) - f(0)}{x} \right) - \left(\lim_{x \to 0} \frac{f(kx) - f(0)}{kx} \right) k = \alpha$$

$$\Rightarrow \begin{cases} \lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x} - \lim\limits_{x\to 0^-} \dfrac{f(kx)-f(0)}{kx}.k = \alpha \\ \lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x} - \lim\limits_{x\to 0^+} \dfrac{f(kx)-f(0)}{kx}.k = \alpha \end{cases}$$

$$= \begin{cases} f'(0^-) - kf'(0^-) = \alpha \\ f'(0^+) - kf'(0^+) = \alpha \end{cases}$$

$$= \begin{cases} (1-k)f'(0^-) = \alpha \\ (1-k)f'(0^+) = \alpha \end{cases}$$

$$= \begin{cases} f'(0^-) = \dfrac{\alpha}{1-k} \\ f'(0^+) = \dfrac{\alpha}{1-k} \end{cases}$$

$$\therefore\ f'(0) = f'(0^-) = f'(0^+) = \frac{\alpha}{1-k}$$

18 A derivable function $f: R^+ \to R$ satisfies the condition $f(x) - f(y) \ge \ln(x/y) + x - y$ for every $x, y \in R^+$. If g denotes the derivative of f then compute the value of the sum $\sum\limits_{n=1}^{100} g\left(\dfrac{1}{n}\right)$.

Sol. $f(x) - f(x) \ge \ell n(x/y) + x - y$

$\Rightarrow f(x) - f(y) \ge \ell nx - \ell ny + x - y$

$$\Rightarrow \frac{f(x)-f(y)}{x-y} \ge \frac{\ell n x - my}{x-y} + 1 \qquad [\text{for } x \ne y]$$

$$\Rightarrow \lim_{x\to y} \frac{f(x)-t(y)}{x-y} \ge \lim_{x\to y} \frac{\ell n\ x - \ell ny}{x-y} + 1$$

$$\Rightarrow \lim_{h\to 0} \frac{f(h+y)-f(y)}{h} \ge \lim_{h\to 0} \frac{m\left(\dfrac{y+h}{y}\right)}{h} + 1$$

$$\Rightarrow f'(y) \ge \lim_{h\to 0} \ell n\left(1+\frac{h}{y}\right)^{1/h} + 1$$

19 If $y = \dfrac{x^2}{2} + \dfrac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ prove that $2y = xy' + \ln y'$. where ' denotes the derivative.

[Sol. $y = \dfrac{x^2}{2} + \dfrac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$

$$y' = x + \frac{1}{2}\left[\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right] + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2}\left[\frac{2x^2+1}{\sqrt{x^2+1}}\right] + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{1}{2\sqrt{x^2+1}}[2(x^2+1)]$$

$$y' = x + \sqrt{x^2+1}$$

Also $2y = x^2 + x\sqrt{x^2+1} + ln(x + \sqrt{x^2+1})$

$= x(x + \sqrt{x^2+1}) + ln(x + \sqrt{x^2+1}) = xy' + ln\, y'$ Hence proved]

20 If $y = \sec 4x$ and $x = \tan^{-1}(t)$, prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$.

[Sol. $y = \frac{1}{\cos 4x} = \frac{1+\tan^2 2x}{1-\tan^2 2x}$ ....(1)

using $\tan x = t$ (given)

$\tan 2x = \frac{2t}{1-t^2}$

substituting in (1)

$$y = \frac{1+\frac{4t^2}{(1-t^2)^2}}{1-\frac{4t^2}{(1-t^2)^2}} = \frac{(1+t^2)^2}{(1-t^2)^2-4t^2} = \frac{(1+t^2)^2}{1-6t^2+t^4}$$

$$\frac{dy}{dt} = \frac{(1-6t^2+t^4)\cdot 2(1+t^2)\cdot 2t-(1+t^2)(4t^3-12t)}{(1-6t^2+t^4)^2}$$

$$= \frac{4t(1+t^2)[(1-6t^2+t^4)-(1+t^2)(t^2-3)]}{\left(1-(t^2+t^4)^2\right)} = \frac{4t(1+t^2)(1-t^2)}{(1-6t^2+t^4)^2} = \frac{4t(1-t^4)}{(1-6t^2+t^4)^2}$$]

21 If $x = \frac{1+lnt}{t^2}$ and $y = \frac{3+2lnt}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

[Sol. $\frac{dx}{dt} = \frac{t-(1+lnt)2t}{t^4} = \frac{t(1-2-lnt)}{t^4} = -\frac{(1+2lnt)}{t^3}$

$$\frac{dy}{dt} = \frac{t\left(\frac{2}{t}\right)-(3+2\ln t)}{t^2} = -\frac{(1+2\ln t)}{t^2}$$

$$\frac{dy}{dx} = \frac{1+2\,ln\,t}{t^2}\cdot\frac{t^3}{1+2\,ln\,t} = t$$

Now L.H.S. $= \frac{3+2lnt}{t}.t = 3 + 2\,lnt$

$$\text{R.H.S.} = \frac{2(1+\ln t)}{t^2} \cdot t^2 + 1 = 3 + 2\ln 2$$

$\Rightarrow$ L.H.S. = R.H.S.]

22 If $y = 1 + \frac{x_1}{x - x_1} + \frac{x_2 \cdot x}{(x-x_1)(x-x_2)} + \frac{x_3 \cdot x^2}{(x-x_1)(x-x_2)(x-x_3)} + \ldots$ upto (n+1) terms then prove that

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + \frac{x_3}{x_3 - x} + \ldots + \frac{x_n}{x_n - x}\right]$$

[Sol. adding term by term

$$y = \frac{x^n}{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}$$

$$y = \frac{x}{(x-x_1)} \cdot \frac{x}{(x-x_2)} \cdot \frac{x}{(x-x_3)} \ldots\ldots \frac{x}{(x-x_n)}$$

$$\ln y = \ln\frac{x}{(x-x_1)} + \ln\frac{x}{(x-x_2)} + \ln\frac{x}{(x-x_3)} + \ldots\ldots + \ln\frac{x}{(x-x_n)}$$

$$\text{now } D\left(\frac{x}{x-x_n}\right) = \frac{x-x_n}{x}\left(\frac{(x-x_n)-x}{(x-x_n)^2}\right) = \frac{1}{x}\left(\frac{x_n}{x_n - x}\right)$$

$$\text{Hence } \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + \ldots + \frac{x_n}{x_n - x}\right]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + \ldots + \frac{x_n}{x_n - x}\right]$$]

23 Suppose $f(x) = \tan\left(\sin^{-1}(2x)\right)$

(a) Find the domain and range of f.

(b) Express $f(x)$ as an algebaric function of x.

(c) Find $f'(1/4)$. [Ans. (a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(-\infty, \infty)$; (b) $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; (c) $\frac{16\sqrt{3}}{9}$]

[Sol. $f(x) = \tan\left(\sin^{-1}(2x)\right)$

(a) for f to be well defined

$$-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2} \quad \left[\because \text{ for } x = \pm\frac{1}{2}, \tan\frac{\pi}{2} \text{ is not defined}\right]$$

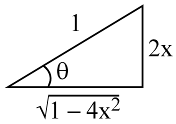
Hence domain is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\sin^{-1}2x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ hence for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ can take all real values.

Hence range of f is $x \in R$

(b) $f(x) = \tan\theta$ where $\sin^{-1}(2x) = \theta \Rightarrow \sin\theta = 2x$

$$f(x) = \frac{2x}{\sqrt{1-4x^2}}$$



(c) $f'(x) = \frac{\sec^2\left(\sin^{-1}(2x)\right)}{\sqrt{1-4x^2}} \cdot 2$

$$f'\left(\frac{1}{4}\right) = \frac{2\sec^2\left(\sin^{-1}\left(\frac{1}{2}\right)\right)}{\sqrt{1-\frac{1}{4}}} = \frac{2\times 2}{\sqrt{3}} \cdot \frac{4}{3} = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$$]

24 If $x = \tan\frac{y}{2} - ln\left[\frac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$. Show that $\frac{dy}{dx} = \frac{1}{2}\sin y(1 + \sin y + \cos y)$.

Sol Put $\tan\frac{y}{2} = t$ $\quad\therefore\quad$ $\sin y = \frac{2t}{1+t^2}, \cos y = \frac{1-t^2}{1+t^2}$

$\therefore \quad 1 + \sin y + \cos y = \frac{2+2t}{1+t^2}$

and $\quad y = 2\tan^{-1} t \quad$ ...(1)

$\therefore \quad \frac{dy}{dt} = \frac{2}{1+t^2} \quad$ ...(2)

Now $\quad x = t - 2\log(1+t) + \log t$

$\therefore \quad \frac{dx}{dt} = 1 - \frac{2}{1+t} + \frac{1}{t} = \frac{t^2+1}{t(t+1)}$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} + \frac{dx}{dt} = \frac{2}{1+t^2} \cdot \frac{t^2+t}{1+t^2}$, by (2) & (3)

or $\quad \frac{dy}{dx} = \frac{2t}{1+t^2} \cdot \frac{1}{2} \frac{2t+2}{1+t^2}$

$= \frac{1}{2}\sin y(1 + \sin y + \cos y)$, by (1)

25 If $y = \operatorname{arc}\cos\sqrt{\frac{\cos 3x}{\cos^3 x}}$ then show that $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}$, $\sin x > 0$.

Sol We have,

$$y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$\therefore \quad \cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$

$\Rightarrow \quad \cos y = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$

$\Rightarrow \quad \cos y = \sqrt{4 - 3\sec^2 x}$

$\Rightarrow \quad \cos^2 y = 4 - 3\left(1 + \tan^2 x\right)$

$\Rightarrow \quad 1 - \cos^2 y = 3\tan^2 x$

$\Rightarrow \quad \sin^2 y = 3\tan^2 x$

$\Rightarrow \quad \sin y = \sqrt{3}\tan x$

Differentiating both side with respect to x, we get, $\cos y \dfrac{dy}{dx} = \sqrt{3}\sec^2 x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos y \cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos^2 x}\sqrt{\frac{\cos^3 x}{\cos 3x}} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

Hence Proved $\dfrac{dy}{dx} = \sqrt{\dfrac{6}{\cos 2x + \cos 4x}}$, $\sin x > 0$.

26 Prove that if $|a_1 \sin x + a_2 \sin 2x + \ldots\ldots + a_n \sin nx| \le |\sin x|$ for $x \in R$, then $|a_1 + 2a_1 + 3a_3 + \ldots\ldots + na_n| \le 1$

[Sol. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \ldots\ldots\ldots + a_n \sin nx$

$f'(x) = a_1 \cos x + 2a_2 \cos 2x + \ldots\ldots + na_n \cos nx$

$f'(0) = a_1 + 2a_2 + \ldots\ldots + na_n$

Hence TPT $|f'(0)| \le 1$

Given $|f(x)| \le |\sin x|$ for $x \in R$

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} \qquad \text{(as } f(0) = 0\text{)}$$

$$|f'(0)| = \lim_{h\to 0}\left|\frac{f(h)}{h}\right| \le \lim_{h\to 0}\left|\frac{\sin h}{h}\right| = 1 \qquad [\text{as } |f(x)| \le |\sin x|\ ]$$

Hence $|f'(0)| \le 1$]

27 Show that the substitution $z = ln\left(\tan\dfrac{x}{2}\right)$ changes the equation $\dfrac{d^2y}{dx^2} + \cot x\dfrac{dy}{dx} + 4y\,\text{cosec}^2 x = 0$ to $(d^2y/dz^2) + 4y = 0$.

Sol Since $x = l\,n \tan\left(\dfrac{x}{2}\right)$

$$\therefore \quad \frac{dz}{dx} = \text{cosec}\,x \quad \text{or} \quad \frac{dx}{dz} = \sin x \qquad \ldots(1)$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx} = \text{cosec}\,x \cdot \dfrac{dy}{dz}$ [From (1)] ...(2)

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\text{cosec}\,x\,\frac{dy}{dz}\right)$$

$$= \operatorname{cosec} x \frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(-\operatorname{cosec} x \cot x\right)$$

$$= \operatorname{cosec} x \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} - \operatorname{cosec} x \cot x \frac{dy}{dz}$$

$$= \operatorname{cosec}^2 x \frac{d^2y}{dz^2} - \operatorname{cosec} x \cot x \frac{dy}{dz} \qquad \text{[From (1)]} \qquad ....(3)$$

But given $\quad \dfrac{d^2y}{dx^2} + \cot x \dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$

$$\operatorname{cosec}^2 x \frac{d^2y}{dz^2} - \operatorname{cosec} x \cot x \frac{dy}{dz} + \cot x \operatorname{cosec} x \frac{dy}{dz} + 4y \operatorname{cosec}^2 x = 0 \qquad \text{[From (2) and (3)]}$$

$$\Rightarrow \quad \operatorname{cosec}^2 x \frac{d^2y}{dz^2} + 4y \operatorname{cosec}^2 x = 0 \qquad \text{or} \qquad \frac{d^2y}{dz^2} + 4y = 0$$

28 Let $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then prove that

(a) f is continuous and differentiable for all x. (b) f' is continuous and differentiable for all x.

[Sol. $f'(x) = \begin{cases} xe^x + e^x = e^x(x+1), & x < 0 \\ 1 + 2x - 3x^2 & x > 0 \end{cases}$

$\lim\limits_{x \to 0^-} f'(x) = 1; \quad \lim\limits_{x \to 0^+} f'(x) = 1$

hence f (x) is continuous hence f is continuous and differentiable at x = 0

Again $f''(x) = \begin{cases} e^x + (x+1)e^x = e^x(x+2), & x < 0 \\ 2 - 6x & x > 0 \end{cases}$

$\lim\limits_{x \to 0^+} f''(x) = \lim\limits_{x \to 0^-} f''(x) = 2 \quad \Rightarrow \quad$ f ' (x) is also continuous and differentiable]

29 Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x) = 0$ and that $f(x) = f(0) + kx$ where k denotes the sum of all the co-factors of the elements in f(0).

[Hint: $f'(x) = \begin{vmatrix} 1 & 1 & 1 \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ 1 & 1 & 1 \\ p+x & q+x & r+x \end{vmatrix} + \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ 1 & 1 & 1 \end{vmatrix}$

$f''(x) = 0$ (obviously – two identical rows)

$f'(x) = k \Rightarrow f(x) = kx + x, \; f(0) = c$

$\Rightarrow \quad f(x) = f(0) + kx$. Note that $f'(x) = k$

$\Rightarrow \quad f'(0) = k = \begin{vmatrix} 1 & 1 & 1 \\ \ell & m & n \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ p & q & r \end{vmatrix} + \begin{vmatrix} a & b & c \\ \ell & m & n \\ 1 & 1 & 1 \end{vmatrix}$

$= \; {}_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23}) + (c_{31} + c_{32} + c_{33})$

= sum of co-factors of elements f(0)]

30 If Y = sX and Z = tX, where all the letters denotes the functions of x and suffixes denotes the differentiation w.r.t. x then prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$

Sol Since $Y = sX$ and $Z = tX$ ...(1)

$\therefore$ $Y_1 = sX_1 + Xs_1$ and $Z_1 = tX_1 + Xt_1$ ...(2)

$\Rightarrow$ $Y_2 = sX_2 + Xs_2 + 2s_1X_1$ and $Z_2 = tX_2 + Xt_2 + 2t_1X_1$ ...(3)

$$L.H.S = \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$\begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + Xs_1 & tX_1 + Xt_1 \\ X_2 & sX_2 + Xs_2 + 2s_1X_1 & tX_2 + Xt_2 + 2t_1X_1 \end{vmatrix}$$ [From (1),(2) and (3)]

Applying $C_2 \to C_2 - sC_1$ and $C_3 \to C_3 - tC_1$

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

Expand w.r.t. first row, then

$$= X \begin{vmatrix} Xs_1 & Xt_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X^3 \begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

Applying $R_2 \to R_2 - 2X_1R_1$ $= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} = R.H.S.$

28 A function $f : R \to R$ is defined as $f(x) = \lim_{n\to\infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}}$ where f is continuous on R. Find the value of a, band c.

Sol. $f(x) = \lim_{n\to\infty} \frac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}}$

$$= \begin{cases} \lim_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{nx}} & ; x < 0 \\ \lim_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{xn}} & ; x = 0 \\ \lim_{n\to\infty} \dfrac{ax^2 + bx + c + e^{nx}}{1 + c.e^{xn}} & ; x > 0 \end{cases}$$

$$= \begin{cases} \dfrac{ax^2+bx+c+0}{1+c.0} & ; \quad x<0 \left(\lim\limits_{n\to\infty} e^{nx}=0\right) \\ \dfrac{c+1}{c+1} & ; \quad x=0 \\ \lim\limits_{n\to\infty} \dfrac{\dfrac{ax^2}{e^{nx}}+\dfrac{bx}{e^{nx}}+\dfrac{c}{e^{nx}}+1}{\dfrac{1+c}{e^{nx}}} & ; \quad x>0 \end{cases}$$

$$\left(\lim_{h\to\infty} e^{hx}=\infty\right)$$

$$= \begin{cases} ax^2+bx+c & ; \quad x<0 \\ 1 & ; \quad x=0 \\ \dfrac{1}{c} & ; \quad x>0 \end{cases}$$

since f(x) is continuous function $\forall\ x \in R$

$\therefore \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = f(0)$

$\Rightarrow \lim\limits_{x\to 0^+}\left(\dfrac{1}{c}\right) = \lim\limits_{x\to 0^-}(ax^2+bx+c) = 1 \qquad \Rightarrow \lim\limits_{x\to 0^+}\dfrac{1}{c} = 1\ \&\ \lim\limits_{x\to 0^-}(ax^2+bx+c)=1$

$\Rightarrow \dfrac{1}{c} = 1 \qquad \Rightarrow a - 0 + 3.0 + c = 1$

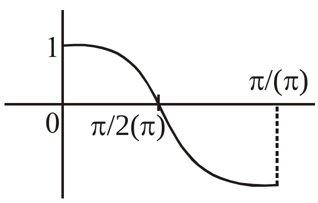
$\therefore c = 1 \qquad \Rightarrow c = 1$

$\therefore c = 1,\ a,\ b \in R$

29 Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x-5|\,[x] & \text{for } x>1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where [x] is the greatest integer not greater than x.

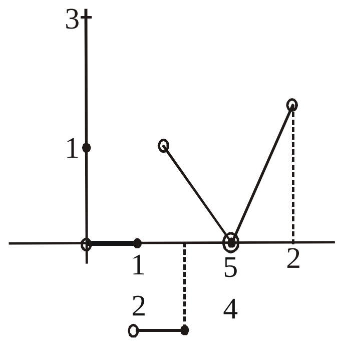
Sol. f(x) = cos πx

$$[\cos \pi x] = \begin{cases} 1 & ; \quad x=0 \\ 0 & ; \quad x<x\le \dfrac{1}{2} \\ -1 & ; \quad \dfrac{1}{2}<x\le 1 \end{cases}$$



$$|4x-5|\,[x] = \begin{cases} |4x-5| ; & 1 < x < 2 \\ 6 ; & x = 2 \end{cases} = \begin{cases} (4x-5) ; & 1 < x < \frac{5}{4} \\ 4x-5 ; & \frac{5}{4} \le x < z \\ 6 ; & x = 2 \end{cases}$$

$$f(x) = \begin{cases} 1 ; & x = 0 \\ 0 ; & 0 < x \le \frac{1}{2} \\ -1 ; & \frac{1}{2} < x \le 1 \\ -(4x-5) ; & 1 < x < \frac{5}{4} \\ 4x-5 ; & \frac{5}{4} \le x < 2 \\ 6 ; & x = 2 \end{cases}$$



function dis at 0, $0, \frac{1}{2}, 1, 2$

30 If f(x) = x + {-x} + [x] , where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [– 2, 2].

Sol. f(x) = x + {–x} + [x]

$\because$ {x} = x – [x]

{–x} = – x – [–x]

f(x) = x + (– x – [– x] + [x])

f(x) = [x] – [–x] < x – (–x) = 2x; x $\in$ I ; [x] – (– [x] – 1) = 1 – 2[x]; x $\notin$ I

$$f(x) = \begin{cases} 2x ; & x \in I \\ 1-2[x] ; & x \notin I \end{cases}$$

$$f(x) = \begin{cases} -4 & ; \quad x = -2 \\ 5 & ; \quad -2 < x < -1 \\ -2 & ; \quad x = -1 \\ 3 & ; \quad -1 < x < 0 \\ 0 & ; \quad x = 0 \\ 1 & ; \quad 0 < x < 1 \\ 2 & ; \quad x = 1 \\ -1 & ; \quad 1 < x < 2 \\ 4 & ; \quad x = 2 \end{cases}$$

so the function is discontinuous at all integers in [–2, 2].

31 Find the locus of (a, b) for which the function $f(x) = \begin{bmatrix} ax - b & \text{for} & x \le 1 \\ 3x & \text{for} & 1 < x < 2 \\ bx^2 - a & \text{for} & x \ge 2 \end{bmatrix}$ is continuous at x = 1 but discontinuous at x = 2.

Sol. conti at x = 1

$a - b = 3$ ...(1)

dis at x = 2

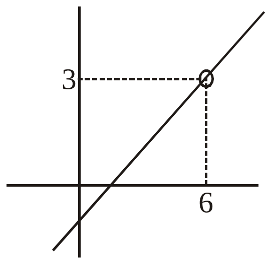
$\neq 4b - a$

$6 \neq 4b - 3 - b$

$6 \neq 3b - 3$

$\boxed{b \neq 3}$

$\boxed{a \neq 6}$



$(a, b) \neq (6, 3)$

$(x, y) \neq (6, 3)$ **Ans**

32 $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for $x > 0$

$= \dfrac{ln(1 + x + x^2) + ln(1 - x + x^2)}{\sec x - \cos x}$ for $x < 0$, if f is continuous at x = 0, find 'a'

now if $g(x) = ln\left(2 - \dfrac{x}{a}\right) \cdot \cot(x - a)$ for $x \neq a$, $a \neq 0$, $a > 0$. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

Sol. Since the function is conti at x = 0 then

$V.F.|_{x=0} = RHL|_{x=0} = LHL|_{x=0}$

$RHL|_{x=0} = \lim\limits_{x \to 0^+} f(x)$

$= \lim\limits_{x \to 0^+} \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$

since the function is conti then

$f(0) = LHL|_{x=0} = RHL|_{x=0}$

$-\ell na = 1$

$$= \lim_{x\to 0^+} \frac{a^{\tan x}(a^{\sin x - \tan x} - 1)}{-1(\sin x - \tan x)}$$

$$\boxed{\text{RHL}|_{x=0} = -\ell na}$$

$$\text{LHL}|_{x=0} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} \frac{\ell n(1+x+x^2) + \ell n(1-x+x^2)}{\sec x - \cos x}$$

$$= \lim_{x\to 0^-} \frac{\ell n((1+x+x^2)\,(1-x+x^2)).\cos x}{1-\cos 2x}$$

put $x = 0 - h$

$$= \lim_{h\to 0} \frac{\ell n(1+h^2+h^4)\cosh}{\sin^2 h}$$

$$= \lim_{h\to 0} (h^2+h^4)\frac{\cosh}{\sin^2 h}$$

$$= \lim_{h\to 0} \left(\frac{h}{\sinh}\right)^2 (1+h^2)\cosh$$

$$\boxed{a = \frac{1}{e}}$$

since $g(x)$ conti at $x = a$

$$g(a) = \lim_{x\to a} g(x)$$

$$= \lim_{x\to a} \ell n\left(2 - \frac{x}{a}\right)\cot(x-a)$$

$$= \lim_{x\to a} \frac{\ell n\left(2-\frac{x}{a}\right)}{\tan(x-a)}$$

put $x = a + h$

$$= \lim_{h\to 0} \frac{\ell n\left(1-\frac{h}{a}\right)}{\left(-\frac{h}{a}\right)}.\frac{h}{\tan}\left(-\frac{1}{a}\right)$$

$$g(a) = -\frac{1}{a}$$

$$\boxed{g\left(\frac{1}{e}\right) = -e}$$

33 Find the value of $\lim\limits_{x\to 0^+} x^{\left(x^x - 1\right)}$. [Ans. 1]

[Sol. $l = \lim\limits_{x\to 0^+} x^{(x^x-1)}$ (0^0 form)

$$\ln l = \lim_{x\to 0} (x^x - 1).\ln x = \text{Limit}_{x\to 0} \frac{\left(e^{x\ln x} - 1\right)}{x\ln x} \ \text{Limit}_{x\to 0} \ x\ln x\,.\ln x$$

$$= \text{Limit}_{x\to 0} \ x(\ln x)^2 \qquad (\text{ as } x\to 0 \ x\ln x \to 0)$$

$$= \text{Limit}_{x\to 0} \frac{(\ln x)^2}{1/x} = \text{Limit}_{x\to 0} -\frac{2\ln x}{x}.x^2 \quad \text{(use Lopital's rule)}$$

$$= \text{Limit}_{x\to 0} -2\ln x.x = 0 \qquad \Rightarrow \qquad l = e^0 = 1$$

34 Given $f(x) = \sum_{r=1}^{n} \tan\left(\frac{x}{2^r}\right) \sec\left(\frac{x}{2^{r-1}}\right)$; $r, n \in N$

$$g(x) = \underset{n \to \infty}{\text{Limit}} \frac{\ell n\left(f(x) + \tan \frac{x}{2^n}\right) - \left(f(x) + \tan \frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan \frac{x}{2}\right)\right]}{1 + \left(f(x) + \tan \frac{x}{2^n}\right)^n}$$

$= k$ for $x = \frac{\pi}{4}$ and the domain of $g(x)$ is $(0, \pi/2)$.

where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that g(x) is continuous at $x = \pi/4$. Also state the points of discontinuity of g (x) in $(0, \pi/4)$, if any.

Sol. $$\tan\frac{x}{2}\sec x = \frac{\sin x/2}{\cos\frac{x}{2}.\cos x} = \frac{\sin\left(x - \frac{x}{2}\right)}{\cos\frac{x}{2}.\cos x} = \frac{\sin x \cos\frac{x}{2} - \cos x \sin\frac{x}{2}}{\cos\frac{x}{2}.\cos x} = \tan x - \tan\frac{x}{2}$$

$$\tan\frac{x}{2}\sec x = \tan x - \tan\frac{x}{2}$$

$$\tan\frac{x}{2^2} . \sec\frac{x}{2} = \tan\frac{x}{2} - \tan\frac{x}{2^2}$$

$$\tan\frac{x}{2^3}.\sec\frac{x}{2^2} = \tan\frac{x}{2^2} - \tan\frac{x}{2^3}$$

.

.

.

$$\tan\frac{x}{2^n}.\sec\frac{x}{2^{n-1}} = \tan\frac{x}{2^{n-1}} - \tan\frac{x}{2^n}$$

$$f(x) = \tan x - \tan\left(\frac{x}{2^n}\right)$$

$$f(x) + \tan\left(\frac{x}{2^n}\right) = \tan x \qquad ...(1)$$

using (1)

$$g(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{\ell n(\tan x) - (\tan x)^n \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1 + (\tan x)^n} & ; \quad x \neq \dfrac{\pi}{4} \\ k & ; \quad x = \dfrac{\pi}{4} \end{cases}$$

$$g(x) = \begin{cases} \lim\limits_{h\to\infty} \dfrac{\ell n(\tan x)}{1+(\tan x)n} & ; \quad x \neq \dfrac{\pi}{4} \\ k & ; \quad x = \dfrac{\pi}{4} \end{cases}$$

k = 0

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & ; \quad x < 1 \\ 1 & ; \quad x = 1 \\ \infty & ; \quad x > 1 \end{cases}$$

$$\lim_{n\to\infty} (\tan x)^n = \begin{cases} 0 & ; \quad x < \dfrac{\pi}{4} \\ 1 & ; \quad x = \dfrac{\pi}{4} \\ \infty & ; \quad x > \dfrac{\pi}{4} \end{cases}$$

35 Let f be continuous on the interval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

Sol. Consider a conti function

$$g(x) = f\left(x + \frac{1}{2}\right) - f(x);\ g \text{ is conti } \forall\, x \in \left[0, \frac{1}{2}\right]$$

Now

$$g(0) = f\left(\frac{1}{2}\right) - f(0) \Rightarrow g(0) = f\left(\frac{1}{2}\right) - f(1)$$

$$g\left(\frac{1}{2}\right) g(1) - f\left(\frac{1}{2}\right) \Rightarrow g(0) = f(1) - f\left(\frac{1}{2}\right)$$

since g is continuous and g(0) and $g\left(\frac{1}{2}\right)$ are of opposite sign hence the equation g(x) = 0 must have at least one root in $\left[0, \frac{1}{2}\right]$.

$\therefore$ for some $c \in \left[0, \frac{1}{2}\right]; g(c) = 0$

$$\Rightarrow f\left(c+\frac{1}{2}\right) = f(c)$$

36 Consider the function $g(x) = \begin{cases} \dfrac{1-a^x + xa^x \ell na}{a^x x^2} & ; \ x<0 \\ \dfrac{2^x a^x - x\ell n 2 - x\ell n a - 1}{x^2} & ; \ x>0 \end{cases}$

where a > 0, find the value of 'a' & 'g(0)' so that the function g(x) is continuous at x = 0.

Sol. $LHL|_{x=0} = \lim_{x\to 0^-} g(x)$

$$= \lim_{x\to 0^-}\left(\frac{1-a^x + xa^x \ell na}{a^x a^2}\right)$$

put x = a – h

$$= \lim_{x\to 0}\left(\frac{1-a^{-h} - ha^{-h}\ell na}{a^{-h}h^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{a^h - 1 - h\ell na}{2h}\right); \frac{0}{0}\text{form}$$

$$= \lim_{h\to 0}\left(\frac{a^h \ell na - 0 - \ell na}{2h}\right); \frac{0}{0} \ \textbf{Ans}$$

$$= \lim_{h\to 0}\left(\frac{a^h(\ell na)^2}{2}\right)$$

$$\boxed{LHL|_{x=0} = \frac{(\ell na)^2}{2}}$$

$RHL|_{x=0} = \lim_{x\to 0^+} g(x)$

$$= \lim_{x\to 0^+}\left(\frac{2^x a^x - x\ell n2 - x\ell na - 1}{x^2}\right)$$

put x = 0 + h

$$= \lim_{h\to 0}\left(\frac{(2a)^h - h\ell n2 - h\ell na - 1}{h^2}\right); \frac{0}{0}\text{form}$$

$$\boxed{RHL|_{x=0} = \frac{(\ell n2a)^2}{2}}$$

since the function is conti

$g(0) = LHL|_{x=0} = RHL|_{x=0}$

$$\frac{(\ell n(2a))^2}{2} = \frac{(\ell na)^2}{2}$$

$(\ell n\ 2a + \ell na)\ (\ell n\ 2a - \ell na) = 0$

$\ell n(2a^2).\ \ell n2 = 0$

$\ell n2a^2 = 0$

$2a^2 = 1,\ a = \pm \frac{1}{\sqrt{2}}\ ;\ a > 0$

$$\boxed{a = \frac{1}{\sqrt{2}}}$$

$$\therefore\ g(0) = \frac{(\ell n2a)^2}{2}$$

$$= \frac{1}{2}\left(\ell n2.\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2}(\ell n\sqrt{2})^2$$

$= \lim_{h\to 0} \frac{(2a)^h \ell n2a - \ell na2}{2h}; \frac{0}{0}$ form $\qquad = \frac{1}{2}\left(\frac{1}{4}(\ell n2)^2\right)$

$= \lim_{h\to 0} \frac{(2a)^h (\ell n2a)^2}{2} \qquad = \frac{1}{8}(\ell n2)^2$

37 A function $f: R \to R$ satisfies the equation $f(x+y) = f(x).\ f(y)$ for all x,y in R and $f(x) \neq 0$ for any x in R. Let the function the differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R. Hence determine $f(x)$.

Sol Given that $f(x+y) = f(x).f(y)$ for all $x \in R$ ...(1)

Putting $x = y = 0$ in (1), we get

$f(0)\{f(0)-1\} = 0 \qquad \Rightarrow \qquad f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$, then $f(x) = f(x+0) = f(x).f(0) = 0$ for all $x \in R$

Which is not true (given $f(x) \neq 0$)

So, $f(0) = 1$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h}$$

$$= f(x)\lim_{h\to 0} \frac{f(h) - 1}{h}$$

$$= f(x)\lim_{h\to 0} \frac{f(x) - f(0)}{h - 0} \qquad (\because\ f(0) = 1)$$

$$= f(x)f'(0) = 2f(x) \qquad (\because\ f'(0) = 2)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = 2$$

Integrating both sides w.r.t.x and taking limit 0 to x

$$\int_0^x \frac{f'(x)}{f(x)} dx = \int_0^x 2\ dx$$

$\Rightarrow \quad \ln f(x) - \ln f(0) = 2x \qquad \Rightarrow \quad \ln f(x) - \ln 1 = 2x$

$\Rightarrow \quad \ln f(x) - 0 = 2x \qquad \therefore \quad f(x) = e^{2x}.$

38 Let f be a function such that $f(x + f(y)) = f(f(x)) + f(y) \quad \forall\ x, y \in R$ and $f(h) = h$ for $0 < h < \varepsilon$ where $\varepsilon > 0$, then determine $f'(x)$ and $f(x)$.

Sol Given $f(x + f(y)) = f(f(x) + f(y))$ ....(1)

Putting $x = y = 0$ in (1), then

$$f(0 + f(0)) = f(f(0)) + f(0) \quad \Rightarrow \quad f(f(0)) = f(f(0)) + f(0)$$

$\therefore \quad f(0) = 0$ ....(2)

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ $\quad (\text{for } 0 < h < \varepsilon)$

$$= \lim_{h\to 0} \frac{f(h+x) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(f(h))}{h} \quad (\text{form } (1))$$

$$= \lim_{h\to 0} \frac{f(h)}{h} \quad (\because f(h) = h)$$

$$= \lim_{h\to 0} \frac{h}{h} = 1 \quad (\because f(h) = h)$$

Integrating both sides with limites 0 to x then $f(x) = x$

$\therefore \quad f'(x) = 1.$

39 Let $f(x) = \begin{cases} -2 & , \quad -3 \le x \le 0 \\ x - 2 & , \quad 0 < x \le 3 \end{cases}$, where $g(x) = f(|x|) + |f(x)|$. Test the differentiability of $g(x)$ in the interval $(-3, 3)$.

Sol From the given function

$$f(|x|) = \begin{cases} -x-2 & \text{for} \quad -3 \le x \le 0 \\ x - 2 & \text{for} \quad 0 < x \le 3 \end{cases} \text{ and } |f(x)| = \begin{cases} 2 & \text{for} \quad -3 \le x \le 0 \\ -x+2 & \text{for} \quad 0 < x \le 2 \\ x-2 & \text{for} \quad 2 < x \le 3 \end{cases}$$

$\therefore \quad g(x) = f(|x|) + |f(x)|$

$$= \begin{cases} -x & \text{for} \quad -3 \le x \le 0 \\ 0 & \text{for} \quad 0 < x \le 2 \\ 2x - 4 & \text{for} \quad 2 < x \le 3 \end{cases}$$

Check the differentiability

At

$x = 0:$ $\quad Lg'(0) = \lim_{h\to 0} \frac{g(0-h) - g(0)}{-h}$

$$= \lim_{h\to 0} \frac{-(0-h) - 0}{-h} = -1$$

$$Rg'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{(0-0)}{h} = 0$$

$\therefore \quad Lg'(0) \neq Rg'(0)$

$\therefore \quad g(x)$ is not differentiable at $x = 0$

Check at

$x = 2: \quad Lg'(2) = \lim_{h \to 0} \frac{g(2-h) - g(2)}{-h}$

$$= \lim_{h \to 0} \frac{0 - 0}{-h} = 0$$

and $\quad Rg'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$

$$= \lim_{h \to 0} \frac{2(2+h) - 4 - 0}{h} = 2$$

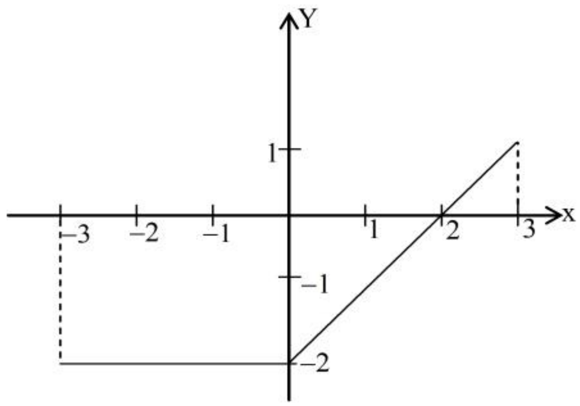
$\therefore \quad Lg'(2) \neq Rg'(2)$

Hence $g(x)$ is not differentiable at $x = 2$.

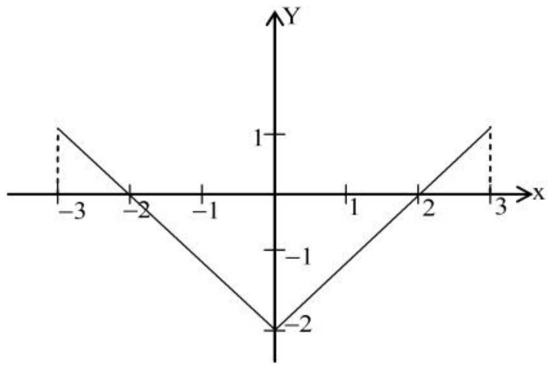
Graphical method :

$$\because \quad f(x) = \begin{cases} -2 & ; \quad -3 \le x \le 0 \\ x - 2 & ; \quad 0 < x \le 3 \end{cases}$$

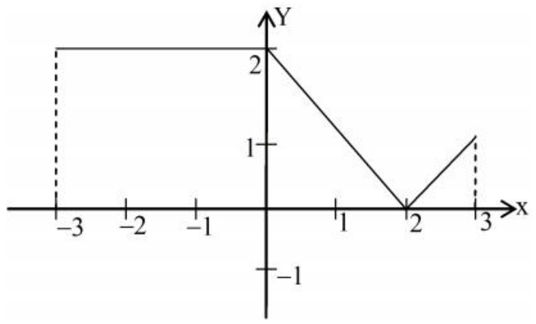
Graph of f(x) :



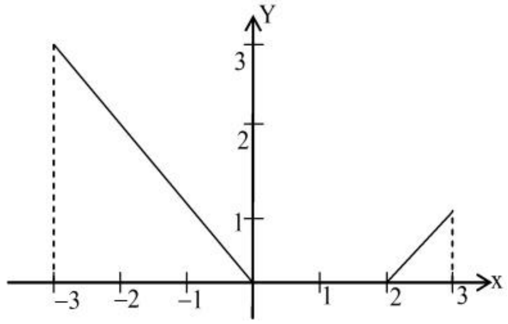
Graph of $f(|x|)$:



Graph of $|f(x)|$:



Graph of $g(x) = |f(x)| + f(|x|)$:



It is clear from the graph that g(x) is not differentiable at $x = 0$ and 2.

40 Let $f: R \to R$ is a real valued function $\forall\ x, y \in R$ such that $|f(x) - f(y)| \le |x - y|^3$.

Prove that $h(x) = \int f(x)dx$ is continuous function of $x \quad \forall \quad x \in R$.

Sol Since $|f(x) - f(y)| \le |x - y|^3 \qquad x \ne y$

$\therefore \qquad \left|\frac{f(x) - f(y)}{x - y}\right| \le |x - y|^2$

Taking lim as $y \to x$, we get

$$\lim_{y \to x}\left|\frac{f(x) - f(y)}{x - y}\right| \le \lim_{y \to x}|x - y|^2$$

$$\Rightarrow \qquad \left|\lim_{y \to x}\frac{f(x) - f(y)}{x - y}\right| \le \left|\lim_{y \to x}(x - y)^2\right|$$

$\Rightarrow \qquad |f'(x)| \le 0 \qquad \Rightarrow \qquad |f'(x)| = 0 \qquad (\because |f'(x)| \ge 0)$

$\therefore \qquad f'(x) = 0 \qquad \Rightarrow \qquad f(x) = c\ (\text{constant})$

$\therefore \qquad h(x) = \int f(x)dx = \int c\ dx = cx + d$ where d is constant of integration.

$\therefore \qquad h(x)$ is a linear function of x which is continuous for all $x \in R$.

41 Let $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$ for all real x and y. If $f'(0)$ exists and equals -1 and $f(0) = 1$,

then find $f(2)$.

Sol Since $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ ....(1)

$$\therefore \quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f\left(\frac{2x+0}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-\frac{f(2x)+f(0)}{2}}{h} \quad \text{[from (1)]}$$

$$=\lim_{h\to 0}\frac{f(2h)-f(0)}{2h-0}$$

$$=f'(0)$$

$$=-1 \quad \forall \; x\in R \quad \text{(given)}$$

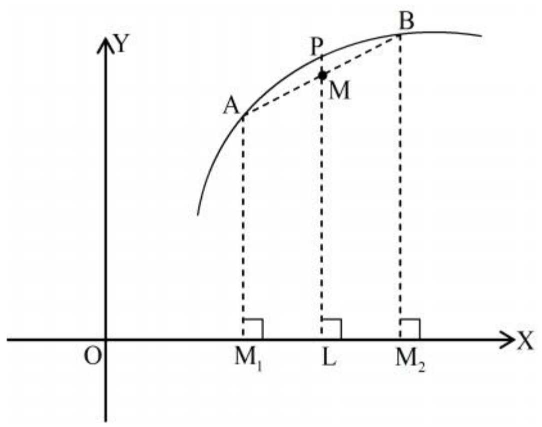
Integrating, we get $f(x)=-x+c$

Putting $x=0$, then $f(0)=0+c=1$ (given)

$\therefore \quad c=1$ then $f(x)=1-x$ $\quad\therefore\quad f(2)=1-2=-1$

Graphical method :

Suppose $A(x,f(x))$ and $B(y,f(y))$ be any two points on the curve $y=f(x)$.



If M is the mid-point of AB then co-ordinates of M are $\left(\frac{x+y}{2},\frac{f(x)+f(y)}{2}\right)$

According to the graph, co-ordinates of P are $\left(\frac{x+y}{2},f\left(\frac{x+y}{2}\right)\right)$ and $PL>ML$

$$\Rightarrow \quad f\left(\frac{x+y}{2}\right)>\frac{f(x)+f(y)}{2}$$

But given $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ which is possible when $P\to M$

i.e. P lies on AB. Hence $y = f(x)$ must be a linear function.

Let $f(x) = ax + b$ $\Rightarrow$ $f(0) = 0 + b = 1$ (given)

and $f'(x) = a$ $\Rightarrow$ $f'(0) = a = -1$ (given)

$\therefore$ $f(x) = -x + 1$ $\therefore$ $f(2) = -2 + 1 = -1$.

42 Let $f\left(\frac{x+y}{n}\right) = \frac{f(x)+f(y)}{n}$ $\forall$ $x, y \in R; n \neq 0, 2$ and if $f'(0) = k$ (A finite quantity) then prove that $f(x) = kx$ $\forall$ $x \in R$.

Sol Given $f\left(\frac{x+y}{n}\right) = \frac{f(x)+f(y)}{n}$ ...(1)

Putting $x = y = 0$, we get $(n-2)f(0) = 0$

$\therefore$ $f(0) = 0$ $(\because\ n - 2 \neq 0)$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\frac{nx+nh}{n}\right)-f\left(\frac{nx+0}{n}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{f(nx)+f(nh)}{n}-\frac{f(nx)+f(0)}{n}}{h} \qquad \text{[from (1)]}$$

$$= \lim_{h\to 0}\frac{f(nh)-f(0)}{nh-0}$$

$\Rightarrow$ $f'(x) = k$

On integrating we get $f(x) = kx + c$

Putting $x = 0$, then $f(0) = 0 + c = 0$ $(\because f(0) = 0)$

$\therefore$ $c = 0$ then $f(x) = kx$.

43 If $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2) = 2$ then determine $y = f(x)$.

Sol $\because$ $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ ...(1)

Differentiating both sides w.r.t. x treating y as constant,

then $f'\left(\frac{x+y}{3}\right)\left(\frac{1}{3}\right) = \frac{2+f'(x)+0}{3}$

Now replacing x by 0 and y by 3x, then

$$f'(x) = f'(0) = c \qquad (\text{ say })$$

At $x = 2$, $f'(2) = c = 2$ (given)

$\therefore$ $f'(x) = 2$

On integrating we get $f(x) = 2x + d$

Putting $x = 0,$ then $f(0) = 0 + d = 2$ [from (1)]

$\therefore \quad f(x) = 2x + 2$

Hence $y = 2x + 2.$

44 If $f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \quad \forall \; x, y \in R$ and $f'(0) = 1$; prove that $f(x)$ is continuous for all $x \in R.$

Sol $\because \quad f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3}$

Differentiating both sides w.r.t. x treating y as constant

$$f'\left(\frac{x+2y}{3}\right)\cdot\frac{1}{3} = \frac{f'(x)+0}{3}$$

and replacing x by 0 and y by $\frac{3x}{2}$

then $\quad f'(x) = f'(0) = 1$ (given)

On integrating, we get

$f(x) = x + d, d$ is constant of integration which is linear function in x and hence it is always continuous function for all x.

45 If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$ and $xy \neq 1$ and $\lim\limits_{x\to 0}\frac{f(x)}{x} = 2,$ find $f\left(\sqrt{3}\right)$ and $f'(-2).$

Sol Given $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

Putting $x = 0, y = 0,$ we get $\quad f(0) = 0 \quad$ ...(1)

And putting $y = -x,$ we get $f(x) + f(-x) = f(0) = 0$

$\therefore \quad f(x) = -f(-x) \quad$ ...(2)

Now $\quad f'(x) = \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x+h) + f(-x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{h}{1+x(x+h)}\right)}{\frac{h}{1+x(x+h)}}\cdot\frac{1}{1+x(x+h)}$$

$$= 2 \cdot \frac{1}{1+x^2} \qquad \left(\because \lim_{x\to 0} \frac{f(x)}{x} = 2\right)$$

$$= \frac{2}{1+x^2}$$

$\therefore \quad f(x) = 2\tan^{-1} x + c \qquad$ or $\qquad f(0) = 2\tan^{-1} 0 + c = 0$

$\Rightarrow \quad 0 = 0 + c \qquad \therefore \quad c = 0$

then $f(x) = 2\tan^{-1} x$

$\therefore \quad f(\sqrt{3}) = 2\tan^{-1}(\sqrt{3}) = \frac{2\pi}{3} \qquad$ and $\qquad f'(-2) = \frac{2}{1+(-2)^2} = \frac{2}{5}$.

46 Let $f(x+y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in R$. If $f(x)$ is differentiable and $f'(0) = \sin\phi$ then prove that $f(x) > 0 \ \forall \ x \in R$.

Sol Given $f(x+y) = f(x) + f(y) + 2xy - 1 \quad \forall \ x, y \in R \qquad ...(1)$

Putting $x = y = 0$ in (1), we get

$$f(0) = 1 \qquad ...(2)$$

$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) + 2xh - 1}{h}$$

$$= \lim_{h\to 0} \frac{f(h) - 1}{h} + \lim_{h\to 0}\left(\frac{2xh}{h}\right)$$

$$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} + \lim_{h\to 0}(2x)$$

$$= f'(0) + 2x$$

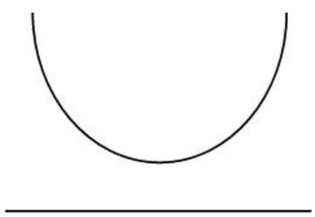
$$= \sin\phi + 2x \qquad (\because f(0) = \sin\phi)$$

Integrating both sides w.r.t. x and taking limit 0 to x, then

$\int_0^x f'(x)\,dx = \int_0^x (\sin\phi + 2x)\,dx$

$\Rightarrow \quad f(x) - f(0) = x\sin\phi + x^2$

$\Rightarrow \quad f(x) = x^2 + x\sin\phi + 1 \qquad (\because f(0) = 1)$



Here coefficient of x^2 is $1 > 0$ and Discriminant

$D = \sin^2\phi - 4 < 0$.

Hence it is clear from graph $f(x) > 0 \quad \forall \ x \in R$.

47 Let f be a one-one function such that $f(x)f(y)+2=f(x)+f(y)+f(xy)\ \forall\ x,y\in R\sim\{0\}$ and $f(0)=1, f'(1)=2$ then prove that $3\int f(x)dx - x(f(x)+2)$ is constant.

Sol We have $f(x)f(y)+2=f(x)+f(y)+f(xy)$ ...(1)

Putting $x=1$ and $y=1$, we get

$$(f(1))^2+2=3f(1)$$

$\therefore$ $f(1)=1,2$ $\Rightarrow$ $f(1)=2$ ...(2)

$f(1)\neq 1$ ($\because f(0)=1$ and f is one-one function)

In (1), replacing y by $\frac{1}{x}$

$\therefore$ $f(x)f\left(\frac{1}{x}\right)+2=f(x)+f\left(\frac{1}{x}\right)+f(1)$

$\Rightarrow$ $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ $(\because f(1)=2)$

$\therefore$ $f(x)=1\pm x^n\ (x\in N)$

$\Rightarrow$ $f'(x)=\pm nx^{n-1}$ $\Rightarrow$ $f'(1)=\pm n=2$

Taking positive sign $\Rightarrow$ $n=2$ then $f(x)=1+x^2$

Now, $3\int f(x)dx - x(f(x)+2)$

$=3\int(1+x^2)dx - x(1+x^2+2)$

$=3\left(x+\frac{x^3}{3}\right)+c-3x-x^3$

$=c=$ constant.

48 If $e^{-xy}f(xy)=e^{-x}f(x)+e^{-y}f(y)\ \forall\ x,y\in R^+$, and $f'(1)=e$, determine $f(x)$.

Sol Given $e^{-xy}f(xy)=e^{-x}f(x)+e^{-y}f(y)$ ...(1)

Putting $x=y=1$ in (1) we get $f(1)=0$ ...(2)

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x\cdot 1)}{h}$

$=\lim_{h\to 0}\frac{e^{x+h}\cdot\left\{e^{-x}f(x)+e^{-1-\frac{h}{x}}f\left(1+\frac{h}{x}\right)\right\}-e^{x}\left(e^{-x}f(x)+e^{-1}f(1)\right)}{h}$

$$= \lim_{h\to 0} \frac{e^h f(x) + e^{x+h-1-\frac{h}{x}} f\left(1+\frac{h}{x}\right) - f(x) - e^{x-1} f(1)}{h}$$

$$= f(x) \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right) + e^{(x-1)} \lim_{h\to 0} \frac{e^{h-\frac{h}{x}} f\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}} \qquad (\because f(1)=0)$$

$$= f(x).1 + e^{x-1}\cdot\frac{f'(1)}{x}$$

$$= f(x) + \frac{e^{x-1}.e}{x} \qquad (\because f'(1)=e)$$

$$f'(x) = f(x) + \frac{e^x}{x} \qquad \Rightarrow \qquad e^{-x} f'(x) - e^{-x} f(x) = \frac{1}{x}$$

$$\Rightarrow \qquad \frac{d}{dx}\left(e^{-x} f(x)\right) = \frac{1}{x}$$

On integrating we have $e^{-x} f(x) = \ln x + c$ at $x = 1, c = 0$

$\therefore \qquad f(x) = e^x \ln x.$

49 Let $f : R \to R$, such that $f'(0) = 1$

and $f(x+y) = f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy \ \forall\, x, y \in R$ then determine $f(x)$.

Sol Given $f(x+y) = f(x) + f(y) + e^{x+y}(x+y) - xe^x - ye^y + 2xy$ ...(1)

Putting $x = y = 0$, we get $f(0) = 0$ ...(2)

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) + e^{x+h}(x+h) - xe^x - he^h + 2xh - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h) + xe^x(e^h - 1) + he^{x+h} - he^h + 2xh}{h}$$

$$= \lim_{h\to 0}\left\{\frac{f(h)}{h} + xe^x\frac{(e^h-1)}{h} + e^{x+h} - e^h + 2x\right\}$$

$$= f'(0) + xe^x.1 + e^x - 1 + 2x$$

$$= 1 + xe^x + e^x + 2x - 1$$

$$= xe^x + e^x + 2x$$

Integrating both sides w.r.t. x with limit 0 to x

$\therefore \qquad f(x) - f(0) = xe^x - e^x + e^x + x^2$

$$f(x) - 0 = xe^x + x^2$$

Hence $f(x) = x^2 + xe^x$

50 Let $f(xy) = x\,f(y) + y\,f(x)$ for all $x, y \in R_+$ and $f(x)$ be differentiable in $(0, \infty)$ then determine $f(x)$.

Sol Given $f(xy) = xf(y) + yf(x)$

Differentiating both sides w.r.t. x treating y as constant,

$$f'(xy).y = f(y) + yf'(x)$$

Putting $y = x$ and $x = 1$, then $f'(x).x = f(x) + xf'(1)$

$$\Rightarrow \quad \frac{xf'(x) - f(x)}{x^2} = \frac{f'(1)}{x} \qquad \Rightarrow \quad \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(1)}{x}$$

Integrating both sides w.r.t. x taking limit 1 to x,

$$\frac{f(x)}{x} - \frac{f(1)}{1} = f'(1)\{\ln x - \ln 1\}$$

$$\Rightarrow \quad \frac{f(x)}{x} - 0 = f'(1)\ln x \qquad (\because f(1) = 0)$$

Hence, $f(x) = f'(1)(x \ln x)$.

51 Let $f(xy) = f(x)f(y)\ \forall\ x, y \in R$ and f is differentiable at $x = 1$ such that $f'(1) = 1$ also $f(1) \neq 0$ then show that f is differentiable for all $x \neq 0$. Hence, determine $f(x)$.

Sol Given $f(xy) = f(x)f(y)$

Putting $x = y = 1$ then we get $f(1) = 1$.

Differentiating both sides w.r.t. x treating y as constant,

$$f'(xy).y = f'(x)f(y)$$

Replacing y by x and x by 1, then

$$f'(x).x = f'(1)f(x)$$

$$\Rightarrow \quad f'(x) = \frac{f(x)f'(1)}{x} = \frac{f(x)}{x} \qquad (\because f'(1) = 1)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating both sides w.r.t. x and taking limit 1 to x, then

$$\int_1^x \frac{f'(x)}{f(x)}dx = \int_1^x \frac{1}{x}dx$$

$$\Rightarrow \quad \ln f(x) - \ln f(1) = \ln x - \ln 1 \qquad (\because f(1) = 1)$$

$$\Rightarrow \quad \ln f(x) - 0 = \ln x - 0 \qquad \therefore \qquad f(x) = x.$$

52 If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all $x, y \in R^+, f(1) = 0$ and $f'(1) = 1$, then find $f(e)$ and $f'(2)$.

Sol Given $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ ...(1)

Replacing x by y and y by x in (1), then

$$2f(y) = f(xy) + f\left(\frac{y}{x}\right) \quad ...(2)$$

Subtract (2) from (1), we get

$$2\{f(x) - f(y)\} = f\left(\frac{x}{y}\right) - f\left(\frac{y}{x}\right) \quad ...(3)$$

Putting $x = 1$ in (1) then $2f(1) = f(y) + f\left(\frac{1}{y}\right) = 0 \quad (\because \; f(1) = 0)$

$\therefore \quad f(y) = -f\left(\frac{1}{y}\right) \qquad \therefore \quad f\left(\frac{y}{x}\right) = -f\left(\frac{x}{y}\right)$ ...(4)

Now from (3) and (4), we get

$$2\{f(x) - f(y)\} = 2f\left(\frac{x}{y}\right)$$

or $$f(x) - f(y) = f\left(\frac{x}{y}\right) \quad ...(5)$$

Now, $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h} \qquad \text{[From (5)]}$$

$$= \lim_{h\to 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x} \cdot x} = \frac{1}{x} f'(1) = \frac{1}{x} \qquad \{\because f'(1) = 1\}$$

$\therefore \quad f'(x) = \frac{1}{x} \qquad \Rightarrow \quad f'(2) = \frac{1}{2}$

and $f(x) = \ln x + \ln c$ for $x = 1$, and $f(1) = \ln 1 + \ln c$

$\Rightarrow \quad 0 = 0 + \ln c \qquad \therefore \quad \ln c = 0$

then $f(x) = \ln x \qquad \therefore \quad f(e) = \ln e = 1$.

. . . . $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, prove that $|a_1 + 2a_2 + ... + na_n| \le 1$.

Sol Given $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

$\therefore \quad p'(x) = 0 + a_1 + 2a_2x + ... + na_nx^{n-1}$

$\Rightarrow \quad p'(1) = a_1 + 2a_2 + ... + na_n$ ...(1)

Now, $\left|p(1)\right| \le \left|e^{1-1}-1\right|$

$$= \left|e^0 - 1\right| = \left|1-1\right| = 0$$

$\Rightarrow \quad \left|p(1)\right| \le 0 \quad \Rightarrow \quad p(1) = 0 \qquad \left(\because \left|p(1)\right| \ge 0\right)$

As $\quad \left|p(x)\right| \le \left|e^{x-1}-1\right|$

we get $\left|p(1+h)\right| \le \left|e^h - 1\right| \ \ \forall\ h > -1, h \ne 0$

$\Rightarrow \quad \left|p(1+h) - p(1)\right| \le \left|e^h - 1\right| \qquad \left(\because p(1) = 0\right)$

$$\Rightarrow \quad \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{e^h-1}{h}\right|$$

Taking limit as $h \to 0$, then

$$\Rightarrow \quad \lim_{h\to 0}\left|\frac{p(1+h)-p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h-1}{h}\right|$$

$$\Rightarrow \quad \left|\lim_{h\to 0}\frac{p(1+h)-p(1)}{h}\right| \le \left|\lim_{h\to 0}\frac{e^h-1}{h}\right|$$

$$\Rightarrow \quad \left|p'(1)\right| \le 1$$

$\Rightarrow \quad \left|a_1 + 2a_2 + \ldots + na_n\right| \le 1 \qquad$ [from (1)]

54 Let $f\left(\frac{xy}{2}\right) = \frac{f(x)f(y)}{2}$ for all real x and y. If $f(1) = f'(1)$, show that $f(x) + f(1-x) =$ constant, for all non-zero real x.

Sol Given $f\left(\frac{xy}{2}\right) = \frac{f(x)f(y)}{2}$

Replacing x by 2x and y by 1, we get

$$2f(x) = f(2x)f(1) \qquad \ldots(1)$$

and,

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{x\left(1+\frac{y}{x}\right)}{2}\right) = \frac{f(x)f\left(1+\frac{y}{x}\right)}{2}, x \ne 0 \qquad \ldots(2)$$

now,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(2x) f\left(1+\frac{h}{x}\right)}{2} - f(x)}{h} \qquad \text{[from (2)]}$$

$$= \lim_{h\to 0} \frac{f(2x) f\left(1+\frac{h}{x}\right) - 2f(x)}{2h}$$

$$= \lim_{h\to 0} \frac{f(2x) f\left(1+\frac{h}{x}\right) - f(2x) f(1)}{2h} \qquad \text{[from (1)]}$$

$$= \frac{f(2x)}{2} \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{x \cdot \frac{h}{x}}$$

$$= \frac{f(2x)}{2x} \cdot f'(1)$$

$$= \frac{2f(x)}{f(1).2x} \cdot f'(1) = \frac{f(x)}{x} \qquad (\because f'(1) = f(1))$$

$$= \frac{f'(x)}{f(x)} = \frac{1}{x}$$

Integrating both sides w.r.t. x, we get

$$\ln f(x) = \ln x + \ln c$$

$\Rightarrow$ $f(x) = cx$ (c is constant > 0)

$\therefore$ $f(x) + f(1-x) = cx + c(1-x) = cx + c - cx = c = \text{constant}.$

55 Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \max\{f(t) : 0 \le t \le x\}, 0 \le x \le 1 = 3 - x, 1 < x \le 2.$

Discuss the continuity and differentiability of the function $g(x)$ in the interval $(0, 2)$.

Sol Given $f(x) = x^3 - x^2 + x + 1$

$\therefore$ $f'(x) = 3x^2 - 2x + 1$

$$= 3\left\{x^2 - \frac{2x}{3} + \frac{1}{3}\right\}$$

$$= 3\left\{\left(x - \frac{1}{3}\right)^2 + \frac{2}{9}\right\} > 0$$

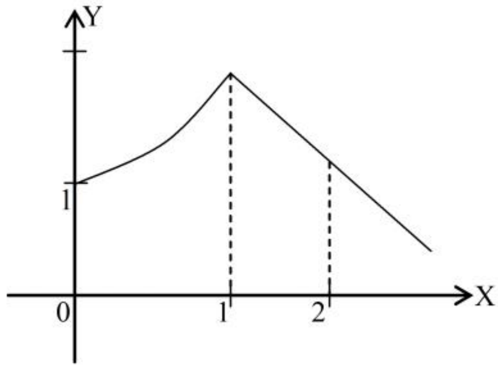
$\therefore$ $f(x)$ is strictly increasing in (0,2)

$\therefore$ maximum value of $f(t)$ in $0 \le t \le x$ is $f(x)$

$\therefore \qquad g(x) = \begin{cases} f(x) & , \quad 0 \le x \le 1 \\ 3 - x & , \quad 1 < x \le 2 \end{cases}$

$= \begin{cases} x^3 - x^2 + x + 1 & , \quad 0 \le x \le 1 \\ 3 - x & , \quad 1 < x \le 2 \end{cases}$

Graph of $g(x)$:



Clearly, $g(x)$ is continuous for all $x \in (0,2)$ and differentiable at all points in this interval except $x = 1$.

56 Let $f(x) = x^3 - 9x^2 + 15x + 6$, and $g(x) = \begin{cases} \min f(t) : 0 \le t \le x & , \quad 0 \le x \le 6 \\ x - 18 & , \quad x > 6 \end{cases}$, then draw the graph of $g(x)$ and discuss the continuity and differentiability of $g(x)$.

Sol $\because \qquad f(x) = x^3 - 9x^2 + 15x + 6,$

$\therefore \qquad f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-1)(x-5)$

If $\qquad f'(x) > 0$ then $x \in (-\infty, 1) \cup (5, \infty)$

and if $f'(x) < 0$ then $x \in (1,5)$

Hence $f(x)$ is increasing in

$(-\infty,1) \cup (5,\infty)$ and decreasing in (1,5).

Now, $\quad f(x) = 6 \qquad \Rightarrow \qquad x^3 - 9x^2 + 15x + 6 = 6$

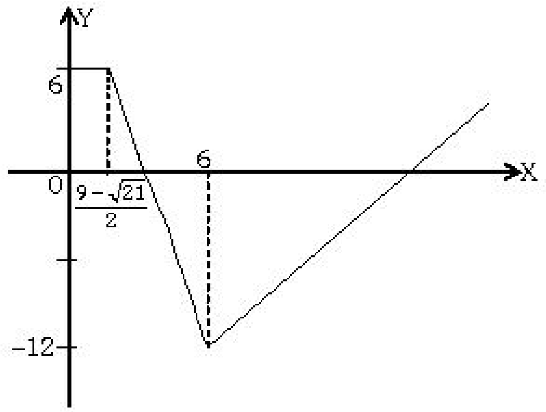
$\Rightarrow \qquad x^3 + 9x^2 + 15x = 0 \qquad \Rightarrow \qquad x(x^2 - 9x + 15) = 0$

$\Rightarrow \qquad x = 0, \dfrac{9 \pm \sqrt{21}}{2}$

$\Rightarrow \qquad x = 0, \dfrac{9 - \sqrt{21}}{2} \qquad \left(x \ne \dfrac{9 + \sqrt{21}}{2}, \because \dfrac{9 - \sqrt{21}}{2} > 6 \right)$

$$\therefore \quad g(x) = \begin{cases} 6 & , \quad 0 \le x < \frac{9-\sqrt{21}}{2} \\ x^3 - 9x^2 + 15x + 6 & , \quad \frac{9-\sqrt{21}}{2} \le x \le 6 \\ x - 18 & , \quad x > 6 \end{cases}$$

Graph of $g(x)$:



Clearly g(x) is continuous in $[0, \infty)$ and differentiable at all points in this interval other than $\frac{9-\sqrt{21}}{2}$ and 6.

57 Let $f(x) = \begin{cases} b\sin^{-1}\left(\frac{x+c}{2}\right) & , \quad -\frac{1}{2} < x < 0 \\ \frac{1}{2} & , \quad x = 0 \\ \frac{e^{ax/2}-1}{x} & , \quad 0 < x < \frac{1}{2} \end{cases}$, If $f(x)$ is differentiable at $x = 0$. Find the value of a also prove that $64b^2 = 4 - c^2$.

Sol $Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{e^{\frac{ah}{2}} - 1}{h} - \frac{1}{2}}{h}$

$$= \lim_{h \to 0} \frac{\frac{a}{2} \cdot \left(\frac{e^{\frac{ah}{2}} - 1}{\frac{ah}{2}}\right) - \frac{1}{2}}{h}$$

at $h \to 0$ numerator must be = 0, then $\frac{a}{2}.1 - \frac{1}{2} = 0$

$\therefore \quad a = 1$

$$\Rightarrow \quad Rf'(0) = \lim_{h\to 0} \frac{\frac{e^{\frac{h}{2}} - 1}{h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{2\left(e^{\frac{h}{2}} - 1\right) - h}{2h^2} = P\ (\text{say}) \qquad \ldots(1)$$

$$\therefore \quad P = \lim_{h\to 0} \frac{2\left(e^{\frac{h}{2}} - 1\right) - h}{2h^2}$$

Replacing h by $-h$ then $P = \lim_{h\to 0} \frac{2\left(e^{-\frac{h}{2}} - 1\right) + h}{2h^2}$ ...(2)

Adding (1) and (2) then $2P = \lim_{h\to 0} \frac{e^{\frac{h}{2}} + e^{-\frac{h}{2}} - 2}{h^2} = \lim_{h\to 0} \frac{e^h - 2e^{\frac{h}{2}} + 1}{h^2 e^{\frac{h}{2}}}$

$$= \lim_{h\to 0} \left(\frac{e^{\frac{h}{2}} - 1}{\frac{h}{2}} \right)^2 \cdot \frac{1}{4e^{\frac{h}{2}}} = \frac{1}{4}$$

$\therefore \quad P = \frac{1}{8} \qquad \Rightarrow \qquad Rf'(0) = \frac{1}{8}$ ...(3)

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{b\sin^{-1}\left(\frac{-h+c}{2}\right) - \frac{1}{2}}{-h}$$

Now, at $h \to 0$ numerator must $be = 0$

$\therefore \quad b\sin^{-1}\left(\frac{c}{2}\right) - \frac{1}{2} = 0$

then,

$$Lf'(0) = b\lim_{h\to 0} \frac{\sin^{-1}\left(\frac{c-h}{2}\right) - \sin^{-1}\left(\frac{c}{2}\right)}{-h}$$

$$= b\lim_{h\to 0} \frac{\sin^{-1}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1 - \frac{c^2}{4}\right)} - \frac{c}{2}\sqrt{\left(1 - \left(\frac{c-h}{2}\right)^2\right)}\right\}}{-h}$$

$$= b\lim_{h\to 0} \frac{\sin^{-1}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1 - \frac{c^2}{4}\right)} - \frac{c}{2}\sqrt{\left(1 - \left(\frac{c-h}{2}\right)^2\right)}\right\}}{\left(\frac{c-h}{2}\right)\sqrt{\left(1 - \frac{c^2}{4}\right)} - \frac{c}{2}\sqrt{\left(1 - \left(\frac{c-h}{2}\right)^2\right)}}$$

$$\cdot\frac{\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}{-h}$$

$$=-b\lim_{h\to 0}\frac{\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}+\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}{h\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}-\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}$$

$$=-b\lim_{h\to 0}\frac{\left(\frac{c-h}{2}\right)^2\left(1-\frac{c^2}{4}\right)-\frac{c^2}{4}\left(1-\left(\frac{c-h}{2}\right)^2\right)}{h\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}+\frac{c}{2}\sqrt{1-\left(\frac{c-h}{2}\right)^2}\right\}}$$

$$=-b\lim_{h\to 0}\frac{(2c-h)(-h)}{4h\left\{\left(\frac{c-h}{2}\right)\sqrt{\left(1-\frac{c^2}{4}\right)}+\frac{c}{2}\sqrt{\left\{1-\left(\frac{c-h}{2}\right)^2\right\}}\right\}}$$

$$=\frac{2bc}{4\left\{c\sqrt{\left(1-\frac{c^2}{4}\right)}\right\}}=\frac{b}{2\sqrt{\left(1-\frac{c^2}{4}\right)}} \qquad ...(5)$$

From (3) and (5),

$$\frac{1}{8}=\frac{b}{2\sqrt{\left(1-\frac{c^2}{4}\right)}}$$

$\Rightarrow \quad 64b^2=4-c^2$

58 Let $\alpha\in R.$ Prove that a function $f:R\to R$ is differentiable at $x=\alpha$ if and only if there is a function $g:R\to R$ which is continuous at α and satisfies $f(x)-f(\alpha)=g(x)(x-\alpha)$ for all $\alpha\in R.$

Sol Let $f:R\to R$ be differentiable at $x=\alpha\in R$, then

$\lim\limits_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}=f'(\alpha)$ exists and finite.

i.e. $\quad Lf'(\alpha)=Rf'(\alpha)=f'(\alpha)$

$$\Rightarrow \quad \lim_{x\to\alpha-}\frac{f(x)-f(\alpha)}{(x-\alpha)}=\lim_{x\to\alpha+}\frac{f(x)-f(\alpha)}{(x-\alpha)}=f'(\alpha)$$

$$\lim_{x\to\alpha-}g(x)=\lim_{x\to\alpha+}g(x)=f'(\alpha) \qquad \{\because f(x)-f(\alpha)=g(x)(x-\alpha)\} \qquad ...(1)$$

Again $f'(\alpha)=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{(x-\alpha)}$

$$=\lim_{x\to\alpha} g(x)=g(\alpha)$$

From (1) and (2), we get $\lim_{x\to\alpha-} g(x)=\lim_{x\to\alpha+} g(x)=g(\alpha)$

L.H.L = R.H.L = V.F.

$\Rightarrow$ $g(x)$ is continuous function at $x=\alpha\in R$.

59 Let $g(x)=0$ if $-e\le x<1$

$$=\left\{1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right\} \text{ if } 1\le x\le e.$$

where {} denotes the fractional part function and

$$f(x)=x\,g(x) \text{ for } g(x)=1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)$$

$$=x\left(g(x)+1\right) \quad \text{otherwise}$$

Discuss the continuity and differentiability of $f(x)$ over its domain.

Sol Given $g(x)=\left\{1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right\}$ for $1\le x\le e$

$$=0 \text{ for } -e\le x<1$$

i.e., $g(x)=1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)-\left[1+\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right]$

$$=\frac{1}{3}\sin\left(\ln x^{2\pi}\right)-\left[\frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right], 1\le x\le e$$

$$=0, -e\le x<1$$

where [.] denotes the greatest integer function.

consider : $1\le x\le e$

$\Rightarrow$ $(1)^{2\pi}\le x^{2\pi}\le e^{2\pi}$ $\Rightarrow$ $\ln(1)\le\ln\left(x^{2\pi}\right)\le\ln\left(e^{2\pi}\right)$

$\Rightarrow$ $0\le\ln\left(x^{2\pi}\right)\le 2\pi$

Case I : If $0\le\ln\left(x^{2\pi}\right)\le\pi$ i.e. $1\le x\le\sqrt{e}$ then $0\le\sin\left(\ln\left(x^{2\pi}\right)\right)\le 1$

$\Rightarrow$ $0\le\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\le\frac{1}{3}$ $\therefore$ $\left[\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right]=0$

$\therefore$ $g(x)=\frac{1}{3}\sin\left(\ln x^{2\pi}\right)$ for $1\le x\le\sqrt{e}$

Case II : If $\pi<\ln\left(x^{2\pi}\right)<2\pi$ i.e., $\sqrt{e}<x<e$ then $-1\le\sin\left(\ln\left(x^{2\pi}\right)\right)<0$

$\Rightarrow$ $-\frac{1}{3}\le\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)<0$ $\therefore$ $\left[\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right]=-1$

$\therefore$ $g(x)=1+\frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)$ for $\sqrt{e}<x<e$

Case III : If $\ln\left(x^{2\pi}\right) = 2\pi \quad \Rightarrow \quad x = e \quad \Rightarrow \quad g(x) = \{1\} = 0$

Combining all cases, we get

$$f(x) = x\left(1 + \frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right) \qquad \text{for } \sqrt{e} < x < e$$

$$= x\left(1 + \frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right) \qquad \text{for } 1 \le x \le \sqrt{e}$$

$$= x(1+0) \qquad \text{for} - e \le x < 1$$

$$= x(1+0) \qquad \text{for } x = e$$

$$\Rightarrow \quad f(x) = x\left(1 + \frac{1}{3}\sin\left(\ln\left(x^{2\pi}\right)\right)\right) \qquad \text{for } 1 \le x \le e$$

$$= x$$

$\therefore$ f is differentiable in $(-e, 1)$ and $(1, e)$

Check the differentiable of $f(x)$ at $x = 1$.

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h) - 1}{-h} = 1$$

and $$Rf'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)\cdot\left(1 + \frac{1}{3}\sin\left(\ln(1+h)^{2\pi}\right)\right) - 1}{h}$$

$$= \lim_{h\to 0}\frac{h + \frac{(1+h)}{3}\sin\left(\ln(1+h)^{2\pi}\right)}{h}$$

$$= \lim_{h\to 0}\left(1 + \frac{(1+h)}{3}\frac{\sin\left\{\ln(1+h)^{2\pi}\right\}}{h}\right)$$

$$= 1 + \lim_{h\to 0}\frac{(1+h)}{3}\ \lim_{h\to 0}\ \frac{\sin\left(\ln(1+h)^{2\pi}\right)}{h}$$

$$= 1 + \lim_{h\to 0}\frac{(1+h)}{3}\lim_{h\to 0}\frac{\sin\{2\pi\ln(1+h)\}}{2\pi\ln(1+h)}\cdot\frac{2\pi\ln(1+h)}{h}$$

$$= 1 + \left(\frac{1+0}{3}\right).1.2\pi.1$$

$$= 1 + \frac{2\pi}{3}\cdot$$

Thus f is not differentiable at $x = 1$.

Hence f is continuous and differentiable for all $x \in$ domain of except not differentiable at $x = 1$.

60 Suppose that f and g are non-constant differentiable real valued functions on R. If for every $x, y \in R, f(x+y) = f(x)f(y) - g(x)g(y),\ g(x+y) = g(x)f(y) + f(x)g(y)$ and $f'(0) = 0$ then prove that $\{f(x)\}^2 + \{g(x)\}^2 = 1 \ \forall \ x \in R.$

Sol We have $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x+0)}{h}$

$$= \lim_{h\to 0} \frac{\{f(x)f(h) - g(x)g(h)\} - \{f(x)f(0) - g(x)g(0)\}}{h}$$

$$= \lim_{h\to 0} \frac{f(x)(f(h) - f(0))}{(h-0)} - \lim_{h\to 0} \frac{g(x)(g(h) - g(0))}{(h-0)}$$

$$= f(x)f'(0) - g(x)g'(0)$$

$$= 0 - g(x)g'(0) \qquad (\because f'(0) = 0)$$

$\therefore$ $f'(x) = -g(x)g'(0)$ ....(1)

and $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{g(x+h) - g(x+0)}{h}$

$$= \lim_{h\to 0} \frac{\{g(x)f(h) + f(x)g(h)\} - \{g(x)f(0) + f(x)g(0)\}}{h}$$

$$= g(x)\lim_{h\to 0} \frac{f(h) - f(0)}{h-0} + f(x)\lim_{h\to 0} \frac{g(h) - g(0)}{h-0}$$

$$= g(x)f'(0) + f(x)g'(0)$$

$$= 0 + f(x)g'(0) \qquad (\because f'(0) = 0)$$

$= f(x)g'(0)$ ....(2)

Multiplying (1) by f(x) and (2) by g(x) and adding we get

$$f(x)f'(x) + g(x)g'(x) = 0$$

or $2f(x)f'(x) + 2g(x)g'(x) = 0$ on integrating we get

$\{f(x)\}^2 + \{g(x)\}^2 = c$ .....(3)

Putting $x = 0, y = 0$ in the given equation then

$$f(0) = \{f(0)\}^2 - \{g(0)\}^2 \qquad \text{and} \qquad g(0) = 2f(0)g(0)$$

or $g(0)\{2f(0) - 1\} = 0$ or $g(0) = 0$ or $f(0) = \frac{1}{2}$

If $g(0) = 0$, then $f(0) = (f(0))^2 - 0$ or $f(0) = 1$

and for $f(0) = \frac{1}{2}, \frac{1}{2} = \left(\frac{1}{2}\right)^2 - (g(0))^2$

$\Rightarrow \quad (g(0))^2 = -\frac{1}{4} \qquad$ (Impossible)

Hence $f(0) = 1$ and $g(0) = 0$ from (3), $\{f(0)\}^2 + \{g(0)\}^2 = c$

$\Rightarrow \quad 1 + 0 = c \qquad \therefore \quad c = 1$

Hence $\{f(x)\}^2 + \{g(x)\}^2 = 1$.

61 Let $f(x)$ be a real valued function not identically zero such that

$f(x + y^n) = f(x) + \{f(y)\}^n$; $\forall$ $x, y \in R$ (where n is odd natural number > 1) and $f'(0) \geq 0$.

Find out the values of $f'(10)$ and $f(5)$.

Sol Given that $f(x + y^n) = f(x) + (f(y))^n$

Putting $x = y = 0 \qquad \Rightarrow \qquad f(0) = 0$

$$\therefore \quad f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h} = \lambda \text{ (say)} \qquad \ldots(1)$$

Also,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(0 + \left(h^{1/n}\right)^n\right) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(0) + \left\{f\left(h^{1/n}\right)\right\}^n - f(0)}{h}$$

$$= \lim_{h \to 0} \left\{\frac{f\left(h^{1/n}\right)}{h^{1/n}}\right\}^n$$

$$= \lambda^n \qquad \text{[from (1)]}$$

From (1) and (2), $\quad \lambda = \lambda^n$

$\therefore \quad \lambda = -1, 0, 1 \qquad (\because$ n is odd and $\lambda \in R)$

$\because \quad f'(0) \geq 0 \qquad (\because \lambda \neq -1)$

$\therefore \quad f'(0) = 0, 1$

Again $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(x + \left(h^{1/n}\right)^n\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x) + \left(f\left(h^{1/n}\right)\right)^n - f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f\left(h^{1/n}\right)}{h^{1/n}}\right)^n \qquad = \lambda^n$$

For $\lambda = 0, f'(x) = 0$

On intergrating we get $f(x) = c$

At $\quad x = 0, f(0) = c = 0 \qquad (\because f(0) = 0)$

$\therefore \quad f(x) = 0$

which is impossible as f(x) is not identically zero, i.e., $f(x) \neq 0$

and for $\lambda = 1 \qquad f'(x) = 1$

On intergrating w.r.t. x and taking limit 0 to x,

then $\int_0^x f'(x)\,dx = \int_0^x 1.dx$

$\Rightarrow \quad f(x) - f(0) = x \qquad \Rightarrow \quad f(x) - (0) = x \qquad (\because f(0) = 0)$

Hence $f(x) = x$ and $f'(x) = 1 \qquad \therefore \quad f'(10) = 1$ and $f(5) = 5.$

62 Let $a_1 > a_2 > a_3$ .... $a_n > 1$; $p_1 > p_2 > p_3$......... $> p_n > 0$; such that $p_1 + p_2 + p_3 + \ldots\ldots + p_n = 1$

Also $F(x) = \left(p_1 a_1^x + p_2 a_2^x + \ldots\ldots + p_n a_n^x\right)^{1/x}$. Compute

(a) $\lim\limits_{x\to 0^+} F(x)$ **(b)** $\lim\limits_{x\to\infty} F(x)$ **(c)** $\lim\limits_{x\to -\infty} F(x)$ [**Ans. (a)** $a_1^{p_1} \cdot a_2^{p_2} \ldots\ldots a_n^{p_n}$; **(b)** a_1 ; **(c)** a_n]

[Sol.

(1) $\lim\limits_{x\to 0^+} F(x) = \lim\limits_{x\to 0^+} \left(p_1 a_1^x + p_2 a_2^x + \ldots\ldots + p_n a_n^x\right)^{1/x} \qquad (1^\infty$ form)

$_1 = e^l$ where $l = \lim\limits_{x\to 0} \dfrac{p_1 a_1^x + p_2 a_2^x + \ldots\ldots + p_n a_n^x - 1}{x} \qquad \left(\dfrac{0}{0}\right)$

using L'Hospital's Rule

$l = \lim\limits_{x\to 0} \left(p_1\, ln\, a_1\, a_1^x + p_2\, ln\, a_2\, a_2^x + \ldots\ldots + p_n\, ln\, a_n\, a_n^x\right)$

$= p_1\, ln\, a_1 + p_2\, ln\, a_2 + \ldots\ldots + p_n\, ln\, a_n$

$= ln\left(a_1^{p_1} \cdot a_2^{p_2} \ldots\ldots a_n^{p_n}\right)$

$\therefore \quad L_1 = e^l = a_1^{p_1} \cdot a_2^{p_2} \ldots\ldots a_n^{p_n}$ **Ans.**

(2) $\lim\limits_{x\to\infty} F(x) = L_2 = \lim\limits_{x\to\infty} \left(p_1 a_1^x + p_2 a_2^x + \ldots\ldots + p_n a_n^x\right)^{1/x} \quad (\infty^0$ form) [only when $a_1\, a_2$ etc. > 1]

$\therefore \quad ln\, L_2 = \lim\limits_{x\to\infty} \dfrac{ln\left(p_1 a_1^x + p_2 a_2^x + \ldots\ldots + p_n a_n^x\right)}{x}$

using L'Hospital's Rule

$$_2 = \lim_{x\to\infty} \frac{\left(p_1 \ln a_1 a_1^x + p_2 \ln a_2 a_2^x + ...+ p_n \ln a_n a_n^x\right)}{p_1 a_1^x + p_2 a_2^x + ...+ p_n a_n^x} \qquad ....(1)$$

dividing by a_1^x and taking limit, we get

$$\lim_{x\to\infty}, \left(\frac{a_2}{a_1}\right)^x, \left(\frac{a_3}{a_2}\right)^x, \text{ etc all vanishes as } x \to \infty$$

$$= \frac{p_1 \ln a_1}{p_1} = \ln a_1$$

hence $\ln L_2 = \ln a_1 \quad \Rightarrow \quad L_2 = a_1$ **Ans.**

(3) $\lim_{x\to -\infty} F(x) = L_3$ (say)

$$\therefore \quad \ln L_3 = \lim_{x\to -\infty} \frac{\left(p_1 \ln a_1 a_1^x + p_2 \ln a_2 a_2^x + ...+ p_n \ln a_n a_n^x\right)}{p_1 a_1^x + p_2 a_2^x + ...+ p_n a_n^x}$$

dividing by $(a_n)^x$ and taking $\lim_{x\to -\infty}, \left(\frac{a_1}{a_n}\right)^x, \left(\frac{a_2}{a_n}\right)^x$ etc vanishes

$$\therefore \quad \ln L_3 = \frac{p_n \ln a_n}{p_n} \quad \Rightarrow \quad L_3 = a_n$$

63 Let $f: R^+ \to R$ be a differentiable function with $f(1) = 3$ and satisfying :

$$\int_1^{xy} f(t)dt = y\int_1^x f(t)dt + x\int_1^y f(t)dt \; ; \quad \forall \quad x, y \in R^+$$

then find $f(x)$.

Sol We have $\int_1^{xy} f(t)dt = y\int_1^x f(t)dt + x\int_1^y f(t)dt$

Differentiating both sides w.r.t. x treating y as constant; we get

$$f(xy).y = yf(x) + \int_1^y f(t)\,dt$$

Putting $x = 1$, we get $yf(y) = yf(1) + \int_1^y f(t)dt$

$$\Rightarrow \quad yf(y) = 3y + \int_1^y f(t)dt \qquad (\because f(1) = 3)$$

Again differentiating both sides w.r.t. y, we get

$$yf'(y) + f(y).1 = 3 + f(y)$$

$$\Rightarrow \quad f'(y) = \frac{3}{y}$$

Integrating both sides w.r.t. y with limit 1 to x then

$$yf'(1) = 3 \ln x - 3 \ln 1$$

$$f(x) - f(1) = 3 \ln x - 3 \ln 1$$

$$\Rightarrow \quad f(x) - 3 = 3 \ln x - 0 \qquad (\because f(1) = 3)$$

$\Rightarrow \quad f(x) = 3 + 3\ln x$

$= 3\ln e + 3\ln x = 3\ln(ex)$

Hence $f(x) = 3\ln(ex)$.

64 Let $f(x^m y^n) = mf(x) + nf(y) \ \forall \ x, y \in R^+$ and $\forall \ m, n \in R$. If $f'(x)$ exists and has the value $\frac{e}{x}$, then find $\lim\limits_{x\to 0}\frac{f(1+x)}{x}$.

Sol $\because \quad f(x^m y^n) = mf(x) + nf(y)$ ....(1)

Putting $x = y = m = n = 1$, then $f(1) = f(1) + f(1)$

$\Rightarrow \quad f(1) = 0$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x\cdot 1)}{h}$$

$$= \lim_{h\to 0}\frac{f\left\{\left(x^{1/m}\right)^m\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\}^n\right\} - f\left\{\left(x^{1/m}\right)^m\left((1)^{1/n}\right)^n\right\}}{h}$$

$$= \lim_{h\to 0}\frac{mf\left(x^{1/m}\right) + nf\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\} - mf\left(x^{1/m}\right) - nf(1)}{h}$$

$$= \lim_{h\to 0}\frac{nf\left\{\left(1+\frac{h}{x}\right)^{1/n}\right\}}{h}$$

$$= \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{x\left(\frac{h}{x}\right)} \qquad \left(\text{Putting } y = 1 \text{ in (1) then } f(x^m) = mf(x)\right)$$

$$\Rightarrow \quad \frac{e}{x} = \frac{1}{x}\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)} \qquad \Rightarrow \qquad \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)} = e$$

Hence $\lim\limits_{h\to 0}\frac{f(1+x)}{x} = e$

65 Let f be a continuous and differentiable function in (x_1, x_2). If $f(x).f'(x) \geq x\sqrt{1-(f(x))^4}$

and $\lim_{x\to x_1^+}(f(x))^2 = 1$ and $\lim_{x\to x_2^-}(f(x))^2 = \frac{1}{2}$ for $x \in (x_1, x_2)$, then prove that $x_1^2 - x_2^2 \ge \frac{\pi}{3}$

(assume that $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ holds everywhere).

Sol Given $f(x)\cdot f'(x) \ge x\sqrt{1-(f(x))^4}$

$\Rightarrow \quad \frac{f(x)f'(x)}{\sqrt{1-(f(x))^4}} - x \ge 0 \qquad$ or $\qquad \frac{2f(x)f'(x)}{\sqrt{1-(f(x))^4}} - 2x \ge 0$

or $\quad \frac{d}{dx}\left\{\sin^{-1}(f(x))^2 - x^2\right\} \ge 0$

$\Rightarrow \quad F(x) = \sin^{-1}(f(x))^2 - x^2$ is a non decreasing function.

$\Rightarrow \quad \lim_{x\to x_1^+} F(x) \le \lim_{x\to x_2^-} F(x)$

$\Rightarrow \quad \lim_{x\to x_1^+}\left\{\sin^{-1}(f(x))^2 - x^2\right\} \le \lim_{x\to x_2^-}\left\{\sin^{-1}(f(x))^2 - x^2\right\}$

$\Rightarrow \quad \frac{\pi}{2} - x_1^2 \le \frac{\pi}{6} - x_2^2 \qquad \Rightarrow \qquad x_1^2 - x_2^2 \ge \frac{\pi}{3}.$

66 Are there any non-constant differentiable functions $f: R \to R$ such that $f(f(f(x))) = f(x) \ge 0 \ \forall \ x \in R$?

Sol Given $f(f(f(x))) = f(x)$ ....(1)

Applying f to both sides of the equation (1), then

$$f(f(f(x))) = f\{f(x)\} \qquad ....(2)$$

If $g(x) = f(f(x)) \ \forall \ x \in R$ then equation (2) can be written as $g(g(x)) = g(x)$; g is also a differentiable function on R and $g(x) \ge 0 \ \forall \ x \in R$.

Then the range $T = g(R)$ of g is an interval in $[0, \infty)$. Let a be the infimum of T.

Since $g(t) = t$ for all $t \in T$ and g is continuous.

$\Rightarrow \quad g(a) = a$

Assume T has more than one element. Choose $\delta > 0$ such that $(a, a+\delta \subseteq T)$.

Then $x \in (a-\delta, a)$

$\Rightarrow \quad g(x) \ge g(a) = a \qquad \therefore \qquad \frac{g(x)-g(a)}{x-a} \le 0$

$\therefore \quad Lg'(a) = \lim_{x\to a-} \frac{g(x)-g(a)}{x-a} \le 0$

$$= \lim_{h\to 0} \frac{g(a-h)-g(a)}{-h} \le 0 \qquad ....(3)$$

For $x \in (a, a+\delta)$ we have $\qquad \dfrac{g(x)-g(a)}{x-a} = 1$

Hence $\qquad Rg'(a) = \lim_{x \to a+} \dfrac{g(x)-g(a)}{x-a} = 1 \qquad$ ....(4)

As g is differentiable at a, therefore (3) and (4) are contradictory. This concludes that T is a single point i.e., g is a constant function,

$$g(x) = c \quad \forall \; x \in R, \qquad \text{(c is constant)}$$

from (1), $\qquad f(c) = f(x) \;\; \forall \;\; x \in R$

This shows that f is a constant function. Thus there is no non-constant differentiable function satisfying (1).

67 Let $f(x) = x^3 - 3x^2 + 6 \;\; \forall \;\; x \in R$ and

$$g(x) = \begin{cases} \max\{f(t) : x+1 \le t \le x+2, -3 \le x < 0\} \\ 1-x, \quad \text{for} \quad x \ge 0 \end{cases}$$

Test continuity of $g(x)$ for $x \in [-3, 1]$.

Sol Since $f(x) = x^3 - 3x^2 + 6$

$\Rightarrow \qquad f'(x) = 3x^2 - 6x$

$\qquad\qquad = 3x(x-2)$

for maximum and minima $f'(x) = 0$

$\therefore \qquad x = 0, 2$

$\qquad f''(x) = 6x - 6$

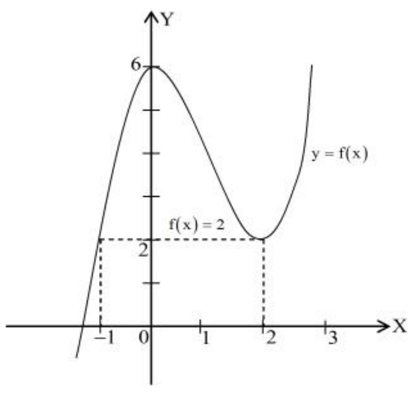
$\qquad f''(0) = -6 < 0 \qquad$ (local maxima at $x = 0$)

$\qquad f''(2) = 6 > 0 \qquad$ (local minima at $x = 2$)

Cut off x-axis $x^3 - 3x^2 + 6 = 0$ has maximum 2 positive and 1 negative real roots.

Cut off y-axis. $F(0) = 6$.

Now graph of $f(x)$ is :



Clearly f(x) is increasing in $(-\infty, 0) \cup (2, \infty)$ and decreasing in (0,2)

$\Rightarrow \quad x+2<0 \quad \Rightarrow \quad x<-2 \quad \Rightarrow \quad -3 \le x < -2$

$\Rightarrow \quad -2 \le x+1 < -1 \quad$ and $\quad -1 \le x+2 < 0$

in both cases f(x) increases (maximum) of $g(x) = f(x+2)$

$\therefore \quad g(x) = f(x+2)\,; -3 \le x < -2 \qquad \ldots(1)$

and if $x+1<0$ and $0 \le x+2 < 2 \quad \Rightarrow \quad -2 \le x < -1$

then $g(x) = f(0)$

Now for $x+1 \ge 0$ and $x+2<2 \quad \Rightarrow \quad -1 \le x < 0, g(x) = f(x+1)$

Hence $$g(x) = \begin{cases} f(x+2) & ; \ -3 \le x < -2 \\ f(0) & ; \ -2 \le x < -1 \\ f(x+1) & ; \ -1 \le x < -0 \\ 1-x & ; \ x \ge 0 \end{cases}$$

Hence g(x) is continuous in the interval $[-3,1]$.

68 $f : [0, 1] \to R$ is defined as $f(x) = \begin{bmatrix} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{bmatrix}$, then prove that

(a) f is differentiable in [0, 1] (b) f is bounded in [0, 1] (c) f' is bounded in [0, 1]

Sol. $$f(x) = \begin{bmatrix} x^3(1-x)\sin\left(\frac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{bmatrix}$$

$$f'(0^+) = \lim_{h\to 0} \frac{h^3(1-h)\sin\frac{1}{h^2} - 0}{h} = 0$$

$$f'(1^-) = \lim_{h\to 0} \frac{(1-h)^3(+h)\sin\frac{1}{(1-h)^2} - 0}{-h} = \lim_{h\to 0} -(1-h)^3 \sin\frac{1}{(1-h)^2} = -\sin 1$$

Hence f is derivable in [0, 1], obviously f is continuous in [0, 1] hence f is bounded

hence $$f'(x) = \begin{bmatrix} (x^3 - x^4)\cos\left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right) + \sin\frac{1}{x^2}(3x^2 - 4x^3) & x \ne 0 \\ 0 \quad \text{if } x = 0 \end{bmatrix}$$

$\lim_{x\to 1^-} = (0) + \sin 1(3-4)$, hence f ' is also bounded.

69 Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x-5|\,[x] & \text{for } x > 1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where [x] is the greatest integer not greater than x.

Sol. $$f(x) = \begin{cases} |4x-5|\,[x] & \text{for } 1 < x \le 2 \\ [\cos \pi x] & \text{for } 0 \le x \le 1 \end{cases} = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \le \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < x \le 1 \\ (5-4x) & \text{if } 1 < x < \frac{5}{4} \\ (4x-5) & \text{if } \frac{5}{4} \le x < 2 \\ 6 & \text{if } x = 2 \end{cases}$$

Clearly f(x) is discont. for x = 0, 1/2, 1 & 2.

70 If f(x) = x + {-x} + [x] , where [x] is the integral part & {x} is the fractional part of x. Discuss the continuity of f in [– 2, 2].

Sol. $f(x) = x + \{-x\} + [x]$

if $n < x < n+1$, then $f(x) = 2n+1$

$\{$as for nonintegral values $\{-x\} = 1 - x + [x]$ and $[x] = n\}$

if $x = n$, then $f(x) = 2n$

Hence $f(x) = \begin{cases} 2n & \text{if } x = n \\ 2n+1 & \text{if } n < x < n+1 \\ 2n+2 & \text{if } x = n+1 \end{cases}$