EXERCISE – 1 (A)

1. **(B)**

$ae = 2$

$a x \frac{1}{2} = 2 \Rightarrow a = 4$

$b^2 = a^2 - a^2e^2 = 16 - 4 = 12$

$\therefore \frac{x^2}{16} + \frac{y^2}{12} = 1$

2. **(B)**

$9x^2 + 5(y-3)^2 = 45$

$\Rightarrow \frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$

$a^2 = 5 \qquad b^2 = 9$

$b^2e^2 = b^2 - a^2$

$9e^2 = 4 \Rightarrow e = \frac{2}{3}$

3. **(B)**

$2\sqrt{1-e^2} \qquad b = \frac{2b}{2} \Rightarrow \sqrt{1-e^2} = \frac{1}{2}$

$e = \frac{\sqrt{3}}{2}$

4. **(C)**

$\frac{2a}{e} = 3 \times (2ae) \Rightarrow e^2 = \frac{1}{3} \Rightarrow e = \frac{1}{\sqrt{3}}$

5. **(B)**

Conceptual

6. **(B)**

$$\frac{\sqrt{(h-6)^2 + (k-7)^2}}{\left|\frac{h+k+2}{\sqrt{2}}\right|} = \frac{1}{\sqrt{3}}$$

$(h-6)^2 + (k+7)^2 = \frac{(h+k+2)^2}{6}$

$6h^2 - 7h + 216 + 6k^2 - 84k + 294 = h^2 + k^2 + 42hk + 4h + 4k$

$5h^2 + 5k^2 - 2hk - 76 - 88k + 506 = 0$

7. **(A)**

$2\sqrt{1-e^2} \;\; b = \frac{2a}{2} \Rightarrow \frac{a^2}{b^2} = 4(1-e^2)$

$b^2 = a^2 - a^2c^2$

$1=4\left(1-e^2\right)-4\left(1-e^2\right)e^2$

$1=4\left(1-e^2\right)^2$

$\Rightarrow 1-e^2=\pm\frac{1}{2}$

$e^2=\frac{1}{2}\Rightarrow e=\frac{1}{\sqrt{2}}$ or $e^2=\frac{3}{2}\Rightarrow e=\sqrt{\frac{3}{2}}$

Hence $e=\frac{1}{\sqrt{2}}$

8. **(B)**

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{9}{a^2}+\frac{1}{b^2}=1$

$\frac{4}{a^2}+\frac{y}{b^2}=1$

$\frac{y}{a^2}+4\left(1-\frac{9}{a^2}\right)=1$

$\frac{-32}{a^2}=-3\Rightarrow a^2=\frac{32}{3}$

$\frac{27}{32}+\frac{1}{b^2}=1\Rightarrow b^2=\frac{32}{5}$

$\therefore\ 3x^2+5y^2=32$

9. **(C)**

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

If major axis is x axis

$b^2=a^2-a^2e^2=\frac{3}{5}a^2$

$\therefore \frac{x^2}{a^2}+\frac{5}{3a^2}y^2=1$

$3x^2+5y^2=3a^2$

Since it passes through $\left(-3,1\right)$

$3x^2+5y^2=32$

If major axis is y axis

$a^2=b^2-b^2e^2=\frac{3}{5}b^2$

$\frac{5x^2}{3b^2}+\frac{y^2}{b^2}=1\Rightarrow 5x^2+3y^2=3b^2$

$\left(-3,1\right)\Rightarrow 5x^2+3y^2=48$

10. **(A)**

$4(x-1)^2+9(y-2)^2=36$

$\therefore \frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}=1$

$a^2=9 \quad b^2=4$

$9e^2=9-4 \Rightarrow e=\frac{\sqrt{5}}{3}$

Latus rectum = $2\sqrt{1-e^2}\,b=4\sqrt{1-\frac{5}{9}}=\frac{8}{3}$

11. **(A)**

Let equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$2\sqrt{1-e^2}\,b=10$

$\Rightarrow b^2-b^2e^2=25$

$b^2=a^2-a^2e^2$

$\Rightarrow a^2\left(1-e^2\right)^2=25$

$2ae=2b \Rightarrow ae=b$

$a^2e^2=a^2-a^2e^2$

$e=\frac{1}{\sqrt{2}}$

$a^2\left(\frac{1}{2}\right)^2=25 \Rightarrow a^2=100$

$b^2=a^2e^2=50$

$\therefore \frac{x^2}{100}+\frac{y^2}{50}=1 \Rightarrow x^2+2y^2=100$

12. **(B)**

$2ae=2b \Rightarrow ae=b$

$a^2e^2=a^2-a^2e^2 \Rightarrow e=\frac{1}{\sqrt{2}}$

13. **(B)**

14. **(B)**

$4(x-1)^2+(y+1)^2=4$

$\Rightarrow (x-1)^2+\frac{(y+1)^2}{4}=4$

$\therefore a^2=1 \qquad b^2=4$

$a^2=b^2\left(1-e^2\right)$

$\frac{1}{4}=1-e^2$

$\Rightarrow e^2=\frac{3}{4} \Rightarrow e=\frac{\sqrt{3}}{2}$

15. **(D)**

Let equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$2ae = 8 \Rightarrow ae = 4$

$\frac{2a}{e}=18 \Rightarrow \frac{a}{e}=9$

$\Rightarrow ae\times\frac{a}{e}=4\times9 \Rightarrow a^2=36$

$e=\frac{2}{3}$

$b^2=a^2\left(1-e^2\right)=36\left(1-\frac{4}{9}\right)=20$

$\frac{x^2}{36}+\frac{y^2}{20}=1 \Rightarrow 5x^2+9y^2=180$

16. **(C)**

$2ae = 6 \Rightarrow ae = 3$

$2b = 8 \Rightarrow b = 4$

$a^2=b^2+a^2e^2=25$

$\therefore\ 5e=3 \Rightarrow e=\frac{3}{5}$

17. **(B)**

Let equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$e=\frac{2}{3}\quad 2b\sqrt{1-e^2}=5$

$\Rightarrow b^2\left(1-\frac{4}{9}\right)=\frac{5}{4} \Rightarrow b^2=\frac{9}{4}$

$a^2\left(1-\frac{4}{9}\right)=\frac{9}{4} \Rightarrow a^2=\frac{81}{20}$

$\therefore \frac{20x^2}{81}+\frac{4y^2}{9}=1$

$\therefore \frac{4x^2}{81}+\frac{4y^2}{45}=\frac{1}{5}$

18. **(B)**

$a^2=36 \qquad b^2=49$

$b^2\left(1-e^2\right)=a^2 \Rightarrow 1-e^2=\frac{36}{49}$

L.R. = $2\times6\times\sqrt{\frac{36}{49}}=\frac{72}{7}$

19. **(A)**

$a^2=64 \qquad b^2=28$

$b^2 = a^2 - a^2e^2$

$\Rightarrow 64e^2 = 64 - 28 = 36$

$e = \frac{3}{4}$

20. **(B)**

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore b = 7$

$\frac{b}{e} = 12 \Rightarrow e = \frac{7}{12}$

$a^2 = b^2\left(1 - e^2\right) = 49\left(1 - \frac{49}{144}\right) = \frac{4655}{144}$

$\therefore \frac{144x^2}{4655} + \frac{y^2}{49} = 1$

$\Rightarrow 144x^2 + 95y^2 = 4655$

21. **(A)**

Equation is $\rightarrow \frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$

centre $= (-1, -2)$

$\therefore$ major axis $x = -1$

$a^2 = b^2\left(1, -e^2\right)$

$\frac{9}{25} = 1 - e^2$

$\therefore e = \frac{4}{5}$

$\therefore \downarrow$ foci $= -2 \pm \frac{4}{5} \times 5 = 2, -6$

Ordinate of Hence $(-1, 2), (-1, -6)$

22. **(B)**

$25(x-3)^2 + 16y^2 = 400$

$\frac{(x-3)^2}{16} + \frac{y^2}{25} = 1$

$\therefore a^2 = 16 \qquad b^2 = 25$

$16 = 25\left(1 - e^2\right)$

$\Rightarrow e = \frac{3}{5}$

23. **(B)**

Conceptual

24. **(A)**

Centre of the ellipse = (3,0)

$\therefore \frac{(x-3)^2}{a^2}+\frac{y^2}{b^2}=1$

$ae = 4 \Rightarrow a = 8$

$b^2 = a^2(1-e^2) = 48 \Rightarrow b = 4\sqrt{3}$

$\therefore \frac{(x-3)^2}{64}+\frac{y^2}{48}=1$

Parametric form : $\left(3+8\cos\theta, 4\sqrt{3}\sin\theta\right)$

25. **(A)**

Putting $(4,-3)$ in $2x^2+5y^2=20$

$2(4)^2+5(-3)^2-20$, which is +ve

Hence it is outside the ellipse

26. **(C)**

Let the line be tangent at $(a\cos\theta, b\sin\theta)$

$\therefore \frac{2x}{a^2}+\frac{2y\,y'}{b^2}=0$

$y' = -\frac{xb^2}{ya^2} = -\frac{ab^2\cos\theta}{a^2 b\sin\theta} = -\frac{b}{a}\cot\theta$

Slope of given line = $-\frac{b}{a}$

$\therefore -\frac{b}{a}\cot\theta = -\frac{b}{a}$

$\Rightarrow \cot\theta = 1 \Rightarrow \theta = 45^o$

27. **(A)**

If $y=1, x=\pm\sqrt{2}$

$2x+4yy'=0 \Rightarrow y' = -\frac{x}{2y}$

$(\sqrt{2},1) \Rightarrow y' = -\frac{9}{\sqrt{2}}$

$\therefore \sqrt{2}y + x = 2\sqrt{2}$

$(-\sqrt{2},1) \Rightarrow y'\frac{1}{\sqrt{2}}$

$\therefore \sqrt{2}y - x = 2\sqrt{2}$

28. **(A)**

Let the tangent be drawn at $(a\cos\theta, b\sin\theta)$

$\therefore$ equation at tangent : $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

x intercept = $a\sec\theta$ y intercept = $b\operatorname{cosec}\theta$

$\therefore a\sec\theta = b\operatorname{cosec}\theta$

$\Rightarrow \tan\theta = \frac{b}{a}$ or $-\frac{b}{a}$

$y = -x\frac{b}{a}\cot\theta + b\,\mathrm{cosec}\,\theta$

$\therefore y = -x \pm \sqrt{b^2 + a^2}$

29. (A)

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

Equation of tangent at $(4\cos\theta, 3\sin\theta) = \frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$

$(2,3) \Rightarrow \frac{\cos\theta}{2} + \sin\theta = 1$

$\Rightarrow \cos^2\theta/2 - \sin^2\theta/2 = 2(\cos\theta/2 - \sin\theta/2)^2$

Either $\cos\frac{\theta}{2} = \sin\frac{\theta}{2}$ or $\cos\frac{\theta}{2} + \sin\frac{\theta}{2} = 2\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)$

$\Rightarrow \theta = 90^{\circ}$ $\qquad 3\sin\frac{\theta}{2} = \cos\frac{\theta}{2}$

$\Rightarrow \tan\frac{\theta}{2} = \frac{1}{3}$

$\tan\theta = \frac{2/3}{1 - \frac{1}{9}} = \frac{2/3}{8/9} = \frac{3}{4}$ or $\tan\theta = \infty$

Equation of tangent = $\frac{3x\cot\theta}{4} + y = 3\,\mathrm{cosec}\,\theta$

$\tan\theta = \infty \Rightarrow y = 3 \times 1 \Rightarrow y = 3$

$\tan\theta = \frac{3}{4} \Rightarrow \frac{3x}{4} \times \frac{4}{3} + y = 3 \times \frac{5}{3}$

$\Rightarrow x + y = 5$

30. (C)

Tangent at $(a\cos\theta, b\sin\theta) = \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

X intercept length = $|a\sec\theta| = h$

Y intercept length = $|b\cos\theta| = k$

$\therefore |\cos\theta| = \frac{a}{h}$

$|\sin\theta| = \frac{b}{k}$

$\therefore \frac{a^2}{h^2} + \frac{b^2}{k^2} = 1$

31. (A)

We need to find tangent at $(2\cos\theta, 2\sqrt{3}\sin\theta)$

$\therefore \cos\theta = \frac{1}{8}\sin\theta = \frac{1}{8\sqrt{3}}$

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$\frac{x}{2}\left(\frac{1}{8}\right) + \frac{y}{2\sqrt{3}}\left(\frac{1}{8\sqrt{3}}\right) = 1$

$3x + y = 48$

32. **(C)**

Tangent at $\left(a\cos\theta, b\sin\theta\right) = \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b}$

$\Rightarrow b\times\cos\theta + ay\sin\theta = ab$

$\therefore \frac{\cos\alpha}{b\cos\theta} = \frac{\sin\alpha}{a\sin\theta} = \frac{p}{ab}$

$\therefore \cos\theta = \frac{a\cos ga}{p}$ $\quad\quad \sin\theta = \frac{b\sin\alpha}{p}$

$\therefore a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$

33. **(C)**

Tangent at $\left(3\cos\theta, 2\sin\theta\right) = \frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$

$\Rightarrow y = 6\operatorname{cosec}\theta - \frac{2}{3}\times\cot\theta$

$\therefore\ m = -\frac{2}{3}\cot\theta$ $\quad\quad c = 6\operatorname{cosec}\theta$

$C = 6 \pm \sqrt{1+\cot^2\theta} = \pm 6\sqrt{1+\frac{9}{4}m^2}$

$= \pm m\sqrt{4+9m^2}$

34. **(C)**

$b^2x^2 + a^2y^2 = a^2b^2$

$b^2x^2 + a^2\left(mx+c\right)^2 = a^2b^2$ should have 2 real roots

$\Rightarrow \left(b^2 + a^2m^2\right)x^2 + 2a^2mcx + a^2\left(c^2 - b^2\right) = 0$

$\therefore 4a^4m^2c^2 - 4\left(b^2 + a^2m^2\right)a^2\left(c^2 - b^2\right) \ge 0$

$a^2m^2c^2 - \left(b^2 + a^2m^2\right)\left(c^2 - b^2\right) \ge 0$

$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 \ge 0$

$a^2m^2 + b^2 - c^2 \ge 0$

35. **(D)**

Tangent at $\left(2\sqrt{2}\cos\theta, 2\sin\theta\right) = \frac{x\cos\theta}{2\sqrt{2}} + \frac{y\sin\theta}{2} = 1$

$\Rightarrow x = 2\operatorname{cosec}\theta - \frac{x\cot\theta}{\sqrt{2}}$

$\therefore \frac{-\cot\theta}{\sqrt{2}} = 4 \Rightarrow \cot\theta = -4\sqrt{2}$

$c = 2\operatorname{cosec}\theta = \pm 2\sqrt{1+\left(-4\sqrt{2}\right)^2}$

$= \pm 2\sqrt{33}$

36. **(B)**

$\frac{4x^2}{5} + \frac{3y^2}{5} = 1$

Equation of tangent at $\left(\frac{\sqrt{5}}{2}\cos\theta, \sqrt{\frac{5}{3}}\sin\theta\right)$

$\frac{2\times\cos\theta}{\sqrt{5}} + \sqrt{\frac{3}{5}}y\sin\theta = 1$

$\Rightarrow 2\times\cos\theta + \sqrt{3}y\sin\theta = \sqrt{5}$

$y = \sqrt{\frac{5}{3}}\operatorname{cosec} gq - \frac{2}{\sqrt{3}}\times\cot\theta$

$\therefore \frac{-2}{\sqrt{3}}\cot\theta = 3 \Rightarrow \cot\theta = \frac{-3\sqrt{3}}{2}$

$\sqrt{\frac{5}{3}}\operatorname{cosec}\theta = \pm\sqrt{\frac{5}{3}}\sqrt{1+\cot^2\theta}$

$= \pm\sqrt{\frac{5}{3}}\sqrt{1+\frac{27}{4}} = \pm\frac{\sqrt{155}}{2\sqrt{3}}$

37. **(A)**

$\frac{x^2}{2} + y^2 = 1$

$2e^2 = (2-1) \Rightarrow e = \frac{1}{\sqrt{2}}$

Abcessa of ends of latus rectum = $\pm ae = \pm 1$

Or dinate of ends of latus rectum = $\pm\frac{1}{\sqrt{2}}$

Tangent at $\left(\sqrt{2}\cos\theta, \sin\theta\right) = \frac{x\cos\theta}{\sqrt{2}} + y\sin\theta$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore T_1 = x + \sqrt{2}y = 2$

$T_2 = x - \sqrt{2}y = 2$

$T_3 = x + \sqrt{2}y = -2$

$T_4 = x - \sqrt{2}y = -2$

$\therefore$ end points of quadrilateral = $(2,0)\, m\, (-2,0)$, $\left(0, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}\right)$

Total area = $4\times2\times\frac{1}{\sqrt{2}}=\frac{8}{\sqrt{2}}$

38. **(A)**
Conceptual

39. **(B)**
Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore\sqrt{a^2e^2+b^2}=K\Rightarrow a=K$

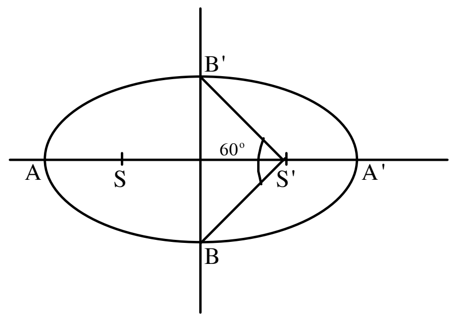
$2ae=2h\Rightarrow ae=h$

$b^2=a^2-a^2e^2=k^2-h^2$

$\therefore\frac{x^2}{k^2}+\frac{x^2}{k^2-h^2}=1$

40. **(B)**
Conceptual

41. **(A)**
$B'S' = BS'\Rightarrow\Delta BB'S'$ is equilateral.

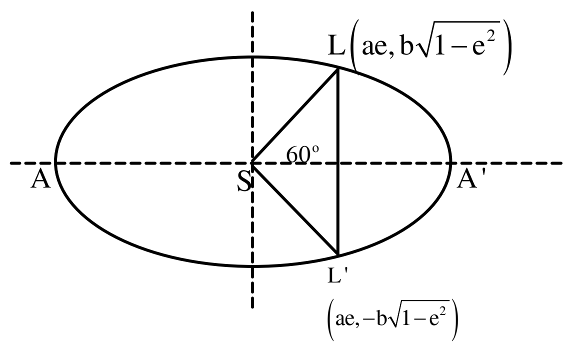


$\therefore 2b=\sqrt{a^2e^2+b^2}$

$\Rightarrow 3b^2=a^2e^2$

$\Rightarrow 3\left(1-e^2\right)=e^2\Rightarrow e=\frac{\sqrt{3}}{2}$

42. **(B)**



$\sqrt{\left(ae^2\right)+\left(b\sqrt{1-e^2}\right)^2}=2b\sqrt{1-e^2}$

$a^2e^2+b^2-b^2e^2=4b^2-4b^2e^2$

$a^2e^2=3b^2-3b^2e^2$

$e^2=3\frac{b^2}{a^2}\left(1-e^2\right)$

$e^2=3\left(1-e^2\right)^2$

$e^2=3-6e^2+3e^4\Rightarrow 3e^4-7e^2+3=0$

$e^2=\frac{7\pm\sqrt{49-36}}{6}$

43. **(A)**

$\frac{x^2}{\cot^2\alpha}+\frac{y^2}{\cos^2\alpha}=1$

$\cot^2\alpha=\frac{\cos^2\alpha}{\sin^2\alpha}\Rightarrow\cot^2\alpha>\cos^2\alpha$

$a^2=\cot^2\alpha\qquad b^2=\cos^2\alpha$

$b^2=a^2\left(1-e^2\right)$

$\left(1-e^2\right)=\sin^2\alpha$

$2|\cos\theta||\sin\alpha|=\frac{1}{2}$

$|\sin 2\alpha|=\frac{1}{2}$

$\therefore \alpha=\frac{\pi}{12}$

44. **(A)**

45. **(A)**

Let the point be (h,k)

Equation of double ordinate $x=\ell$

$y=\pm b\sqrt{1-\frac{h^2}{a^2}}=\pm y_1$

Divide in the ratio 1 : 2 : $\frac{2y_1-y_1}{3}=\frac{y_1}{3}$

$\therefore k=\frac{b}{3}\sqrt{1-\frac{h^2}{a^2}}$

$\therefore \frac{9k^2}{b^2}+\frac{h^2}{a^2}=1$

46. **(B)**

Tangent at $\left(\sqrt{5}\cos\theta,\sqrt{3}\sin\theta\right)=$

$\frac{x\cos\theta}{\sqrt{5}}+\frac{y\sin\theta}{\sqrt{3}}=1$

$a^2e^2=a^2-b^2\Rightarrow 5e^2=2\Rightarrow e^2=\frac{2}{5}$

$\therefore$ foci $=\left(\sqrt{2},0\right)$ and $\left(-\sqrt{2},0\right)$

$$\therefore S_1F_1=\left|\frac{\frac{\sqrt{2}}{\sqrt{5}}\cos\theta-1}{\sqrt{\frac{\cos^2\theta}{5}+\frac{\sin^2\theta}{3}}}\right|=\left|\frac{\sqrt{2}\cos\theta-\sqrt{5}}{\sqrt{3\cos^2\theta+5\sin^2\theta}}\sqrt{2}\right|$$

$$=\left|\frac{\left(\sqrt{3}\right)\left(\sqrt{2}\cos\theta-\sqrt{5}\right)}{\sqrt{5-2\cos^2\theta}}\right|$$

$$S_2F_2=\left|\frac{-\sqrt{\frac{2}{5}}\cos\theta-1}{\frac{\cos^2\theta}{5}+\frac{\sin^2\theta}{3}}\right|=\sqrt{3}\left|\frac{\sqrt{2}\cos\theta+\sqrt{5}}{\sqrt{5-2\cos^2\theta}}\right|$$

$$\therefore S_1F_1\times S_2F_2=3\left|\frac{2\cos^2 gq-5}{5-2\cos^2\theta}\right|=3$$

47. **(C)**
Conceptual

48. **(C)**
Let the eccentric angles be θ and $\theta+90^{o}$

$P_1=(a\cos\theta, b\sin\theta)$

$P_2=(-a\sin\theta, b\cos\theta)$

$\therefore a\ell\cos\theta+mb\sin\theta+n=0$

al sin gq + mb cos θ + n = 0

$mb\cos\theta-al\sin\theta+n=0$

$\therefore \frac{\cos\theta}{mbn+aln}=\frac{-\sin gq}{aln-mbn}=\frac{1}{a^2l^2-m^2b^2}$

$\therefore \cos\theta=\frac{mbn+aln}{a^2l^2+m^2b^2}\sin\theta=\frac{aln-mbn}{a^2l^2+m^2b^2}$

$\therefore \frac{(aln+mbn)^2}{(a^2l^2+m^2b^2)^2}+\frac{(aln-mbn)^2}{(a^2l^2+m^2b^2)^2}=1$

$\therefore 2a^2l^2n^2+2m^2b^2n^2=(a^2l^2+m^2b^2)^2$

$\Rightarrow 2n^2(a^2l^2+m^2b^2)=(a^2l^2+m^2b^2)^2$

$\Rightarrow 2n^2=a^2l^2+m^2b^2$

49. **(A)**
Positive end of a latus rectum = $\left(ae, b\sqrt{1-e^2}\right)$

$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0\Rightarrow y'=-\frac{xb^2}{ya^2}$

$\therefore$ slope of normal = $\frac{ya^2}{xb^2}=\frac{a^2b\sqrt{1-e^2}}{ae\,b^2}$

$=\frac{a\sqrt{1-e^2}}{be}$

$\left(y-b\sqrt{1-e^2}\right)=\frac{a\sqrt{1-e^2}}{be}(x-ae)$

$b=a\sqrt{1-e^2}$

$\therefore y-a(1-e^2)=\frac{x}{e}-a$

$y+ae^2=\frac{x}{e}\Rightarrow x-ey-ae^3=0$

50. **(B)**
Conceptual and proved in q. 46.

51. **(B)**
Tangent ar $\left(3\sqrt{2}\cos\theta, 4\sqrt{2}\sin\theta\right)=$

$$\frac{x\cos\theta}{3\sqrt{2}}+\frac{y\sin\theta}{4\sqrt{2}}=1$$

$$4x\cos\theta+3y\sin\theta=12\sqrt{2}$$

$$\therefore -\frac{4\cos\theta}{3\sin\theta}=\frac{-4}{3}\Rightarrow\theta=\frac{\pi}{4}\text{ or }\frac{3\pi}{4}$$

$$\therefore 4x+3y=24$$

$x=0\Rightarrow y=8$ $\qquad$ $y=0\Rightarrow x=6$

Area = $\frac{1}{2}\times 8\times 6=24$

52. **(D)**

$2a=8\Rightarrow a=4$ meters

$2ae=4\Rightarrow ae=2\Rightarrow e=\frac{1}{2}$

$\therefore b^2=a^2-a^2e^2=16\times\frac{3}{4}=12$

$b=\sqrt{12}$ meters

$\therefore$ Area = $\pi ab=\pi\times 4\times 2\sqrt{3}=8\sqrt{3}\pi m^2$

53. **(C)**

Tangents at the end points of a focal chord intersect at the directrix of ellipse

$$x=\pm\frac{a^2}{\sqrt{a^2-b^2}}$$

54. **(D)**

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

$CS=4e$ $\qquad$ major axis = 8

$\sqrt{16-9}=\sqrt{7}$

$$\therefore\frac{CS}{\text{major axis}}=\frac{\sqrt{7}}{8}$$

55. **(D)**

$\therefore$ Ellipse passes through $(3,0)$ and $(0,4)$

Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore a^2=9\ \&\ b^2=16$

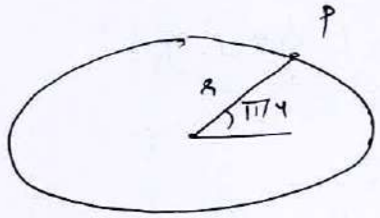
$\therefore b^2e^2=b^2-a^2$

$\Rightarrow e^2=\frac{7}{16}$

$\therefore e=\frac{\sqrt{7}}{4}$

Ellipse 2(A)

(1)



let P be $(r\cos\pi/4, r\sin\pi/4)$

or $P\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right)$

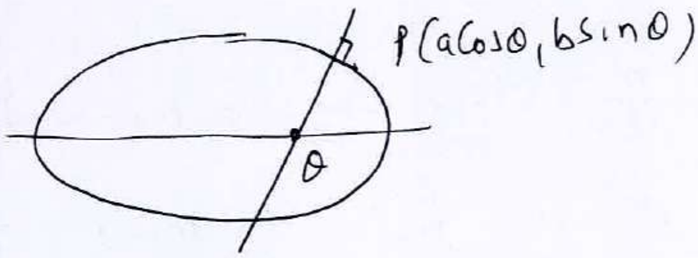
put P on ellipse

$$\frac{r^2}{8} + \frac{r^2}{4} = 1$$

$$3r^2 = 8 \Rightarrow \boxed{r = \frac{2\sqrt{2}}{\sqrt{3}}}$$

Q2

sol



$r = a - ae\cos\theta$

$r_1 = a + ae\cos\theta$

The Eqn of normal is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2e^2$$

$$x = \frac{a^2e^2}{a\sec\theta} = ae^2\cos\theta$$

$\therefore Q(ae^2\cos\theta, 0)$

$$PQ = \sqrt{(ae^2\cos\theta - a\cos\theta)^2 + b^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\theta\left(\frac{b^4}{a^4}\right) + b^2\sin^2\theta}$$

$$= \frac{b}{a}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{b}{a}\sqrt{a^2 - (b^2 - a^2)\cos^2\theta}$$

$$= \frac{b}{a}\sqrt{r_1 r}$$

③ $m = 1/2$, tangent to ellipse is

$$y = \frac{1}{2}x \pm \sqrt{4 \times \frac{1}{4} + b^2}$$

$$2y = x \pm \sqrt{b^2+1}$$

Compare

$$\Rightarrow \quad b^2 + 1 = 4$$

$$\boxed{b = \pm\sqrt{3}}$$

The other common tangent will have slope $-1/2$

$\therefore \quad b = \sqrt{3}, \quad x + 2y + 4 = 0$

Q4 let m be the slope of common tangent

$\therefore$ Comparing 'c' for both tangents gives.

$$c^2 = (a^2+b^2)m^2 + b^2 = a^2 m^2 + (a^2+b^2)$$

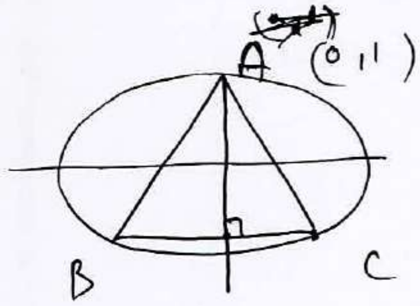
$$\Rightarrow \quad b^2 m^2 + b^2 = a^2 + b^2$$

$$\Rightarrow \quad m = \frac{\pm a}{b}$$

$y = mx \pm \sqrt{a^2 m^2 + (a^2+b^2)}$ put m to get

option (B)

(5) (2)



let length be L

Coordinate of C is

$x = L\cos 60^\circ = L/2$

$y = 1 - L\sin 60^\circ = 1 - \frac{\sqrt{3}L}{2}$

put in Ellipse

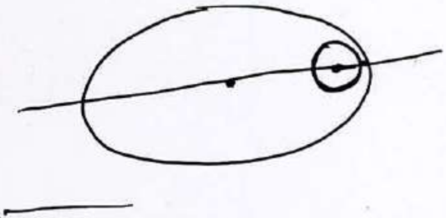
$\frac{L^2}{16} + \left(1 - \frac{\sqrt{3}L}{2}\right)^2 = 1$

$\frac{L^2}{16} + 1 + \frac{3}{4}L^2 - \sqrt{3}L = 1$

$\frac{13L^2}{16} = \sqrt{3}L$

$\Rightarrow L = \frac{16\sqrt{3}}{13}$

(6)



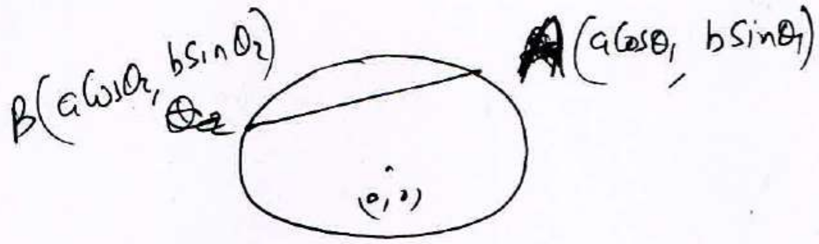
The circle can only touch @ vertex

$e = \sqrt{1 - \frac{16}{25}} = 3/5$ i.e. $r = a - ae$

$r = 5(1-e)$

$= 5(1 - 3/5) = 2$

(7)



$$\text{Slope } OA = \frac{b}{a}\tan\theta_1$$

$$\text{Slope } OB = \frac{b}{a}\tan\theta_2$$

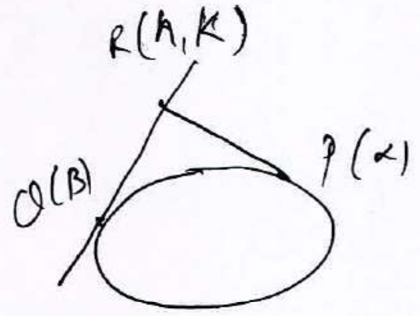
$$\therefore m_1 m_2 = \frac{b^2}{a^2} \times \tan\theta_1 \tan\theta_2$$

$$= -1$$

$$\therefore OA \perp OB$$

Q8

Sol



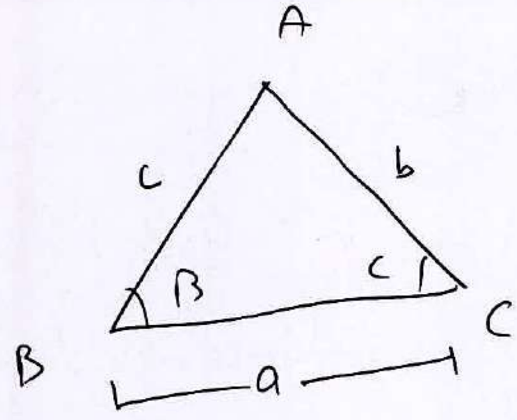
$\alpha + \beta = \text{Constant}$

Also, $h = \dfrac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$, $k = \dfrac{b\sin\frac{\alpha+\beta}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}$

$$\frac{h}{k} = \frac{a}{b}\cot\left(\frac{\alpha+\beta}{2}\right) = \text{Constant}$$

$\Rightarrow$ Locus A straight line

(9)



We know $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

divide

$\Rightarrow \frac{s-b}{s-c} = \text{constant}$

Componendo & dividendo

$$\frac{2s-(b+c)}{b-c} = \text{constant}$$

$\Rightarrow$ $\boxed{b-c = \text{constant}}$

or $|AB - AC| = \text{constant}$

$\Rightarrow$ A is hyperbola

(10) given $m_1 m_2 = c$

Assume locus point to be (h, k)

put (h, k) in eqn of a tangent

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$(k - mh)^2 = a^2m^2 + b^2$

clearly roots are m_1 & m_2

put $m_1 m_2 = c$

$$\frac{k^2 - b^2}{h^2 - a^2} = c$$

$$c(x^2 - a^2) = y^2 - b^2$$

(11) it is exclusive prop of hyperbola

(12) (3)

$$2h = a(\cos\theta - \sin\theta)$$

$$2k = b(\cos\theta + \sin\theta)$$

Square & add

$$\left(\frac{2h}{a}\right)^2 + \left(\frac{2k}{b}\right)^2 = 2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$$

(13) The Chord of Contact from (α, β) is

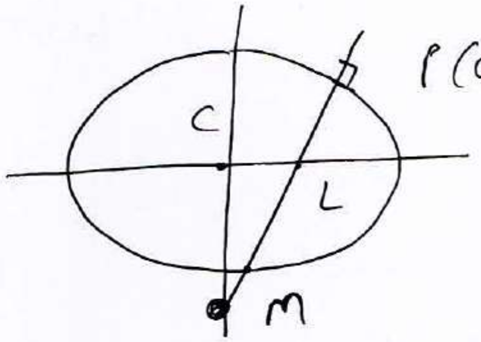
$T = 0$ $\qquad \frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1$

$\parallel$ touches circle

$\Rightarrow$ $\boxed{p = r}$

$$\frac{1}{\sqrt{\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4}}} = c \Rightarrow \boxed{\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}}$$

(14)



Normal @ P is $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2e^2$

Coordinate of L is $(ae^2\cos\theta, 0)$

" M is $\left(0, -\frac{a^2e^2}{b}\sin\theta\right)$

$= a^2 CL^2 + b^2 CM^2$

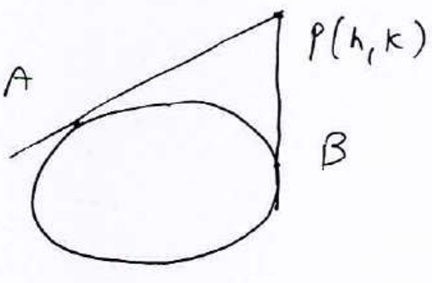
$= a^2(a^2e^4\cos^2\theta) + b^2\left(\frac{a^2e^4}{b^2}\sin^2\theta\right)$

$= a^2e^2[\ \cos^2\theta + \sin^2\theta]$

$= (a^2-b^2)$ [~~$(a^2-b^2)\cos^2\theta + \sin^2\theta$~~]

= ~~(a^2-b^2) [~~ option (B)

(15)



let $A(a\cos\alpha, b\sin\alpha)$

$B(a\cos\beta, b\sin\beta)$

Given

$b\sin\alpha + b\sin\beta = c$

or $\sin\alpha + \sin\beta = c/b$

$\Rightarrow$ $\left| 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = c/b \right|$

here $\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 1/2$

Intersection pt is

$$h = \frac{5\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \quad k = \frac{3\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right) = 1$$

$$\Rightarrow \left(\frac{h^2}{25} + \frac{k^2}{9}\right) \sout{\cos^2\left(\frac{\alpha-\beta}{2}\right) = 1} = \frac{1}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$

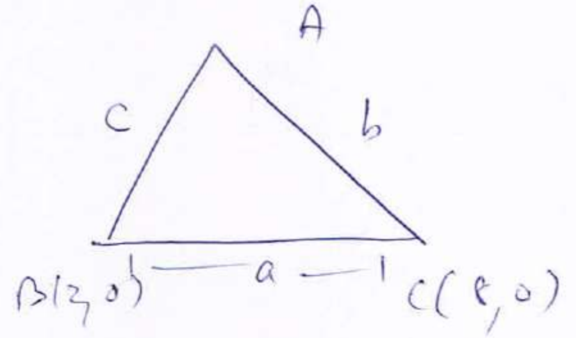
Also,

$$k = \frac{3\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\frac{2k}{3} = \frac{h^2}{25} + \frac{k^2}{9}$$

Ans $9x^2 + 25y^2 = 150y$

(16)



$a = 6$

$\tan B/2 \tan C/2 = 1/4$

Just like question 9

$$\frac{s-a}{s} = \tan B/2 \tan \frac{C}{2} = \frac{1}{4}$$

$$1 - \frac{a}{s} = \frac{1}{4}$$

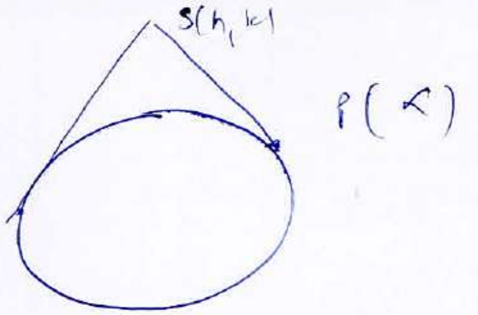
$$\Rightarrow \frac{a}{s} = \frac{3}{4} \Rightarrow \frac{s}{a} = \frac{4}{3}$$

$$\Rightarrow b + c = 10$$

E : Centre (5,0) & foci (2,0) & (8,0)

Ans. (B)

(17)



$\alpha - \beta = 60^\circ$

$$h = \frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\frac{\alpha-\beta}{2}}$$

$$k = \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

put $\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right) = 1$

$$\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right] = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4/3$$

(18) let hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

∴ circle $x^2 + y^2 = a^2e^2$

∴ tangent $x\cos\theta + y\sin\theta = ae$

let the pole be (x_1, y_1)

∴ Pole must be T = 0

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

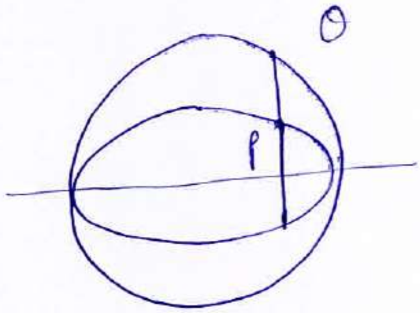
Compare with $x\cos\theta + y\sin\theta = ae$

$$\frac{\cos\theta}{\frac{x_1}{a^2}} = \frac{\sin\theta}{\frac{-y_1}{b^2}} = ae$$

$\cos^2\theta + \sin^2\theta = 1$

$$\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} = \frac{1}{a^2e^2}$$ (Ellipse)

(19)



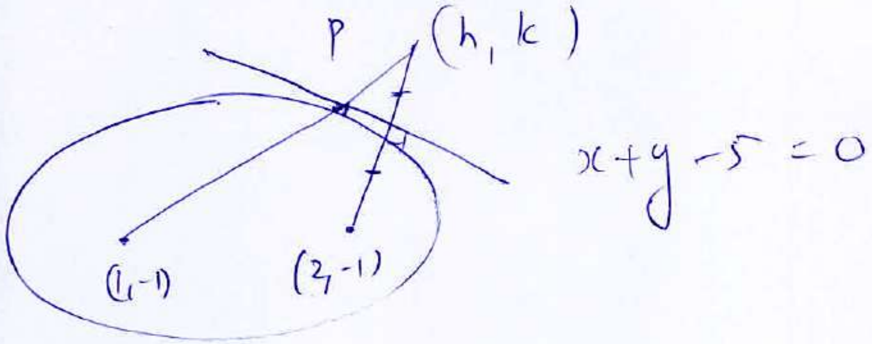
Normal @ P ≡ $ax\sec\theta - by\csc\theta = a^2e^2$

Normal @ Q ≡ $x\sin\theta - y\cos\theta = 0$

Solve $(ay - by) = \sin\theta\,(a^2e^2)$

$$y = \frac{\sin\theta \times 16}{2} = 8\sin\theta$$

$\Rightarrow x = 8\cos\theta$

(20) 

(1)

Image of (2,-1) is

$$\frac{h-2}{1} = \frac{k+1}{1} = -2\left(\frac{-4}{2}\right)$$

$$h-2 = k+1 = 4$$

$I(6,3)$

$\therefore$ line (1,-1) & (6,3) intersects $x+y-5=0$ at point P

$$\frac{y-3}{x-6} = \frac{4}{5}$$

$$5y-15 = 4x-24$$

$$\boxed{4x-5y-9=0}$$

solve 2 lines to get

$$\left(\frac{34}{9}, \ 11/9\right)$$

(21) The parallelogram PQRS is cyclic

$\Rightarrow$ PQRS is rectangle

$\Rightarrow \angle QPR = 90°$

$\Rightarrow$ P is on Director Circle

$\therefore$

$$16 + b^2 = 25$$

$$b = 3$$

$$\therefore e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

A (C)

(22) chord of Contact is

$$\frac{8x}{4} + \frac{27y}{9} = 1$$

$$2x + 3y = 1$$

Homogenize with ellipse

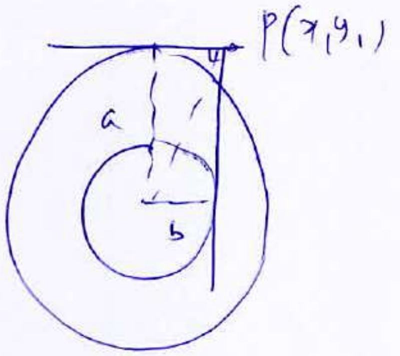
$$\frac{x^2}{4} + \frac{y^2}{9} = (2x+3y)^2$$

Angle @ origin

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\boxed{\tan\theta = \frac{48\sqrt{6}}{435}}$$

(23)



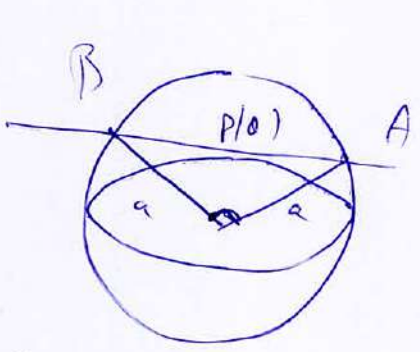
$OP^2 = a^2+b^2$

$$\boxed{x_1^2+y_1^2 = a^2+b^2}$$

A (B)

(24) Property it touches ~~directrix~~ Auxillary Circle

(25)



⊥ distance of AB from (0,0) is $\frac{a}{\sqrt{2}}$

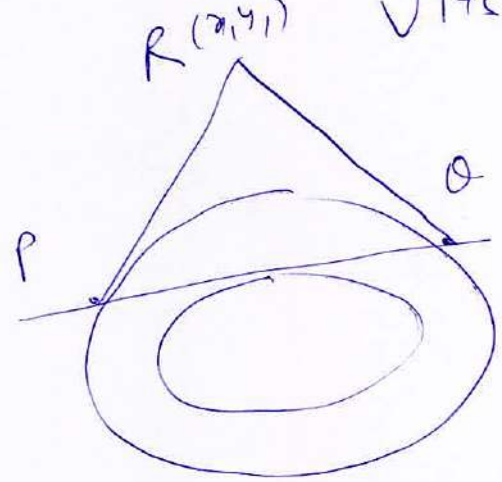
$$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$P = \frac{a}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} = \frac{a^2}{2}$$

rearrange to get

$$e = \frac{1}{\sqrt{1+5\sin^2\theta}}$$

(27)



$\therefore$ PQ is chord of contact

$$\therefore \frac{xx_1}{20} + \frac{yy_1}{5} = 1$$

This line is tangent to smaller ellipse

$$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{1} = 1$$

$$\frac{5\cos\theta}{x_1} = \frac{5\sin\theta}{y_1} = 1$$

$$\cos\theta = \frac{x_1}{5}, \quad \sin\theta = \frac{y_1}{5}$$

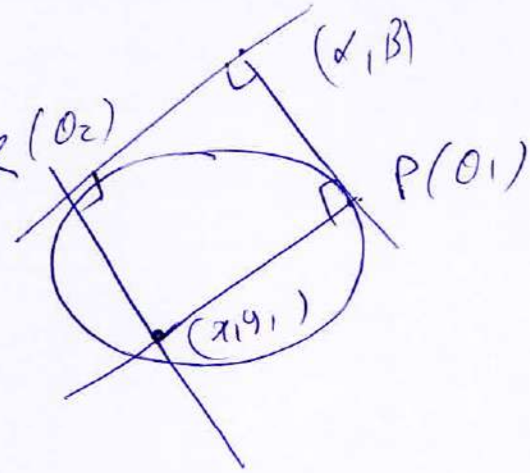
$$\boxed{x_1^2 + y_1^2 = 25}$$

which is Director circle of ~~the~~ smaller ellipse

$\Rightarrow \angle PRQ = 90°$

(28)

(8)



$$\alpha = \frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \quad \beta = \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}$$

Also, since rectangle Equate mid point.

$$\frac{x_1+\alpha}{a} = a(\cos\theta_1 + \cos\theta_2) = 2a\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}$$

$$y_1+\beta = b(\sin\theta_1 + \sin\theta_2) = 2b\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}$$

divide $\dfrac{x_1+\alpha}{y_1+\beta} = \dfrac{\alpha}{\beta}$

$\Rightarrow$ $\boxed{\alpha y_1 = \beta x_1}$

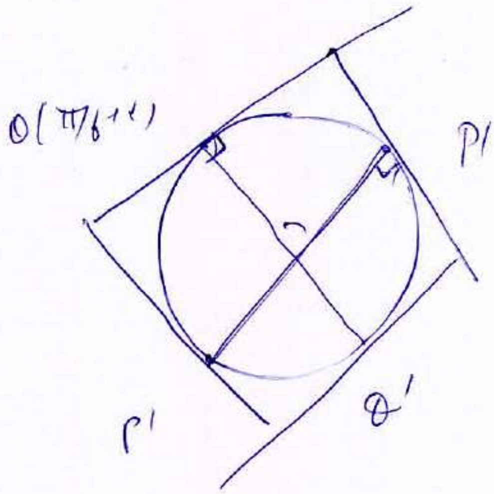
(29) Incentre Coordinates are

$h = ae\cos\theta, \quad k = \dfrac{be\sin\theta}{1+e}$

$$\cos^2\theta + \sin^2\theta = \frac{x^2}{(ae)^2} + \frac{y^2}{\left(\frac{be}{1+e}\right)^2} = 1$$

Which is an ellipse.

(31) Project everything to circle



Area of ∥gm is

~~$\frac{1}{6} P_1P_2 \sin\theta$~~

$\dfrac{P_1P_2}{\sin\pi/6} = 16$

∴ Area in ellipse scenario is

$16 \times \dfrac{1}{\sqrt{2}} = 8\sqrt{2}$

(32) line joining and 2 points is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\pi/6$$

or

$$\frac{x}{a}\cos(\theta)+\frac{y}{b}\sin(\theta)=\sqrt{3}/2$$

or

$$\frac{x\cos\theta}{\left(\frac{\sqrt{3}a}{2}\right)}+\frac{y\sin\theta}{\left(\frac{\sqrt{3}b}{2}\right)}=1$$

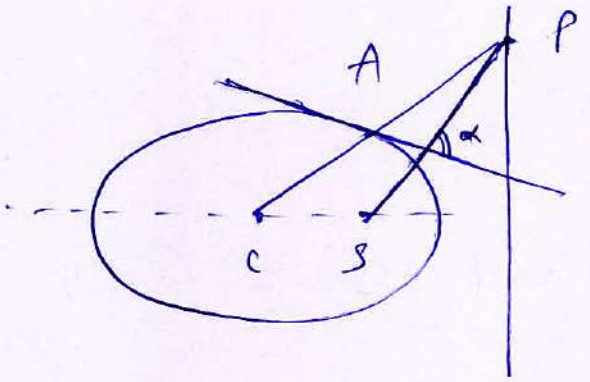
: touches ellipse

$$\frac{x^2}{\left(\frac{\sqrt{3}a}{2}\right)^2}+\frac{y^2}{\left(\frac{\sqrt{3}b}{2}\right)^2}=1$$

or

$$\frac{x^2}{9}+\frac{y^2}{4}=\frac{3}{4}$$

(33)



let $P\left(\frac{a}{e_1}, y_1\right)$

Slope PS $= \frac{y_1}{-ae+\frac{a}{e}} = \frac{-ey_1}{a(e^2-1)} = \frac{aey_1}{b^2} = m_1$

~~line PC ⇒ put this in Ellipse Eqⁿ~~

line PC is $\frac{y}{x} = \frac{ey_1}{a}$ put this in ellipse

$$\frac{x^2}{a^2} + \frac{x^2e^2y_1^2}{b^2} = 1$$

$$x = \frac{ab}{\sqrt{b^2+a^2e^2y_1^2}} \Rightarrow y = \frac{ey_1b}{\sqrt{b^2+a^2e^2y_1^2}}$$

∴ slope of tangent is $-\frac{b^2}{a^2} \times \frac{ab}{ey_1b} = \frac{-b^2}{aey_1} = m_2$

∴ we see

$$\boxed{m_1m_2 = -1}$$

(34) Normal @ $P(\theta)$

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2e^2$$

$G(ae^2\cos\theta, 0)$

$\therefore g(-ae^2\sin\theta, 0)$

$$PG = \sqrt{(a\cos\theta - ae^2\cos\theta)^2 + (b\sin\theta)^2}$$

$$= \sqrt{\frac{b^4\cos^2\theta}{a^2} + b^2\sin^2\theta}$$

$$= \frac{b}{a}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\& \; Og = \sqrt{(a\sin\theta - ae^2\sin\theta)^2 + b^2\cos^2\theta}$$

$$= \frac{b}{a}\sqrt{b^2\sin^2\theta + a^2\cos^2\theta}$$

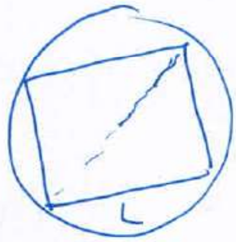
$$PG^2 + Og^2 = \frac{b^2}{a^2}(a^2 + b^2) = b^2\left(1 + \frac{b^2}{a^2}\right) = b^2(2 - e^2)$$

$$\text{or} \quad a^2(1 - e^2)(2 - e^2)$$

Q35

Sol^n The square will be inscribed in ~~A~~ director circle

$\therefore r = \sqrt{a^2+b^2} = \frac{L}{\sqrt{2}}$



~~$a^2+b^2 =$~~

$\sqrt{2a^2+2b^2} = L$

$L = \sqrt{14+11} = 5$

(36)

$mn = m + n$

$(m-1)(n-1) = 1$

$\Rightarrow$ $m = n = 2$ only

Chord of contact of $S \equiv 4x^2 + 9y^2 = 36$ w.r.t $P(2,2)$

is $T = 0$

$4x(2) + 9y(2) = 36$

$$\boxed{4x + 9y = 18}$$

(37)

$$\sum_{i=1}^{10} (SP_i)(S'P_i) = 2560$$

$10 \cdot b^2 = 2560$

$b = 16, \quad a = 20$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{3}{5}$

(38) Let latus-rectum of parabola is, $l = 4a'$

Given, $2a' = 2ae \Rightarrow \boxed{a' = ae}$

& $4a' = \frac{2b^2}{a} \Rightarrow \frac{b^2}{a} = 2a'$

$\Rightarrow b^2 = 2a^2 e$

$\Rightarrow a^2(1-e^2) = 2a^2 e$

$e^2 + 2e - 1 = 0$

$$\boxed{e = \sqrt{2} - 1}$$

(39) $(x+y)^2 + y^2 = 1$

Any point on ellipse is $P(\cos\theta - \sin\theta, \sin\theta)$

& centre of ellipse is $O(0,0)$

$\Rightarrow OP = \sqrt{1 - \sin 2\theta + \sin^2\theta}$

$OP = \sqrt{\frac{3 - (2\sin 2\theta + \cos 2\theta)}{2}}$

$OP_{max} = r = \sqrt{\frac{3+\sqrt{5}}{2}}$

$= \sqrt{\frac{2}{3-\sqrt{5}}}$

(40) homogenizing $x^2 + 2xy + (2+\sin^2\alpha)y^2 = 1$

using $PQ \equiv y - \sqrt{m}\,x = 1$, gives

$x^2 + 2xy + (2+\sin^2\alpha)y^2 - (y - \sqrt{m}\,x)^2 = 1$

joint eqn. of OP & OQ when

~~PQ~~ : PQ. $OP \perp OQ \Rightarrow$

$(1-m) + (2 + \sin^2\alpha - 1) = 0$

$m = \sin^2\alpha + 2 \Rightarrow \boxed{m \in [2,3]}$

ELLIPSE

EXERCISE 2(C)

Q.1 [04]

Given equations are $\frac{x^2}{16}+\frac{y^2}{9}=1$ & $\frac{x^2}{9}+\frac{y^2}{16}=1$

Subtracting the two equations gives $x=\pm y$

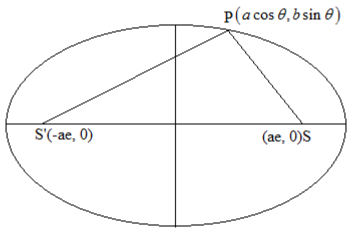
Now $\frac{x^2}{16}+\frac{y^2}{9}=1$ & $x^2=y^2 \Rightarrow x^2=y^2=\frac{144}{25}$

Or $x=\pm\frac{12}{5}$ & $y=\pm\frac{12}{5}$.

Points of intersection $=\left(\pm\frac{12}{5},\pm\frac{12}{5}\right)$

Hence number of points = 4

Q.2 [03]



$SP=a-ae\cos\theta,\ S'P=a+ae\cos\theta$ & $SS'=2ae$.

Now coordinates of in center will be

$$\left(\frac{(a\cos\theta)2ae+ae(a+ae\cos\theta)-ae(a-ae\cos\theta)}{2ae+a+ae\cos\theta+a-ae\cos\theta},\frac{2aeb\sin\theta}{2a+2ae}\right)$$

Or $\left(\frac{2a^2\cos\theta+a^2e+a^2e^2\cos\theta-a^2e+a^2e^2\cos\theta}{2a+2ae},\frac{be\sin\theta}{1+e}\right)$

Or $\left(\frac{2a^2e\cos\theta(1+e)}{2a(1+e)},\frac{be\sin\theta}{1-e}\right)$. Now let $h=\frac{2a^2e\cos\theta(1+e)}{2a(1+e)}$ & $k=\frac{be\sin\theta}{1-e}$,

then eliminating θ gives $\frac{h^2}{a^2e^2}+\frac{(1+e)^2k^2}{b^2e^2}=1$

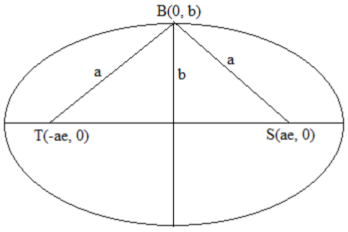
$e=\frac{3}{5} \Rightarrow 1-\frac{a^2}{b^2}=\frac{9}{25}$ or $\frac{b^2}{a^2}=\frac{16}{25}$

Required Locus : $\frac{25x^2}{9a^2}+\frac{64y^2}{9b^2}=1$

Now $\lambda=\sqrt{1-\frac{9b^2}{64a^2}}$ or $\lambda=\sqrt{1-\frac{1}{4}}$

Hence $4\lambda^2 = 3$.

Q.3 [02]



ST = 2*ae* & SB = TB = *a*.

As the triangle is equilateral hence 2*ae* = *a*.

Or *e* = ½.

Hence 4*e* = 2

Q.4 [01]

Let the equation of ellipse be $\frac{(x-1)^2}{64} + \frac{(y+1)^2}{b^2} = 1$.

As it passes through (1, 3) hence $\frac{(1-1)^2}{64} + \frac{(3+1)^2}{b^2} = 1$ or $b^2 = 16$.

Now length of latus rectum = $\frac{2b^2}{a} = \frac{32}{8} = 4$, therefore $\frac{l}{4} = 1$

Q.5 [02]

Let the equation of any tangent be $y = mx + \sqrt{a^2m^2 + b^2}$, where $a = 67$ & $b = 33$

$\Rightarrow y = mx + \sqrt{(67)^2 m^2 + (33)^2}$.

Also two points on minor axis at a distance of $10\sqrt{34}$ from origin will be

$P\left(0, 10\sqrt{34}\right)$ & $Q\left(0, -10\sqrt{34}\right)$.

Sum of squares of perpendiculars on the tangent from P & Q

$$= \frac{\left(10\sqrt{34} + \sqrt{(67)^2 m^2 + (33)^2}\right)^2 + \left(10\sqrt{34} - \sqrt{(67)^2 m^2 + (33)^2}\right)^2}{(1+m^2)}$$

$$\Rightarrow 4489\lambda = \frac{\left(3400 + (67)^2 m^2 + (33)^2\right)^2}{(1+m^2)} \quad \text{or} \quad 4489 \times 2 = 4489 \times \lambda$$

Hence $\lambda = 2$.

Q.6 [02]

Given a = 5, b = 4

Hence $e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$ and $ae = 3$

$SP = \text{radius of circle} = a - ae\cos\theta$

$= 5-3\cos\theta,\ \theta\in\left[0,\frac{\pi}{2}\right].$

$r_{\max} = 5-3 = 2$.

Q.7 [07]

Let equation of common tangent be $y = mx + c$.

For being tangent to ellipse, $c^2 = 25m^2+4$ & for circle $c^2 = 16\left(m^2+1\right)$.

$25m^2+4 = 16m^2+16 \Rightarrow m = \pm\frac{2}{\sqrt{3}}$.

Equation of one of the common tangents : $y=\frac{2}{\sqrt{3}}x+4\sqrt{1+\frac{4}{3}}$

Required area $=\frac{1}{2}\times\frac{4\sqrt{7}}{\sqrt{3}}\times 2\sqrt{7}$

$=\frac{\sqrt{3}L}{4}=7$.

Q.8 [01]

Equation of tangent : $5x\sec\theta - 4y\,\text{cosec}\,\theta = 9$.

Distance of this line from (0, 0), $l=\frac{9}{\sqrt{25\sec^2\theta+16\cos ec^2\theta}}$.

Now $25\sec^2\theta+16\cos ec^2\theta = 41+25\tan^2\theta+16\cot^2\theta$

By A.M.$\geq$ G.M., $25\tan^2\theta+16\cot^2\theta\geq 40$

Hence $25\sec^2\theta+16\cos ec^2\theta\geq 81$

Therefore $\frac{9}{\sqrt{25\sec^2\theta+16\cos ec^2\theta}}\leq 1.$

Q.9 [01]

Let $P(a\cos\theta, b\sin\theta)$ be any general point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Equation of tangent at this point will be given by $\frac{x\cos\theta}{a}+\frac{b\sin\theta}{a}=1$

at $x=0,\ y=\frac{b}{\sin\theta}$ & at $y=0,\ x=\frac{a}{\cos\theta}$

$$\therefore\ \frac{a^2}{CP^2}+\frac{b^2}{CQ^2}=\frac{a^2}{\left(\frac{a}{\cos\theta}\right)^2}+\frac{b^2}{\left(\frac{b}{\sin\theta}\right)^2}$$

Hence $\frac{a^2}{CP^2}+\frac{b^2}{CQ^2}$ = 1.

Q.10 [04]

The equation of normal at (x',y') is $\frac{x-x'}{x'}a^2=\frac{y-y'}{y'}b^2$

If it passes through (h, k), then $y'^2\{a^2(h-x')+b^2x'\}^2=b^4k^2x'^2$ ....(i)

But $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1\ \ or\ \ y'^2=\frac{b^2}{a^2}(a^2-x'^2)$ ....(ii)

Value of y'^{2} from eq.(ii), putting in eq. (i), we get

$$\frac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'^2\}=b^4k^2x'^2$$

$$\Rightarrow\frac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-x^2)\}=b^4k^2x'^2$$

Arrange above as fourth degree equation in x', then roots of the above equation are x_1,x_2,x_3,x_4. Now

$$x_1.x_2.x_3.x_4=-\frac{2ha^2(a^2-b^2)}{-(a^2-b^2)^2}=\frac{2ha^2}{(a^2-b^2)}$$ ....(iii)

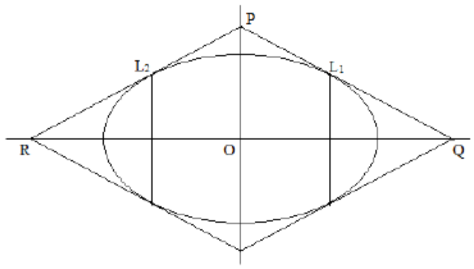
Also $\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=\frac{\sum x_1x_2x_3}{x_1.x_2.x_3.x_4}$

$$= \frac{\dfrac{2a^4h(a^2-b^2)}{-(a^2-b^2)^2}}{\dfrac{a^6h^2}{-(a^2-b^2)^2}} = \frac{2(a^2-b^2)}{a^2h} \qquad \text{..........(iv)}$$

Multiplying (iii) and (iv), we get $(x_1+x_2+x_3+x_4)\times\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

Q.11 [03]

Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. L_1 & L_2 are two of the extremities of latus recta. Now



equation of tangents at $L_1\left(ae,\frac{b^2}{a}\right)$ & $L_2\left(-ae,\frac{b^2}{a}\right)$

will be $\frac{x(ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$ & $\frac{x(-ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$.

Solving these equations we get P as (0, a).

Also Q & R are $\left(\frac{a}{e},0\right)$ & $\left(-\frac{a}{e},0\right)$.

Now area of triangle PQR $=\frac{1}{2}\times\frac{2a}{e}\times a=\frac{a^2}{e}$

Therefore area of quadrilateral = $\frac{2a^2}{e}$

Now $A=\frac{2a^2}{e}\Rightarrow\frac{A}{9}=3$

Q.12 [05]

The man is running on an ellipse in which $2a = 10$ & $2ae = 8$.

Now $b^2=a^2-a^2e^2$ gives $b = 3$.

Area of ellipse = $\pi ab=15\pi$.

Q.13 [03]

Let the common tangent be $y = mx + c$.

For circle, $c^2 = 16\left(1+m^2\right)$ & for ellipse, $c^2 = 25m^2 + 4$

slope of common tangent $= \sqrt{\dfrac{16-4}{25-16}} = \dfrac{2}{\sqrt{3}}$

Equation of the tangent : $y = \dfrac{2x}{\sqrt{3}} + 4\sqrt{1+\dfrac{4}{3}}$ or $y = \dfrac{2x}{\sqrt{3}} + \dfrac{4\sqrt{7}}{\sqrt{3}}$

$x -$ intercept $= -2\sqrt{7}$ & $y -$ intercept $= \dfrac{4\sqrt{7}}{\sqrt{3}}$

Reqd. Length $= \sqrt{28 + \dfrac{112}{3}} = \dfrac{14}{\sqrt{3}}$.

Q.14 [04]

Equation of chord joining $P\left(\theta_1\right) \& Q\left(\theta_2\right)$: $\dfrac{x}{a}\cos\dfrac{\theta_1+\theta_2}{2} + \dfrac{y}{b}\sin\dfrac{\theta_1+\theta_2}{2} = \cos\dfrac{\theta_1-\theta_2}{2}$

or $\dfrac{2x}{a}\cos\dfrac{\theta_1+\theta_2}{2} + \dfrac{2y}{b}\sin\dfrac{\theta_1+\theta_2}{2} = 1$. {given that $\theta_1 - \theta_2 = \dfrac{2\pi}{3}$ }

Let the point of intersection of tangents at $P\left(\theta_1\right) \& Q\left(\theta_2\right)$ be R(h, k).

Chord of contact of R : $\dfrac{hx}{a^2} + \dfrac{ky}{b^2} = 1$.

Comparing the two equations gives $\dfrac{h}{2a} = \cos\dfrac{\theta_1+\theta_2}{2}$ & $\dfrac{k}{2b} = \sin\dfrac{\theta_1+\theta_2}{2}$

Square and add to get $\dfrac{h^2}{4a^2} + \dfrac{k^2}{4b^2} = 1$.

Required locus is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 4$.

Q.15 [04]

Given : $C \equiv (2,-3)$, $S \equiv (3,-3)$, $A \equiv (4,-3)$

Now a = CA = 2 & ae = CS = 1.

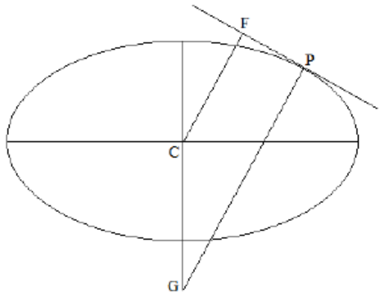
Hence $b = a\sqrt{1-e^2} = \sqrt{3}$.

Equation of ellipse : $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{3} = 1$ or $3x^2 + 4y^2 - 12x + 24y + 36 = 0$.

Q.16 [01]

Standard result : $\frac{1}{e_1^{\,2}}+\frac{1}{e_2^{\,2}}=1$.

Q.17 [07]



Given ellipse is $\frac{x^2}{49}+\frac{y^2}{25}=1$

Let any point be $P(7\cos\theta, 5\sin\theta)$

Equation of tangent P : $\frac{x\cos\theta}{7}+\frac{y\sin\theta}{5}=1$

Perpendicular from center, $CF = \frac{1}{\sqrt{\frac{\cos^2\theta}{49}+\sqrt{\frac{\sin^2\theta}{25}}}}$ or $\frac{35}{\sqrt{25\cos^2\theta+49\sin^2\theta}}$

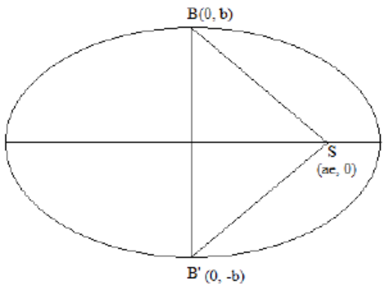
Equation of normal at Q : $7x\sec\theta-5y\,\text{cosec}\,\theta=24$

Now at $x=0,\ y=-\frac{24}{5}\sin\theta$ Hence $G\equiv\left(0,-\frac{24}{5}\sin\theta\right)$

Hence $PG=\sqrt{49\cos^2\theta+\left(5\sin\theta+\frac{24}{5}\sin\theta\right)^2}$ or $\frac{7}{5}\sqrt{25\cos^2\theta+49\sin^2\theta}$

$\therefore \sqrt{CF.PG} = 7$.

Q.18 [01]



Given $m_{SB}\times m_{SB'} = -1$

$\Rightarrow\left(\frac{b-0}{0-a}\right).\left(\frac{a+b}{ae-0}\right)=-1$

$\Rightarrow\frac{b^2}{a^2e^2}=-1$

$\frac{b^2}{a^2}=\frac{1}{2}\Rightarrow e=\sqrt{1-\frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$.

Q.19 [09]

If the equation of tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, is $y = mx + c$, then $c^2 = 25m^2 + 9$

Now given equation is $y = 2x + \lambda$ so $m = 2$ & $c = \lambda$.

$\therefore\ \lambda^2 = 25\times 4 + 9$ or $\lambda^2 - 100 = 9$.

Q.20 [04]

Standard fact : In general from any point four normals can be drawn to an ellipse.

The equation of normal at (x', y') is $\frac{x-x'}{x'}a^2 = \frac{y-y'}{y'}b^2$

If it passes through (h, k), then $y'^2\{a^2(h-x')+b^2x'\}^2 = b^4k^2x'^2$ ....(i)

But $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$ *or* $y'^2 = \frac{b^2}{a^2}(a^2-x'^2)$ ....(ii)

Value of y'^2 from eq.(ii), putting in eq. (i), we get

$$\frac{b^2}{a^2}(a^2-x'^2)\{a^2h+(b^2-a^2)x'^2\} = b^4k^2x'^2$$

$$\Rightarrow \frac{b^2}{a^2}(a^2-x'^2)\{a^4h^2+(b^2-a^2)^2x'^2+2a^2hx'(b^2-x^2)\} = b^4k^2x'^2$$

Arrange above as fourth degree equation in x', then roots of the above equation are x_1, x_2, x_3, x_4.

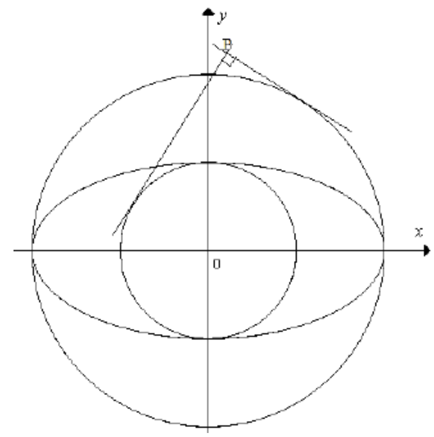
ELLIPSE

Exercise – 3

Q.1 $ae = b\ \ \&\ \ a^2 - a^2e^2 = b^2 \Rightarrow a^2 = 2a^2e^2$

$\Rightarrow e = \frac{1}{\sqrt{2}}$.

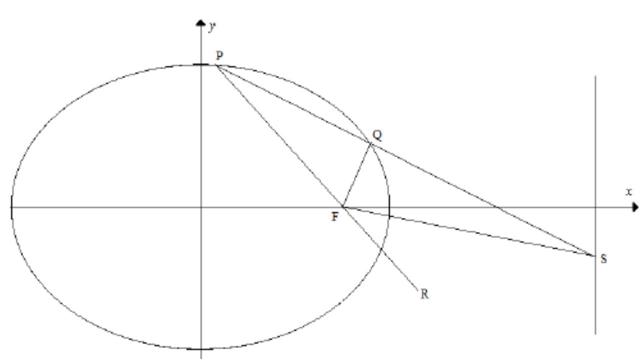
Q.2 Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then



the circle on major axis as diameter will be
$x^2 + y^2 = a^2$ and
the circle on minor axis as diameter will be
$x^2 + y^2 = b^2$
Any tangent with slope m to former circle will be
$y = mx + a\sqrt{1+m^2}$ or $y - mx = a\sqrt{1+m^2}$ and
a perpendicular tangent to the later circle will be
$y = -\frac{1}{m}x + b\sqrt{1+\frac{1}{m^2}}$ or $x + my = b\sqrt{1+m^2}$
From the two equations we get $m = \frac{ay - bx}{by + ax}$.
Substituting this value of m in former equation of tangent gives
$x(by+ax) + y(ay-bx) = a\sqrt{(by+ax)^2 + (ay-bx)^2}$
or $x^2 + y^2 = a^2 + b^2$.

Q.3 Vertices of the rectangle will be $P(\theta), Q(\pi-\theta), R(\pi+\theta)\ \&\ S(-\theta)$.
Hence sides will be of length $2a\cos\theta\ \&\ 2b\sin\theta$.
Perimeter = $2a\cos\theta + 2b\sin\theta \le 2\sqrt{a^2+b^2}$
Area = $4ab\cos\theta\sin\theta \le 2ab$.

Q.4 Let eccentric angles of P & Q be
$\alpha\ \&\ \beta$, then equation of PQ will be



$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$
Hence coordinates of S will be
$\left(\frac{a}{e}, \frac{b\left(e\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right)}{e\sin\frac{\alpha+\beta}{2}}\right)$

Now slope of FS = $\dfrac{b\left(e\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right)}{a\left(1-e^2\right)\sin\frac{\alpha+\beta}{2}} = \dfrac{a\left(e\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}\right)}{b\sin\frac{\alpha+\beta}{2}}$

Slope of QF = $m_1 = \dfrac{b\sin\beta}{a\cos\beta-ae}$ & Slope of PR = $m_2 = \dfrac{b\sin\alpha}{a\cos\alpha-ae}$

Let slope of bisector of angle QFR be m, then

$$\frac{m_1-m}{1+m_1m} = \frac{m-m_2}{1+mm_2} \Rightarrow \left(m_1+m_2\right)m^2+2\left(1-m_1m_2\right)m-\left(m_1+m_2\right)=0$$

Now $m_1+m_2 = \dfrac{b\sin\alpha}{a\cos\alpha-ae}+\dfrac{b\sin\beta}{a\cos\beta-ae}$

$$= \frac{ab\sin\left(\alpha+\beta\right)-abe\left(\sin\alpha+\sin\beta\right)}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

$$= \frac{2ab\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

and $1-m_1m_2 = 1-\left(\dfrac{b\sin\alpha}{a\cos\alpha-ae}\right)\left(\dfrac{b\sin\beta}{a\cos\beta-ae}\right)$

$$= \frac{a^2\cos\alpha\cos\beta-a^2e\left(\cos\alpha+\cos\beta\right)+a^2e^2-b^2\sin\alpha\sin\beta}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

$$= \frac{\left(a^2-b^2\right)\cos^2\frac{\alpha-\beta}{2}+\left(a^2+b^2\right)\cos^2\frac{\alpha+\beta}{2}-2a^2e\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}+a^2e^2-a^2}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

$$= \frac{a^2\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)^2-b^2\sin^2\frac{\alpha+\beta}{2}}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$$

Let $a\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)=p$ & $b\sin\frac{\alpha+\beta}{2}=q$

then $m_1+m_2 = \dfrac{2pq}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$ & $1-m_1m_2 = \dfrac{p^2-q^2}{a^2\left(\cos\alpha-e\right)\left(\cos\beta-e\right)}$

Now $pqm^2+\left(p^2-q^2\right)m-pq=0 \Rightarrow m=\dfrac{p}{q}$ & $-\dfrac{q}{p}$

$$\Rightarrow m = \frac{a\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)}{b\sin\frac{\alpha+\beta}{2}} \;\;\&\;\; -\frac{b\sin\frac{\alpha+\beta}{2}}{a\left(\cos\frac{\alpha+\beta}{2}-e\cos\frac{\alpha-\beta}{2}\right)}$$

Hence one of the bisectors of angle QFR is FS.

Q.5 As the ellipse is touching x – axis hence $b^2 = 12$.

(Product of perpendiculars from the foci on any tangent is b^2)

Also $2ae = \sqrt{(-1-5)^2 + (2-6)^2} \Rightarrow a^2e^2 = 13$

Now $a^2 = a^2e^2 + b^2 = 25$

Hence $e = \frac{\sqrt{13}}{5}$.

Q.6 Let radii of the given circles w_1 & w_2 be r_1 & r_2 and that of w be r.

Now AC = $r_1 - r$ & BC = $r_2 + r$, then

AC + BC = $r_1 + r_2$

Hence locus of C is an ellipse foci at A & B and major axis = $r_1 + r_2$.

Q.7 Normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point P(θ) will be

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2.$$

Distance from origin = $\frac{a^2 - b^2}{\sqrt{a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta}}$

Now $a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta = a^2 \tan^2\theta + b^2 \cot^2\theta + a^2 + b^2$

Further by A.M. $\geq$ G.M.

$a^2 \tan^2\theta + b^2 \cot^2\theta \geq 2ab$

Hence $a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta \geq a^2 + b^2 + 2ab$

$$\Rightarrow \frac{a^2 - b^2}{\sqrt{a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta}} \leq a - b.$$

Q.8 Tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point P(θ) will be

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

It will meet the coordinate axes at $A(a\sec\theta, 0)$ & $B(0, b\operatorname{cosec}\theta)$.

Coordinates of midpoint of AB will be

$x = \frac{a\sec\theta}{2}$ & $y = \frac{b\operatorname{cosec}\theta}{2}$

Eliminating θ gives the required locus as $\frac{a^2}{4x^2} + \frac{b^2}{4y^2} = 1$.

Q.9 Let P be $(a\cos\theta, b\sin\theta)$.

Now F is (ae, 0) hence PF = $a(1 - e\cos\theta)$

Radius of circle on PF as diameter = $\frac{a(1 - e\cos\theta)}{2}$ and center : $\left(\frac{a\cos\theta + ae}{2}, \frac{b\sin\theta}{2}\right)$

Also for auxiliary circle radius = a and center : (0, 0)

Distance between the centers = $\sqrt{\left(\frac{a\cos\theta+ae}{2}\right)^2+\left(\frac{b\sin\theta}{2}\right)^2}$ |

$=\frac{1}{2}\sqrt{a^2\cos^2\theta+2a^2e\cos\theta+a^2e^2+b^2\sin^2\theta}$

$=\frac{1}{2}\sqrt{a^2\cos^2\theta+2a^2e\cos\theta+a^2e^2+b^2-b^2\cos^2\theta}$

$=\frac{1}{2}\sqrt{a^2e^2\cos^2\theta+2a^2e\cos\theta+a^2}$

$=\frac{a(1+e\cos\theta)}{2}=a-\frac{a(1-e\cos\theta)}{2}$

= difference of radii.

Hence circle with PF as diameter touches the auxiliary circle.

Q.10 Let the common tangent be $y = mx + c$.

For being a tangent to the ellipse : $c^2=a^2m^2+b^2$

For being a tangent to the circle : $c^2=r^2\left(m^2+1\right)$

Hence $a^2m^2+b^2=r^2\left(m^2+1\right)$

$\Rightarrow m=\pm\sqrt{\frac{r^2-b^2}{a^2-r^2}}=\tan\theta$

Now parametric coordinates of a point at a distance p from F(ae, 0) on the line RS || PQ will be

$(ae+p\cos\theta, p\sin\theta)$

Substituting these coordinates in the equation of the circle gives

$(ae+p\cos\theta)^2+(p\sin\theta)^2=r^2$ or $p^2+(2ae\cos\theta)p+a^2e^2-r^2=0$

Roots of this equation will be SF and QF.

As SF & QF are measured in opposite directions from F, hence

RS = difference of roots

$\Rightarrow RS=\frac{\sqrt{4a^2e^2\cos^2\theta-4\left(a^2e^2-r^2\right)}}{2}$

$\Rightarrow RS=\sqrt{r^2-a^2e^2\sin^2\theta}$

Now $\tan^2\theta=\frac{r^2-b^2}{a^2-r^2}\Rightarrow\sin^2\theta=\frac{r^2-b^2}{a^2e^2}$

$\Rightarrow RS=b$.

Q.11 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point P(θ) will be

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Homogenizing the equation of the auxiliary circle using the equation of tangent gives

$$x^2 + y^2 = a^2\left(\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b}\right)^2 \quad \text{or}$$

$$\left(b^2\sin^2\theta\right)x^2 - \left(2ab\sin\theta\cos\theta\right)xy + \left(b^2 - a^2\sin^2\theta\right)y^2 = 0$$

As this pair of straight lines subtends a right angle at the origin hence

$$\text{coefficient of } x^2 + \text{coefficient of } y^2 = 0$$

$$\Rightarrow b^2\sin^2\theta + b^2 - a^2\sin^2\theta = 0$$

$$\Rightarrow \frac{a^2 - b^2}{b^2} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow \frac{a^2 - b^2}{a^2} = \frac{1}{1+\sin^2\theta}$$

$$\Rightarrow e = \frac{1}{\sqrt{1+\sin^2\theta}}$$

Q.12 Let P be $\left(a\cos\theta, b\sin\theta\right)$.

Also F_1 & F_2 are $(ae, 0)$ & $(-ae, 0)$

$F_1F_2 = 2ae$, $PF_1 = a\left(1 - e\cos\theta\right)$ & $PF_2 = a\left(1 + e\cos\theta\right)$

Now $(PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta$.

Further Tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P(θ) will be $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

Hence $d = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

Now $4a^2\left(1 - \frac{b^2}{d^2}\right) = 4a^2\left(1 - \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2}\right)$

$= 4\left(a^2 - b^2\right)\cos^2\theta = 4a^2e^2\cos^2\theta$.

Q.13 Let the extremities of the two semi-diameters be $P(\alpha)$ & $Q(\beta)$, then

$\frac{b\sin\alpha}{a\cos\alpha} \times \frac{b\sin\beta}{a\cos\beta} = -1$ (as the diameters are mutually perpendicular)

$$\Rightarrow b^2\sin\alpha\sin\beta + a^2\cos\alpha\cos\beta = 0$$

$$\Rightarrow b^2\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right) + a^2\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right) = 0$$

$$\Rightarrow \left(a^2 + b^2\right)\cos^2\frac{\alpha-\beta}{2} + \left(a^2 - b^2\right)\cos^2\frac{\alpha+\beta}{2} = a^2 \ \ldots\text{(i)}$$

Now the chord PQ will be

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

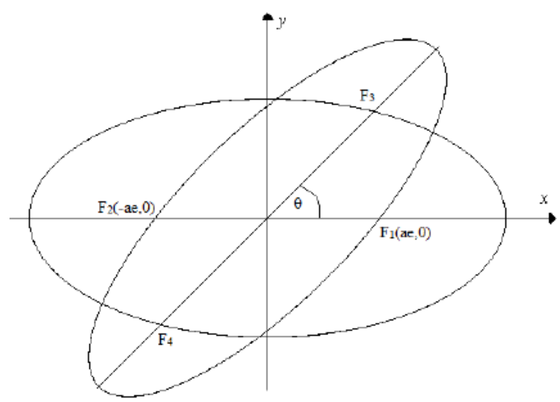
Distance of PQ from the origin is $d = \dfrac{ab\cos\frac{\alpha-\beta}{2}}{\sqrt{b^2\cos^2\frac{\alpha+\beta}{2} + a^2\sin^2\frac{\alpha+\beta}{2}}}$

$$= \frac{ab\cos\frac{\alpha-\beta}{2}}{\sqrt{a^2 - \left(a^2-b^2\right)\cos^2\frac{\alpha+\beta}{2}}}$$

From (i), $d = \dfrac{ab\cos\frac{\alpha-\beta}{2}}{\sqrt{\left(a^2+b^2\right)\cos^2\frac{\alpha-\beta}{2}}} = \dfrac{ab}{\sqrt{\left(a^2+b^2\right)}}$

Hence PQ touches the circle having radius $\dfrac{ab}{\sqrt{\left(a^2+b^2\right)}}$ and center at the origin.

Q.14 Let length of major axis and eccentricity of one ellipse be 2a & e and that of second ellipse be 2a & e'.



Now $F_1F_3 + F_2F_3 = 2a$

Further if eq. of F_3F_4 is $y = x\tan\theta$, then F_3 & F_4 will be

$\left(ae'\cos\theta, ae'\sin\theta\right)$ & $\left(-ae'\cos\theta, -ae'\sin\theta\right)$

Now $\sqrt{\left(ae + ae'\cos\theta\right)^2 + \left(ae'\sin\theta\right)^2}$

$+\sqrt{\left(ae - ae'\cos\theta\right)^2 + \left(ae'\sin\theta\right)^2} = 2a$

$\Rightarrow \sqrt{a^2e^2 + 2a^2ee'\cos\theta + a^2e^{2}{}'} = 2a - \sqrt{a^2e^2 - 2a^2ee'\cos\theta + a^2e'^2}$

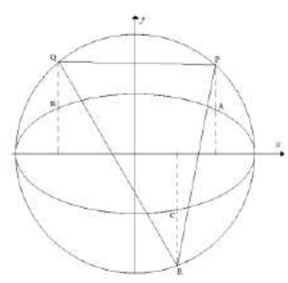
$\Rightarrow a - aee'\cos\theta = \sqrt{a^2e^2 - 2a^2ee'\cos\theta + a^2e^{2}{}'}$

$\Rightarrow 1 + \left(ee'\cos\theta\right)^2 = e^2 + e'^2$

$\Rightarrow \cos^2\theta = \dfrac{e^2 + e'^2 - 1}{e^2e'^2}$

Q.15 Let the vertices of the triangle be $A(\alpha), B(\beta)$ & $C(\gamma)$.

Consider the triangle formed by corresponding points on auxiliary circle as shown in the adjoining figure.



Now

$A_{ABC} = \dfrac{1}{2}\left\|\begin{matrix}1 & a\cos\alpha & b\sin\alpha \\ 1 & a\cos\beta & b\sin\beta \\ 1 & a\cos\gamma & b\sin\gamma\end{matrix}\right\|$ & $A_{PQR} = \dfrac{1}{2}\left\|\begin{matrix}1 & a\cos\alpha & a\sin\alpha \\ 1 & a\cos\beta & a\sin\beta \\ 1 & a\cos\gamma & a\sin\gamma\end{matrix}\right\|$

Hence $A_{ABC} = \frac{b}{a} A_{PQR}$.

Now Area of triangle PQR will be maximum if its equilateral

i.e. α, β, γ differ by $\frac{2\pi}{3}$.

Now centroid of triangle ABC is

$$\left(\frac{a\cos\alpha + a\cos\left(\alpha + \frac{2\pi}{3}\right) + a\cos\left(\alpha + \frac{4\pi}{3}\right)}{3}, \frac{b\sin\alpha + b\sin\left(\alpha + \frac{2\pi}{3}\right) + b\sin\left(\alpha + \frac{4\pi}{3}\right)}{3} \right) \equiv (0,0)$$

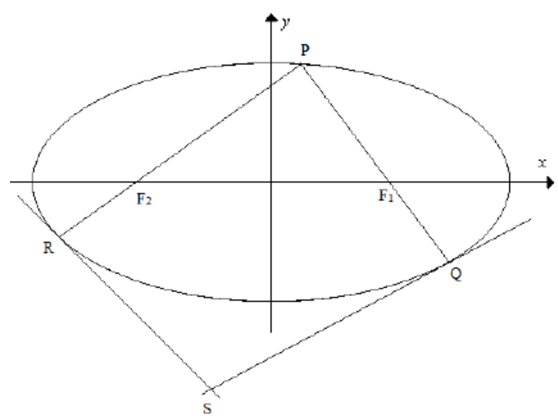
Q.16 Let S be (h, k) and P, Q & R be $(a\cos\theta, b\sin\theta)$, $(a\cos\alpha, b\sin\alpha)$ & $(a\cos\beta, b\sin\beta)$.

Now $h = a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$ & $k = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$

$\Rightarrow h = a\frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$ & $k = b\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$ …(i)

Also $\tan\frac{\alpha}{2}\tan\frac{\theta}{2} = \frac{e-1}{e+1}$ & $\tan\frac{\beta}{2}\tan\frac{\theta}{2}$

$\Rightarrow \tan\frac{\alpha}{2} = \left(\frac{e-1}{e+1}\right)^2 \tan\frac{\beta}{2}$ …(ii)



From (i) & (ii)

$$\frac{h}{a} = \frac{(e+1)^2 - (e-1)^2\tan^2\frac{\beta}{2}}{(e+1)^2 + (e-1)^2\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \tan^2\frac{\beta}{2} = \left(\frac{e+1}{e-1}\right)^2\left(\frac{a-h}{a+h}\right)$$

$$\& \ \frac{k}{b} = \frac{(e^2+1)2\tan\frac{\beta}{2}}{(e+1)^2 + (e-1)^2\tan^2\frac{\beta}{2}} \Rightarrow \frac{h^2}{a^2} + \frac{(1+e^2)^2}{(1-e^2)^2}\frac{k^2}{b^2} = 1$$

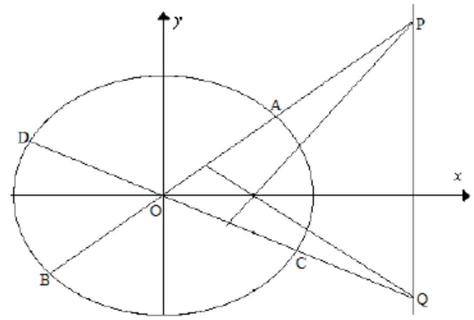
Q.17 Let AB & CD be $y = \frac{mb}{a}x$ & $y = -\frac{b}{ma}x$, then coordinates of P and Q will be

$\left(\frac{a}{e}, \frac{mb}{e}\right)$ & $\left(\frac{a}{e}, -\frac{b}{em}\right)$.

Altitude from P on OQ : $y-\frac{mb}{e}=\frac{ma}{b}\left(x-\frac{a}{e}\right)$

Altitude from O on PQ : y = 0

Hence orthocenter : $\left(\frac{a^2-b^2}{ae},0\right)$ i.e. $(ae,0)$



Q.18 Let three of the sides of quadrilateral be $\frac{x}{a}\cos\frac{\alpha_i+\alpha_{i+1}}{2}+\frac{y}{b}\sin\frac{\alpha_i+\alpha_{i+1}}{2}=\cos\frac{\alpha_i-\alpha_{i+1}}{2}$ for i = 1, 2, 3.

Let direction of three given sides be given by $m_i=-\frac{b}{a}\tan\frac{\alpha_i+\alpha_{i+1}}{2}$ for i = 1, 2, 3.

Clearly direction of fourth side will be uniquely defined as $m_4=-\frac{b}{a}\tan\frac{\alpha_4+\alpha_1}{2}$.

Q.19 Let the equations of AB & CD be $y=m_1x+c_1$ & $y=m_2x+c_2$.

Any curve passing through A, B, C & D will be

$b^2x^2+a^2y^2-a^2b^2+\lambda(m_1x-y+c_1)(m_2x-y+c_2)=0$

If this equation represents a circle, then

$b^2+\lambda m_1m_2=a^2+\lambda$ (coeff. Of x^2 = coeff. Of y^2) ...(i)

$m_1+m_2=0$ (coeff. Of xy = 0) ...(ii)

Clearly from (ii), AB & CD are equally inclined to coordinate axes.

Q.20 Let $P(\theta),Q\left(\frac{\pi}{2}+\theta\right)$ & $R(\pi+\theta),S\left(\frac{3\pi}{2}+\theta\right)$ represent the end points of diameters PQ & RS.

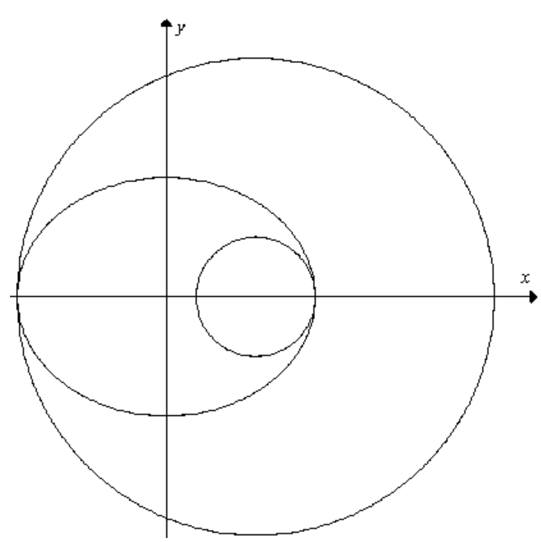
Clearly $PQ^2+RS^2=2(a^2+b^2)$.

Q.21 If any circle having center at (ae, 0) and radius r is touching the ellipse at a point $P(\theta)$, then normal $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ must pass through (ae, 0) and also distance of $(a\cos\theta,b\sin\theta)$ from (ae, 0) must be r. Hence

(i) $\frac{a^2e}{\cos\theta}-\frac{0}{\sin\theta}=a^2-b^2\Rightarrow\cos\theta=\frac{1}{e}$ (ii)

$a(1-e\cos\theta)=r$

From (i) & (ii) it is clear that no normal except major axis can pass through the focus as



e < 1 and hence 1/e > 1 but $\cos\theta \ngtr 1$.
The required circles must be touching the ellipse at end points of major axis.
Refer the adjoining figure.
$R_1 = a - ae$ & $R_2 = a + ae$.
$R_1 R_2 = a^2 - a^2e^2 = b^2$.

Q.22 Let coordinates of R be (h, k).

equation of chord of contact of $\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$ w.r.to R will be

$$\frac{hx}{a(a+b)} + \frac{ky}{b(a+b)} = 1 \ldots(i)$$

Equation of tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any point will be

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \ldots(ii)$$

Comparing (i) & (ii) gives $h = (a+b)\cos\theta, k = (a+b)\sin\theta$

Eliminating θ gives $h^2 + k^2 = (a+b)^2$.

Hence R lies on director circle of $\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$.

$\therefore \angle PRQ = \frac{\pi}{2}$.

Q.23 Feet of perpendiculars from the foci on any tangent lie on the auxiliary circle hence M, N lie on auxiliary circle.
Tangent to the ellipse at P i.e. MN will be chord of contact of Q(h, k) w.r.to the auxiliary circle.

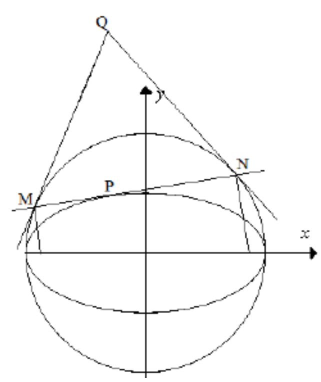
Tangent at P : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Chord of contact of Q : $hx + ky = a^2$.
Comparing the two equations gives

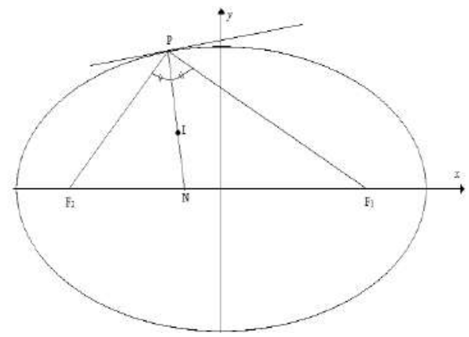
$$h = a\cos\theta, k = \frac{a^2}{b}\sin\theta.$$

Clearly P & Q have same x coordinate.
Further eliminating θ gives required locus as

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2}{b^2}.$$



Q.24 Let P be $(a\cos\theta, b\sin\theta)$.



Also F_1 & F_2 are $(ae, 0)$ & $(-ae, 0)$

$F_1F_2 = 2ae,\ PF_1 = a(1-e\cos\theta)$ & $PF_2 = a(1+e\cos\theta)$

Now in-center will be

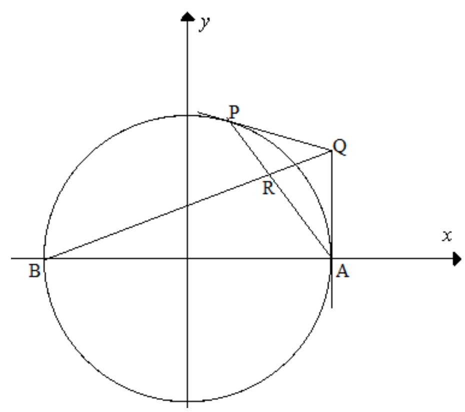
$$x = \frac{a(1+e\cos\theta)\times ae + a(1-e\cos\theta)\times(-ae) + 2ae\times a\cos\theta}{2a+2ae}$$

$$\& \ \ y = \frac{a(1+e\cos\theta)\times 0 + a(1-e\cos\theta)\times o + 2ae\times b\sin\theta}{2a+2ae}$$

$$\Rightarrow x = ae\cos\theta, y = \frac{be\sin\theta}{1+e} \ \text{ or } \ \cos\theta = \frac{x}{ae}, \sin\theta = \frac{y(1+e)}{be}$$

$$\Rightarrow \frac{x^2}{a^2e^2} + \frac{y^2(1+e)^2}{b^2e^2} = 1.$$

Q.25 Let the given circle be $x^2 + y^2 = a^2$ and A & B be along x – axis.



Also let P be $(a\cos\theta, a\sin\theta)$

Now tangent at A is x = a and tangent at P is $x\cos\theta + y\sin\theta = a$.

Coordinates of Q will be $\left(a, a\tan\frac{\theta}{2}\right)$

coordinates of B are (-a, 0).

Now equation of AP : $x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2} = a\cos\frac{\theta}{2}$

and equation of BQ : $x\sin\frac{\theta}{2} - 2y\cos\frac{\theta}{2} = -a\sin\frac{\theta}{2}$

Eliminate θ to get required locus as $x^2 + 2y^2 = a^2$.

Q.26 Let PQ & RS be any two mutually perpendicular diameter with eccentric angles of extremities being $P(\alpha), Q(\pi+\alpha), R(\beta)$ & $S(\pi+\beta)$.

Now slope of PQ = $\frac{b}{a}\tan\alpha$ & slope of RS = $\frac{b}{a}\tan\beta$

As $PQ \perp RS \ \therefore \ \tan\alpha\tan\beta = -\frac{a^2}{b^2}$.

Further $PQ = 2\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ & $RS = 2\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$

Hence $\frac{1}{PQ^2} + \frac{1}{RS^2} = \frac{1}{4(a^2\cos^2\alpha + b^2\sin^2\alpha)} + \frac{1}{4(a^2\cos^2\beta + b^2\sin^2\beta)}$

$$= \frac{1+\tan^2\alpha}{4(a^2 + b^2\tan^2\alpha)} + \frac{1+\tan^2\beta}{4(a^2 + b^2\tan^2\beta)}$$

$$\text{but } \tan^2\beta = \frac{a^4}{b^4\tan^2\alpha}$$

$$\Rightarrow \frac{1}{PQ^2} + \frac{1}{RS^2} = \frac{1+\tan^2\alpha}{4\left(a^2+b^2\tan^2\alpha\right)} + \frac{b^4\tan^2\alpha + a^4}{4a^2b^2\left(b^2\tan^2\alpha + a^2\right)}$$

$$= \frac{a^2b^2\left(1+\tan^2\alpha\right) + b^4\tan^2\alpha + a^4}{4a^2b^2\left(b^2\tan^2\alpha + a^2\right)}$$

$$= \frac{\left(a^2+b^2\tan^2\alpha\right)\left(a^2+b^2\right)}{4a^2b^2\left(b^2\tan^2\alpha + a^2\right)} = \frac{a^2+b^2}{4a^2b^2}$$

Q.27 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(\theta)$ will be $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

Foot of perpendicular from the origin on this tangent will be given by

$$\frac{x-0}{b\cos\theta} = \frac{y-0}{a\sin\theta} = \frac{ab}{b^2\cos^2\theta + a^2\sin^2\theta}$$

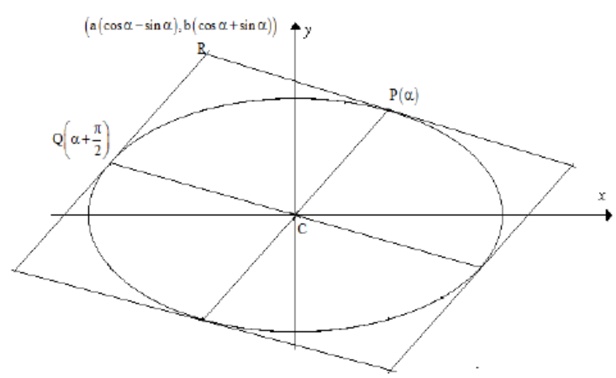
$$\Rightarrow \frac{x}{ab} = \frac{b\cos\theta}{b^2\cos^2\theta + a^2\sin^2\theta} \;\&\; \frac{y}{ab} = \frac{a\sin\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\Rightarrow \frac{x^2}{a^2b^2} + \frac{y}{a^2b^2} = \frac{1}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$\Rightarrow \frac{x}{ab^2\left(\frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}\right)} = \cos\theta \;\&\; \frac{y}{a^2b\left(\frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}\right)} = \sin\theta$$

$$\Rightarrow \frac{x^2}{a^2b^4} + \frac{y^2}{a^4b^2} = \left(\frac{x^2}{a^2b^2}+\frac{y}{a^2b^2}\right)^2.$$

Q.28 Given $P(\alpha) \,\&\, Q\left(\alpha+\frac{\pi}{2}\right)$



Now point of intersection of tangents

|at $P(\alpha) \,\&\, Q\left(\alpha+\frac{\pi}{2}\right)$:

$R\left(a(\cos\alpha - \sin\alpha), b(\cos\alpha + \sin\alpha)\right)$

Required area = $8 \times A_{CPR}$

$$= 8 \times \frac{1}{2} \left\| \begin{matrix} 1 & 0 & 0 \\ 1 & a\cos\alpha & b\sin\alpha \\ 1 & a(\cos\alpha - \sin\alpha) & b(\cos\alpha + \sin\alpha) \end{matrix} \right\|$$

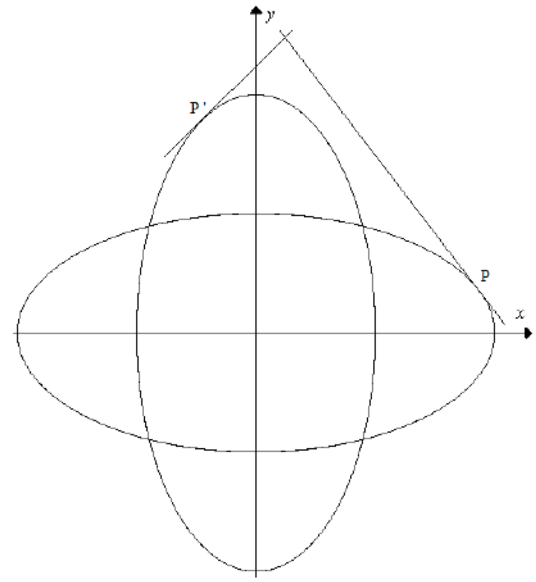
= 4ab

Q.29 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(\theta)$ will be

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \ldots(i)$

After rotation the ellipse will become

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$



Tangent to this ellipse at $P'\left(\frac{\pi}{2}+\theta\right)$ will be

$-\frac{x\sin\theta}{b}+\frac{y\cos\theta}{a}=1 \ldots(ii)$

From (i) & (ii) we get

$\cos\theta=\frac{a(x+y)}{x^2+y^2}$ & $\sin\theta=\frac{b(y-x)}{x^2+y^2}$

Eliminating q gives required locus as

$a^2(x+y)^2+b^2(x-y)^2=(x^2+y^2)^2.$

Q.30 Coordinates of P are $(a\cos\alpha, b\sin\alpha)$ and those of the focus S are (ae , 0)

Hence slope of SP, $\tan\beta=\frac{b\sin\alpha}{a\cos\alpha-ae}$.

$$\Rightarrow \frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{b\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}{a\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}-ae}$$

$$\Rightarrow \frac{\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}}{1-\left(\frac{1+e}{1-e}\right)\tan^2\frac{\alpha}{2}}$$

$$\Rightarrow \tan\frac{\beta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}.$$

Similar we can prove for the other focus.

Q.31 Let CP & CR be any two mutually perpendicular diameter with eccentric angles of extremities being $P(\alpha)\&R(\beta)$.

Now slope of CP = $\frac{b}{a}\tan\alpha$ & slope of CR = $\frac{b}{a}\tan\beta$

As $CP \perp CR \;\therefore\; \tan\alpha\tan\beta = -\frac{a^2}{b^2}$

$\Rightarrow b^2\sin\alpha\sin\beta + a^2\cos\alpha\cos\beta = 0.$

Further $d_1 = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ & $d_2 = \sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$

Hence $d_1d_2 = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$

$$\Rightarrow d_1d_2 = \frac{\sqrt{a^2 + b^2\tan^2\alpha}\sqrt{a^2 + b^2\tan^2\beta}}{\sqrt{1+\tan^2\alpha}\sqrt{1+\tan^2\beta}}$$

$$= \sqrt{a^4\cos^2\alpha\cos^2\beta + b^4\sin^2\alpha\sin^2\beta + a^2b^2\left(\cos^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta\right)}$$

$$= \sqrt{\begin{array}{l} a^4\cos^2\alpha\cos^2\beta + 2a^2b^2\cos\alpha\cos\beta\sin\alpha\sin\beta + b^4\sin^2\alpha\sin^2\beta \\ +a^2b^2\left(\cos^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta\right) - 2a^2b^2\cos\alpha\cos\beta\sin\alpha\sin\beta \end{array}}$$

$$= \sqrt{\left(a^2\cos\alpha\cos\beta + b^2\sin\alpha\sin\beta\right)^2 + a^2b^2\left(\cos\alpha\sin\beta - \sin\alpha\cos\beta\right)^2}.$$

Hence $d_1d_2 = ab\left|\sin(\alpha - \beta)\right|$.

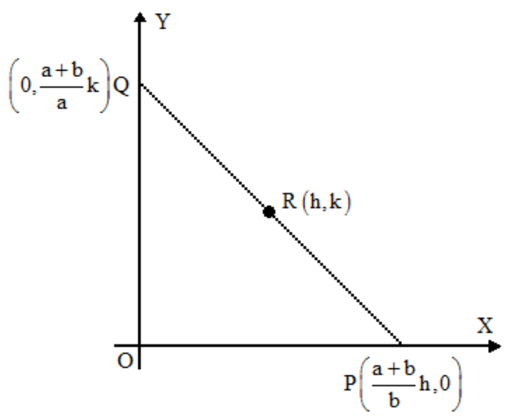
Q.32 From given information let

$\frac{PR}{QR} = \frac{a}{b}$ & $PQ = a + b$.

Hence $\sqrt{\left(\frac{a+b}{b}h\right)^2 + \left(\frac{a+b}{a}k\right)^2} = a + b$

$\Rightarrow \frac{h^2}{b^2} + \frac{k^2}{a^2} = 1.$

Required locus is an ellipse.



Q.33 Let $y = mx + \sqrt{a^2m^2 + b^2}$ &

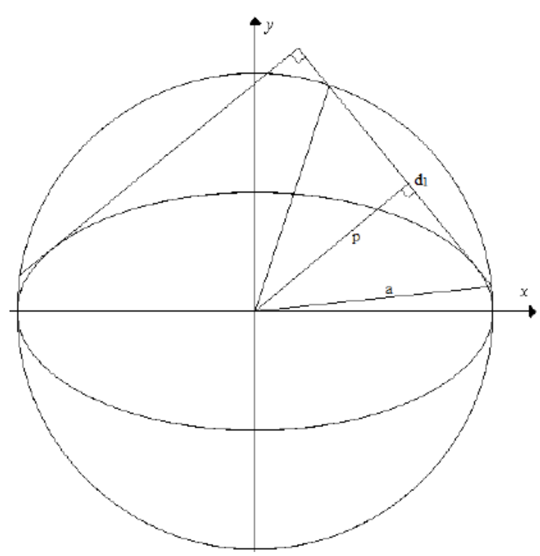
$my = -x + \sqrt{a^2 + b^2m^2}$ be two mutually perpendicular tangents drawn to the ellipse from any point.

Now if the auxiliary circle cuts off a chord of length d_1 on first tangent, then

$d_1^2 = 4\left(a^2 - p^2\right)$, where p is perpendicular distance of the chord from center of the auxiliary circle i.e. (0, 0).

$$\Rightarrow d_1^2 = 4\left(a^2 - \frac{a^2m^2 + b^2}{m^2 + 1}\right) = 4\left(\frac{a^2 - b^2}{m^2 + 1}\right)$$

Similarly for the other tangent



$$\Rightarrow d_2^{\,2} = 4\left(a^2 - \frac{a^2 + b^2m^2}{m^2+1}\right) = 4\left(\frac{(a^2-b^2)m^2}{m^2+1}\right)$$

Hence $\Rightarrow d_1^{\,2} + d_2^{\,2} = 4\left(\frac{a^2-b^2}{m^2+1}\right) + 4\left(\frac{(a^2-b^2)m^2}{m^2+1}\right) = (2ae)^2$.

Q.34 Let P be $(a\cos\alpha, b\sin\alpha)$, then M & N will be $(a\cos\alpha, 0)$ & $(0, b\sin\alpha)$.

Equation of MN will be $\frac{x}{a\cos\alpha} + \frac{y}{b\sin\alpha} = 1$

Comparing with $\frac{Ax}{\cos\alpha} + \frac{By}{\sin\alpha} = A^2 - B^2$ (normal to $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$) gives

$\frac{A^2-B^2}{A} = a$ & $\frac{A^2-B^2}{B} = b$

$\Rightarrow A = \frac{ab^2}{b^2-a^2}, B = \frac{a^2b}{b^2-a^2}$.

Hence MN is normal to the ellipse $\frac{x^2}{\left(\frac{ab^2}{b^2-a^2}\right)^2} + \frac{y^2}{\left(\frac{a^2b}{b^2-a^2}\right)^2} = 1$.

Q.35 Let 'd' be the length of referred diameter with one end point at $(a\cos\alpha, b\sin\alpha)$, then $d^2 = 4(a^2\cos^2\alpha + b^2\sin^2\alpha)$.

As given $d^2 = \frac{8a^2b^2}{a^2+b^2}$

$\Rightarrow a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{2a^2b^2}{a^2+b^2}$

$\Rightarrow a^4\cos^2\alpha + b^4\sin^2\alpha = a^2b^2(\cos^2\alpha + \sin^2\alpha)$

$\Rightarrow (b^2-a^2)b^2\sin^2\alpha = a^2(b^2-a^2)\cos^2\alpha$

$\Rightarrow \tan\alpha = \pm\frac{a}{b}$

Q.36 Let the common tangent be y = mx + c.

For $\frac{x^2}{9} + \frac{y^2}{4} = 1$: $c^2 = 9m^2 + 4$ …(i)

For $y^2 = 4x$: $c = \frac{1}{m}$ …(ii)

From (i) & (ii), $9m^2 + 4 = \frac{1}{m^2}$ or $9m^4 + 4m^2 - 1 = 0$

$$\Rightarrow m = \pm\sqrt{\frac{\sqrt{13}-2}{9}}$$

Hence common tangents are $\left(\sqrt{13}-2\right)x \pm 3\sqrt{\sqrt{13}-2}\,y = 9$.

Q.37 Let the end points P & Q of conjugate diameters be $P(\theta)$ & $Q\left(\frac{\pi}{2}+\theta\right)$.

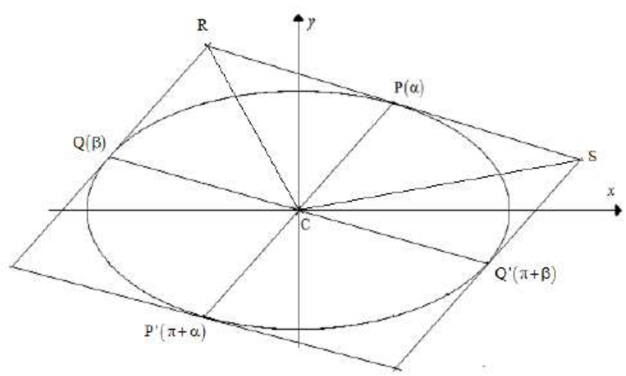
Now point of intersection of tangents at $P \,\&\, Q$:

$$R\left(a\frac{\cos\left(\theta+\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}}, b\frac{\sin\left(\theta+\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}}\right) \text{ i.e. } \frac{x}{a} = \cos\theta - \sin\theta, \frac{y}{b} = \cos\theta + \sin\theta$$

Eliminating θ (square and add) gives required locus as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

Q.38 A parallelogram(rectangle) having its vertices lying on an ellipse will be such that the points corresponding to its vertices on auxiliary circle will form a rhombus(square). As diagonals of a rhombus(square) are mutually perpendicular hence diagonals of the referred parallelogram(rectangle) will be conjugate diameters of the ellipse.

Q.39 Given $P(\alpha) \& Q(\beta)$



Now point of intersection of tangents at $P(\alpha) \& Q(\beta)$:

$$R\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

Also point of intersection of tangents at $P(\alpha) \& Q(\pi+\beta)$:

$$S\left(-a\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}, b\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\right)$$

Required area = $4\times\left(A_{CPR}+A_{CPS}\right)$

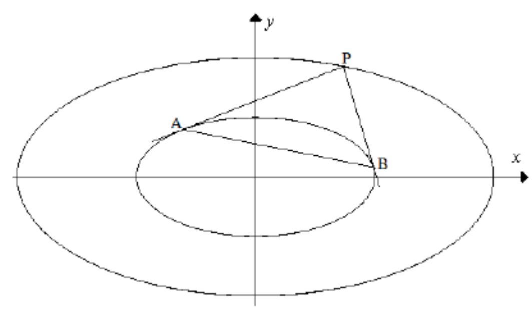
$$= 4\times\frac{1}{2}\left\|\begin{matrix}1 & 0 & 0\\ 1 & a\cos\alpha & b\sin\alpha\\ 1 & a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} & b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\end{matrix}\right\| + 4\times\frac{1}{2}\left\|\begin{matrix}1 & 0 & 0\\ 1 & a\cos\alpha & b\sin\alpha\\ 1 & -a\dfrac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}} & b\dfrac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\end{matrix}\right\|$$

$$= 4ab\left|\frac{\sin\frac{\alpha+\beta}{2}\cos\alpha - \sin\alpha\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right| + 4ab\left|\frac{\cos\frac{\alpha+\beta}{2}\cos\alpha + \sin\alpha\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\right|$$

$$= 4ab\left|\tan\frac{\alpha-\beta}{2}\right| + 4ab\left|\cot\frac{\alpha-\beta}{2}\right|$$

$$= 4ab\left|\operatorname{cosec}(\alpha-\beta)\right|.$$

Q.40 We have to find locus of centroid of triangle formed by tangents of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ and their respective chord of contact.



Let any point P on the later ellipse be $(2a\cos\theta, 2b\sin\theta)$

Also let A & B be $(a\cos\alpha, b\sin\alpha)$ & $(a\cos\beta, b\sin\beta)$

Now point of intersection of tangents at A & B is $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$, hence

$$2\cos\theta = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, 2\sin\theta = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \Rightarrow \sec^2\frac{\alpha-\beta}{2} = 4 \quad \ldots(i)$$

Now centroid of ΔPAB will be

$$G\left(\frac{a(\cos\alpha+\cos\beta+2\cos\theta)}{3}, \frac{b(\sin\alpha+\sin\beta+2\sin\theta)}{3}\right)$$

$$\Rightarrow G\left(\frac{2a\left(\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos\theta\right)}{3},\frac{2b\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin\theta\right)}{3}\right)$$

From (i) we get $G\left(\frac{2a\cos\theta\left(2\cos^2\frac{\alpha-\beta}{2}+1\right)}{3},\frac{2b\sin\theta\left(2\cos^2\frac{\alpha-\beta}{2}+1\right)}{3}\right)$

$\Rightarrow G(a\cos\theta, b\sin\theta)$

Clearly G lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Q.41 Point of intersection of tangents at $P(\alpha)\&Q(\beta)$: $R\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Equation of PQ : $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

If PQ passes through (ae, 0), then $e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$

Hence R becomes $\left(\frac{a}{e}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

Clearly R lies on the directrics.

Q.42 Normal at any point $(x_1\, y_1)$ to the ellipse $bx^2+a^2y^2=a^2b^2$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Now normal at $\left(ae, \frac{b^2}{a}\right)$ will be $\frac{a^2x}{ae}-\frac{b^2y}{\frac{b^2}{a}} = a^2-b^2$

$\Rightarrow \frac{a}{e}x - ay = a^2-b^2$

If it passes through (0 , b), then $-ab = a^2-b^2$

$\Rightarrow b^2-a^2=ab$

$\Rightarrow a^2(1-e^2)-a^2=a^2\sqrt{1-e^2} \Rightarrow e^4+e^2-1=0$.

Q.43 Let the common tangent be $y = mx + c$.

For being a tangent to the ellipse : $c^2 = a^2m^2 + b^2$

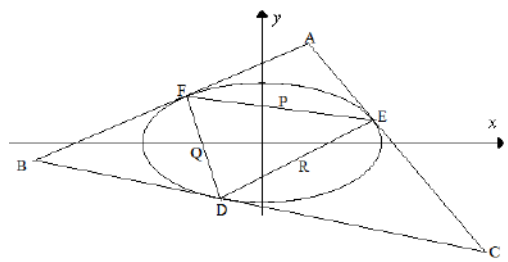
For being a tangent to the circle : $c^2 = r^2\left(m^2+1\right)$

Hence $a^2m^2 + b^2 = r^2\left(m^2+1\right)$

$\Rightarrow m = \pm\sqrt{\dfrac{r^2-b^2}{a^2-r^2}} = \tan\theta$

Clearly as b < r < a, hence there exists a value of m for every value of r.

Q.44 Let E & F be $\left(a\cos\alpha, b\sin\alpha\right)$ & $\left(a\cos\beta, b\sin\beta\right)$



Coordinates of A : $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Also Let P be (h, k)

As P is midpoint of EF hence

EF : $\dfrac{hx}{a^2} + \dfrac{ky}{b^2} = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} \ldots$(i)

But by equation of chord joining E & F

EF : $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2} \ldots$(ii)

Comparing (i) & (ii) gives $\dfrac{h}{a\cos\frac{\alpha+\beta}{2}} = \dfrac{k}{b\sin\frac{\alpha+\beta}{2}} = \dfrac{\frac{h^2}{a^2}+\frac{k^2}{b^2}}{\cos\frac{\alpha-\beta}{2}}$

$\Rightarrow \dfrac{h}{\frac{h^2}{a^2}+\frac{k^2}{b^2}} = \dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{k}{\frac{h^2}{a^2}+\frac{k^2}{b^2}} = \dfrac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$

$\Rightarrow \cos^2\dfrac{\alpha-\beta}{2} = \dfrac{b^2h^2+a^2k^2}{a^2b^2}$ & $\tan\dfrac{\alpha+\beta}{2} = \dfrac{ak}{bh}$ or $\cos^2\dfrac{\alpha+\beta}{2} = \dfrac{b^2h^2}{b^2h^2+a^2k^2}$

$\Rightarrow h = a\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}, k = b\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

Now eq. of AP : $ay\cos\dfrac{\alpha+\beta}{2} = bx\sin\dfrac{\alpha+\beta}{2}$

Clearly AP, BQ & CR will be concurrent at the origin.

Q.45 Given parabola is $y^2 = \left(4a\cos\alpha\right)x$.

Let the feet of normals be $P\left(t_1\right), Q\left(t_2\right)$ & $R\left(t_3\right)$.

As the normals are concurrent at a point (h, $b\sin\alpha$) lying on $y = b\sin\alpha$, Hence $t_1+t_2+t_3=0,\ t_1t_2+t_2t_3+t_3t_1=\dfrac{2a\cos\alpha-h}{a\cos\alpha}$ & $t_1t_2t_3=\dfrac{b\sin\alpha}{a\cos\alpha}$ ...(i)

Now point of intersection of tangents at P & Q : $A\left(a\cos\alpha\, t_1t_2, a\cos\alpha\left(t_1+t_2\right)\right)$

and point of intersection of tangents at Q & R : $B\left(a\cos\alpha\, t_2t_3, a\cos\alpha\left(t_2+t_3\right)\right)$

Slopes of respective altitudes : $-t_3$ & $-t_1$

Hence altitude from A : $y-a\cos\alpha\left(t_1+t_2\right)=-t_3\left(x-a\cos\alpha\, t_1t_2\right)$

and altitude from B : $y-a\cos\alpha\left(t_2+t_3\right)=-t_1\left(x-a\cos\alpha\, t_2t_3\right)$

Solving together gives the orthocenter as

$x=-a\cos\alpha$ & $y=\left(t_1+t_2+t_3+t_1t_2t_3\right)a\cos\alpha$

From (i), $x=-a\cos\alpha$ & $y=b\sin\alpha$.

Eliminating α gives required locus as $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

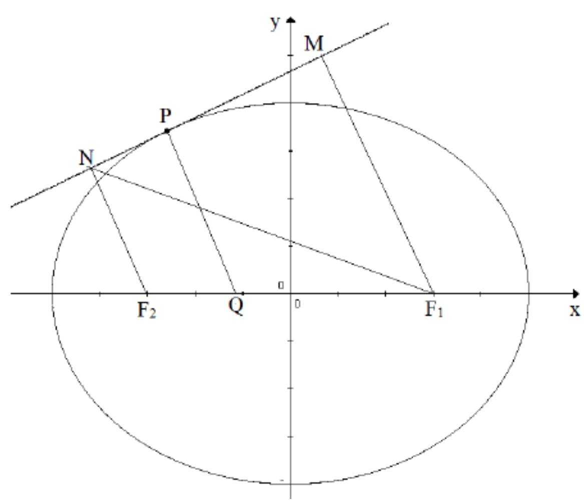
Q.46 Let P be $\left(a\cos\alpha, b\sin\alpha\right)$

Now tangent at P $\dfrac{x\cos\alpha}{a}+\dfrac{y\sin\alpha}{b}=1$ and

normal at P $\dfrac{ax}{\cos\alpha}-\dfrac{by}{\sin\alpha}=a^2-b^2$

Coordinates of Q : $\left(\dfrac{a^2-b^2}{a}\cos\alpha,\ 0\right)$

Midpoint of PQ : $R\left(\dfrac{a\left(e^2+1\right)\cos\alpha}{2},\dfrac{b\sin\alpha}{2}\right)$



Further foot of perpendicular on tangent at P from (–ae, 0) will be given by

$$\frac{x+ae}{b\cos\alpha}=\frac{y}{a\sin\alpha}=\frac{ab\left(e\cos\alpha+1\right)}{b^2\cos^2\alpha+a^2\sin^2\alpha}$$

But $b^2\cos^2\alpha+a^2\sin^2\alpha=a^2\left(1-e^2\cos^2\alpha\right)$, hence

$$\frac{x+ae}{b\cos\alpha}=\frac{y}{a\sin\alpha}=\frac{b}{a\left(1-e\cos\alpha\right)}$$

$$\Rightarrow N\left(\frac{a\left(\cos\alpha-e\right)}{\left(1-e\cos\alpha\right)},\frac{b\sin\alpha}{\left(1-e\cos\alpha\right)}\right)$$

Now prove that F_1, N & R are collinear.

Q.47 Any tangent to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with slope m : $y=mx+\sqrt{a^2m^2+b^2}$

Any tangent to $\dfrac{x^2}{a^2+k}+\dfrac{y^2}{b^2+k}=1$ with slope $-\dfrac{1}{m}$: $my+x=\sqrt{a^2+k+\left(b^2+k\right)m^2}$

From the two equations we get

$\left(x^2-a^2\right)m^2-2xym+y^2-b^2=0$ and

$\left(y^2-b^2-k\right)m^2+2xym+x^2-a^2-k=0$

Comparing the two quadratic equations for common values of m gives

$x^2-a^2=-y^2+b^2+k$ or $x^2+y^2=a^2+b^2+k$.

Q.48 Let P & Q be $(a\cos\alpha, b\sin\alpha)$ & $(-a\sin\alpha, b\cos\alpha)$

Circles on OP & OQ as diameters will be

$x(x-a\cos\alpha)+y(y-b\sin\alpha)=0$ & $x(x+a\sin\alpha)+y(y-b\cos\alpha)=0$

or $x^2+y^2=ax\cos\alpha+by\sin\alpha$ & $x^2+y^2=by\cos\alpha-ax\sin\alpha$

Square and add to eliminate α and get the required locus as

$2\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$.

Q.49 Given that center is at (1, 2) and focus is at (6, 2), hence major axis is along y = 2 and minor axis along x = 1.

Also ae = 5.

Equation of ellipse : $\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1$

As it passes through (4, 6) hence

$\frac{(4-1)^2}{a^2}+\frac{(6-2)^2}{b^2}=1 \Rightarrow \frac{9}{a^2}+\frac{16}{a^2\left(1-e^2\right)}=1$

$\Rightarrow \frac{9}{a^2}+\frac{16}{a^2-25}=1$

$\Rightarrow a^4-50a^2+225=0 \Rightarrow a^2=45$ & $b^2=20$.

Q. 50 Let P be $(a\cos\theta, a\sin\theta)$.

Tangent to the given circle at P : $x\cos\theta+y\sin\theta=a$

Tangent at A(a, 0) : x = a

Point of intersection of the two tangents : $T\left(a, \frac{a(1-\cos\theta)}{\sin\theta}\right)$

Now B is (–a, 0), hence

equation of BT : $y=\frac{1-\cos\theta}{2\sin\theta}(x+a)$ i.e. $x-2y\cot\frac{\theta}{2}+a=0 \ldots$(i)

Equation of AP : $y=\frac{\sin\theta}{\cos\theta-1}(x-a)$ i.e. $x+y\tan\frac{\theta}{2}-a=0 \ldots$(ii)

From (i) & (ii), eliminating $\tan\frac{\theta}{2}$ gives $\frac{x^2}{2a^2}+\frac{y^2}{a^2}=1$.

Now $e=\sqrt{\frac{2a^2-a^2}{2a^2}}=\frac{1}{\sqrt{2}}$.

Q.51 Tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point P(α) will be

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1.$$

Homogenizing the equation of the auxiliary circle using the equation of tangent gives

$$x^2+y^2=a^2\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}\right)^2 \quad \text{or}$$

$$\left(b^2\sin^2\alpha\right)x^2-\left(2ab\sin\alpha\cos\alpha\right)xy+\left(b^2-a^2\sin^2\alpha\right)y^2=0$$

As this pair of straight lines subtends a right angle at the origin hence

$$\text{coefficient of } x^2+\text{coefficient of } y^2=0$$

$$\Rightarrow b^2\sin^2\alpha+b^2-a^2\sin^2\alpha=0$$

$$\Rightarrow \frac{a^2-b^2}{b^2}=\frac{1}{\sin^2\alpha}$$

$$\Rightarrow \frac{a^2-b^2}{a^2}=\frac{1}{1+\sin^2\alpha}$$

$$\Rightarrow e=\frac{1}{\sqrt{1+\sin^2\alpha}}$$

Q.52 Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

As it passes through (– 3, 1) & (2, –2) hence

$$\frac{9}{a^2}+\frac{1}{b^2}=1 \;\&\; \frac{4}{a^2}+\frac{4}{b^2}=1$$

$$\Rightarrow a^2=\frac{32}{3} \;\&\; b^2=\frac{32}{5}.$$

Required ellipse is $3x^2+5y^2=32$.

Q.53 Let the two points on major axis be P(c, 0) & (-c, 0)

Further let equation of chord passing through (c, 0) be

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)|$$

$$\Rightarrow \frac{c}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+0=\cos\left(\frac{\alpha-\beta}{2}\right) \quad \text{Or} \quad \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{c}$$

Taking componendo and dividendo

$$\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}=\frac{a+c}{a-c} \quad \text{Or} \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}$$

Similarly for the chord passing through (-c, 0) we will get $\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=\frac{c+a}{c-a}$

Hence $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1$.

Q.54 Equation of normal at $P(\alpha)$: $2ax\sin\alpha-2by\cos\alpha=\left(a^2-b^2\right)\sin2\alpha$

Equation of normal at $Q(\beta)$: $2ax\sin\beta-2by\cos\beta=\left(a^2-b^2\right)\sin2\beta$

Equation of normal at $R(\gamma)$: $2ax\sin\gamma-2by\cos\gamma=\left(a^2-b^2\right)\sin2\gamma$

As the normals are concurrent, hence

$$\begin{vmatrix}2a\sin\alpha & -2b\cos\alpha & \left(a^2-b^2\right)\sin2\alpha\\ 2a\sin\beta & -2b\cos\beta & \left(a^2-b^2\right)\sin2\beta\\ 2a\sin\gamma & -2b\cos\gamma & \left(a^2-b^2\right)\sin2\gamma\end{vmatrix}=0 \text{ or}$$

$$-4ab\left(a^2-b^2\right)\begin{vmatrix}\sin\alpha & \cos\alpha & \sin2\alpha\\ \sin\beta & \cos\beta & \sin2\beta\\ \sin\gamma & \cos\gamma & \sin2\gamma\end{vmatrix}=0.$$

Q.55 The circle touching the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $\left(ae,\pm\frac{2b^2}{a}\right)$ will have its center on x – axis.

Now normals to the ellipse at $\left(ae,\pm\frac{2b^2}{a}\right)$ will be

$$\frac{x-ae}{ae}a^2=\frac{y-\frac{2b^2}{a}}{\frac{2b^2}{a}}b^2 \quad \& \quad \frac{x-ae}{ae}a^2=\frac{y+\frac{2b^2}{a}}{-\frac{2b^2}{a}}b^2$$

Solving these gives center of the circle as $\left(ae-\frac{b^2e}{a},0\right)$ or $\left(ae^3,0\right)$

Further radius = $\sqrt{\left(ae^3-ae\right)^2+\frac{b^4}{a^2}}=a\left(1-e^2\right)\sqrt{e^2+1}$.

Q.56 Let the required tangent be $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$

Now given $\frac{4}{\cos\theta}=\frac{3}{\sin\theta}$

Hence $\cos\theta = \pm\frac{4}{5}$ & $\sin\theta = \pm\frac{3}{5}$

Required lines are $x + y = \pm 5$.

Q.57 Let the point of intersection of tangents be (h, k), then the corresponding chord of contact will be $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$.

Homogenizing equation of ellipse using this equation of chord gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{hx}{a^2} + \frac{ky}{b^2}\right)^2$$

As this pair of lines subtends a right angle at the origin hence

Coefficient of x^2 + coefficient of y^2 = 0

$$\Rightarrow \frac{h^2 - a^2}{a^4} + \frac{k^2 - b^2}{b^4} = 0$$

Hence the required locus is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$.

Q.58 Tangents to $\frac{x^2}{18} + \frac{y^2}{32} = 1$ with slope $-\frac{4}{3}$ will be

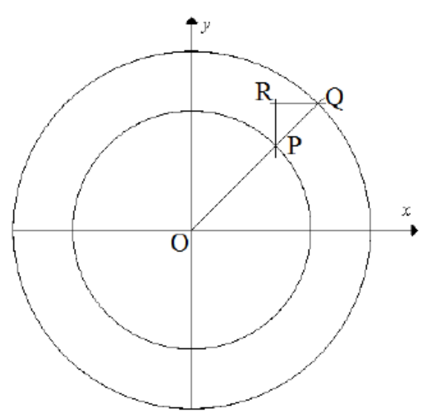
$$y = -\frac{4}{3}x \pm \sqrt{18 \times \frac{16}{9} + 32} \text{ i.e. } 4x + 3y = \pm 8.$$

Now legs of the triangle OAB will be 2 & $\frac{8}{3}$

hence area = $\frac{1}{2} \times 2 \times \frac{8}{3} = \frac{8}{3}$.

Q.59 Let P be $(a\cos\alpha, a\sin\alpha)$ & Q be $(b\cos\alpha, b\sin\alpha)$,

then R will be $(a\cos\alpha, b\sin\alpha)$



Eliminating a between x & y coordinates of R gives required locus as

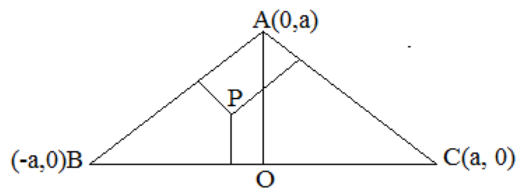
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Clearly locus of R is an ellipse touching the two circles {Touching inner circle at $(0, \pm a)$ and outer circle at $(\pm b, 0)$}

Also If the foci of this ellipse lie on the inner circle, then a = be.

But $e^2 = 1 - \frac{a^2}{b^2} \Rightarrow e = \frac{a}{b} = \frac{1}{\sqrt{2}}$.

Q.60 Let P be (h, k)



Distance of P from BC = |k|

Equations of AB & AC are

$x - y = -a$ & $x + y = a$.

Distances of P from AB & AC

$\frac{|h-k+a|}{\sqrt{2}}$ & $\frac{|h+k-a|}{\sqrt{2}}$.

Further for any point inside the triangle $x + y < a$ & $x - y + a > 0$.

As given $k^2 = \frac{(k-a)^2 - h^2}{4}$

Hence required locus is $x^2 + 3y^2 + 2ay - a^2 = 0$ or $\frac{x^2}{3} + \left(y + \frac{a}{3}\right)^2 = \frac{10a^2}{27}$

Clearly the locus is an ellipse passing through B & C and e = $\sqrt{\frac{2}{3}}$.

Q.61 Let midpoint of the chord be (h, k), them by T = S_1 equation of the chord will be

$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$.

As this chord is drawn from (a, -b) and (h, k) lies on x + y = b, hence

$\frac{ah}{a^2} + \frac{-b(b-h)}{b^2} = \frac{h^2}{a^2} + \frac{(b-h)^2}{b^2}$

$\Rightarrow (a^2+b^2)h^2 - (3a^2b + ab^2)h + 2a^2b^2 = 0$

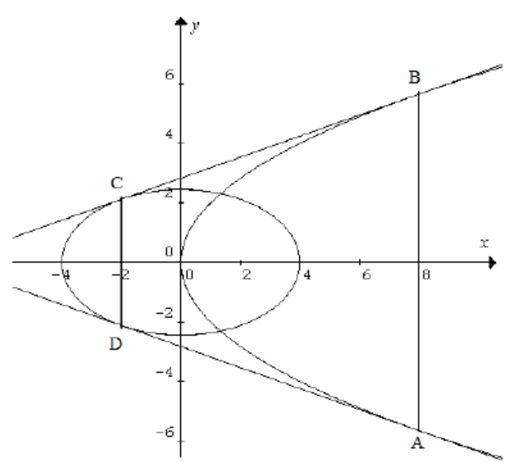
For two distinct values of h, discriminant > 0

$\Rightarrow (3a+b)^2 > 8(a^2+b^2)$

$\Rightarrow (a-7b)(a+b) > 0$

$\Rightarrow a > 7b$

Q.62 Let $y = mx + c$ be the common tangent.



For $\frac{x^2}{16} + \frac{y^2}{6} = 1$, $c^2 = 16m^2 + 6 \ldots$(i)

and for $y^2 = 4x$, $c = \frac{1}{m} \ldots$(ii)

From (i) & (ii) we get

$\frac{1}{m^2} = 16m^2 + 6$ or $16m^4 + 6m^2 - 1 = 0$

Hence $m = \pm\frac{1}{2\sqrt{2}}$.

Therefor the common tangents are

$x \pm 2\sqrt{2}y + 8 = 0$.

Now tangent to $y^2 = 4x$ at a point (h, k) will be $2x - ky + 2h = 0$.

Comparing this with the equations of common tangents gives A & B as $\left(8, \pm 2\sqrt{2}\right)$.

Similarly tangent to $\frac{x^2}{16}+\frac{y^2}{6}=1$ will be $3hx+8ky=48$.

Comparing this with the equations of common tangents gives C & D as $\left(-2, \pm \frac{3\sqrt{2}}{2}\right)$

Hence the quadrilateral ABCD is a trapezium as shown.

AB = $4\sqrt{2}$, CD = $3\sqrt{2}$ & distance between AB & CD = 10.

Required area = $\frac{1}{2}\times\left(4\sqrt{2}+3\sqrt{2}\right)\times 10 = 35\sqrt{2}$.

Q.63 Equation of normal at P(θ) : $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2e^2$

Coordinates of G : $\left(ae^2\cos\theta, 0\right)$

Coordinates of g : $\left(0, -\frac{a^2e^2}{b}\sin\theta\right)$

Hence $CG^2 = a^2e^4\cos^2\theta$ & $Cg^2 = \frac{a^4e^4}{b^2}\sin^2\theta$.

Now $a^2\left(CG^2\right)+b^2\left(Cg^2\right)=a^4e^4\cos^2\theta+a^4e^4\sin^2\theta$

$\Rightarrow a^2\left(CG^2\right)+b^2\left(Cg^2\right)=\left(a^2-b^2\right)^2$.

Further $CG = ae^2\cos\theta = e^2\left(a\cos\theta\right)=e^2\left(CN\right)$.

Q.64 Any point on a line of slope tan θ, passing through S(ae, 0) at a distance r from S will be $\left(ae+r\cos\theta, r\sin\theta\right)$.

These coordinates will satisfy equation of the ellipse for two value of |r|

i.e. $|r_1|$ = PA & $|r_2|$ = PB,

where A & B are points of intersection of this chord with the ellipse.

Substituting these coordinates in the equation of ellipse gives

$b^2\left(ae+r\cos\theta\right)^2+a^2\left(r\sin\theta\right)^2=a^2b^2$ or

$\left(a^2\sin^2\theta+b^2\cos^2\theta\right)r^2+\left(2ab^2e\cos\theta\right)r+a^2b^2\left(e^2-1\right)=0$

Now length of chord will be $|r_1| + |r_2|$ i.e. $|r_1 - r_2|$.

Hence length = $\sqrt{\left(r_1+r_2\right)^2-4r_1r_2}$

$$= \frac{2ab\sqrt{b^2e^2\cos^2\theta - (e^2-1)(a^2\sin^2\theta + b^2\cos^2\theta)}}{a^2\cos^2\theta + b^2\sin^2\theta}$$

$$= \frac{2ab\sqrt{b^2e^2\cos^2\theta - a^2(e^2-1)(1-e^2\cos^2\theta)}}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2ab^2}{a^2\cos^2\theta + b^2\sin^2\theta}.$$

Q.65 Let P be $(a\cos\theta, b\sin\theta)$, then equation of tangent will be

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Coordinates of T : $(a\sec\theta, 0)$

Coordinates of N : $(a\cos\theta, 0)$

Circle on TN as diameter : $(x - a\cos\theta)(x - a\sec\theta) + y^2 = 0$

or $x^2 + y^2 - a(\cos\theta + \sec\theta)x + a^2 = 0$

Now $g_1g_2 + f_1f_2 = a(\cos\theta + \sec\theta)\times 0 + 0\times 0 = 0$ & $\frac{c_1 + c_2}{2} = \frac{a^2 - a^2}{2} = 0$

Clearly the two circles are orthogonal**.**

Q.66 Let the tangents from $T(x_1, y_1)$ be TP & TQ and normals at these points be NP & NQ, N being (h, k).

But (x_1, y_1) lies on the director circle.

Also the tangents and normals will form a cyclic quadrilateral hence (h, k) will also lie on the director circle.

Further the director circle will be drawn on TN as diameter as $\angle TPN = \angle TQN = \frac{\pi}{2}$.

Hence $\frac{h + x_1}{2} = \frac{k + y_1}{2} = 0$ i.e. $\frac{h}{x_1} = \frac{k}{y_1}$.

Q.67 Let the moving point be (h, k).

The eq. of chord of contact will be $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$

As it is touching $x^2 + y^2 = c^2$, hence its distance from (0, 0) must be c.

$$\Rightarrow \frac{1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} = c \text{ or } \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{c^2}$$

Hence the required locus is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$.

Q.68 Let y = mx + c be the common tangent, then

for $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$, $c^2 = a_1^2m^2 + b_1^2$ or $c^2 - a_1^2m^2 = b_1^2$ ...(i) for $\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=1,\ c^2=a_2{}^2m^2+b_2{}^2$ or $c^2-a_2{}^2m^2=b_2{}^2$ …(ii) and

for $\frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=1,\ c^2=a_3{}^2m^2+b_3{}^2$ or $c^2-a_3{}^2m^2=b_3{}^2$ …(iii)

Now for (i), (ii) & (iii) to have a simultaneous solution in (c, m)

$$\begin{vmatrix} a_1^2 & b_1^2 & 1 \\ a_2^2 & b_2^2 & 1 \\ a_3^2 & b_3^2 & 1 \end{vmatrix}=0.$$

Q.69 Let coordinates of P be $(a\cos\theta, b\sin\theta)$

Now equation of normal at P : $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

The normal meets x – axis at G, hence

Coordinates of G : $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$

Also let coordinates of Q be (h, k), then P is midpoint of GQ

$$\Rightarrow a\cos\theta=\frac{\frac{a^2-b^2}{a}\cos\theta+h}{2}, b\sin\theta=\frac{0+k}{2}$$

$$\Rightarrow h=\left(\frac{a^2+b^2}{a}\right)\cos\theta, k=2b\sin\theta$$

Eliminating θ gives $\Rightarrow \frac{h^2}{\left(\frac{a^2+b^2}{a}\right)^2}+\frac{k^2}{4b^2}=1.$

Hence the required locus is the ellipse $\frac{x^2}{\left(\frac{a^2+b^2}{a}\right)^2}+\frac{y^2}{4b^2}=1.$

Clearly eccentricity is $\sqrt{\frac{\left(\frac{a^2+b^2}{a}\right)^2-4b^2}{\left(\frac{a^2+b^2}{a}\right)^2}}$ or $\frac{a^2-b^2}{a^2+b^2}$.

Now tangent to given ellipse at P : $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

and tangent to the locus at Q : $\frac{x\cos\theta}{\frac{a^2+b^2}{a}}+\frac{y\sin\theta}{2b}=1$

$$\Rightarrow \sin\theta = \frac{2b^3}{(b^2-a^2)y}, \ \cos\theta = \frac{a(a^2+b^2)}{(a^2-b^2)x}$$

Eliminating q gives the locus as $\dfrac{4b^6}{y^2} + \dfrac{a^2(a^2+b^2)^2}{x^2} = (a^2-b^2)^2$.

Q.70 Let P & Q be $(a\cos\theta, b\sin\theta)$ & $(a\cos\theta, a\sin\theta)$

Tangent to the ellipse at P : $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

Coordinates of T : $(a\sec\theta, 0)$

Equation of QT : $\dfrac{y}{x - a\sec\theta} = \dfrac{a\sin\theta}{a\cos\theta - a\sec\theta}$ or $x\cos\theta + y\sin\theta = a$

Clearly QT is tangent to the auxiliary circle.

Q.71 Let coordinates of P be $(a\cos\theta, b\sin\theta)$

Slope of PA : $\dfrac{b\sin\theta}{a(\cos\theta-1)}$

Slope of PA' : $\dfrac{b\sin\theta}{a(\cos\theta+1)}$

Equation of line perpendicular to PA, passing through P :

$$y - b\sin\theta = -\frac{a(\cos\theta-1)}{b\sin\theta}(x - a\cos\theta)$$

Point where it meets x – axis : $Q\left(\dfrac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta-1)}{a(\cos\theta-1)}, 0\right)$

Equation of line perpendicular to PA', passing through P :

$$y - b\sin\theta = -\frac{a(\cos\theta+1)}{b\sin\theta}(x - a\cos\theta)$$

Point where it meets x – axis : $R\left(\dfrac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta+1)}{a(\cos\theta+1)}, 0\right)$

Now $\ell(QR) = \dfrac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta+1)}{a(\cos\theta+1)} - \dfrac{b^2\sin^2\theta + a^2\cos\theta(\cos\theta-1)}{a(\cos\theta-1)}$

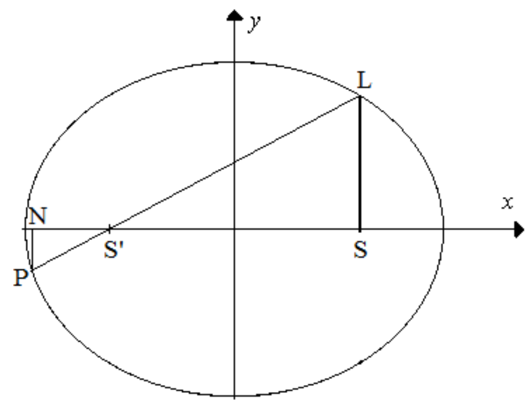
$$\Rightarrow \ell(QR) = \frac{b^2\sin^2\theta((\cos\theta-1)-(\cos\theta+1))}{a(\cos^2\theta-1)} \Rightarrow \ell(QR) = \frac{2b^2}{a}.$$

Q.72 Let eccentric angles of L & P be θ & α.

Now LP is a focal chord, hence $\tan\dfrac{\theta}{2}\tan\dfrac{\alpha}{2} = \dfrac{1+e}{1-e}$.

Comparing $\left(ae, \frac{b^2}{a}\right)$ with $(a\cos\theta, b\sin\theta)$ gives $\cos\theta = e$ & $\sin\theta = \frac{b}{a}$

$\Rightarrow \tan\frac{\theta}{2} = \frac{a(1-e)}{b} \Rightarrow \tan\frac{\alpha}{2} = \frac{b(1+e)}{a(1-e)^2}$, hence



$$\therefore PN = |b\sin\alpha| = \left|\frac{\frac{2b^2(1+e)}{a(1-e)^2}}{1+\frac{b^2(1+e)^2}{a^2(1-e)^4}}\right|$$

$$= \left|\frac{2ab^2(1-e)^2(1+e)}{a^2(1-e)^4 + b^2(1+e)^2}\right| = 2a\frac{(1-e)^2(1+e)^2}{(1-e)^3+(1+e)^3}$$

$$= a\frac{(1-e^2)^2}{1+3e^2}$$

Q.73 Let P be $(a\cos\theta, b\sin\theta)$

Tangent to the ellipse at P : $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Length of perpendicular on this tangent from S(ae, 0) will be

$$p = \frac{ab - abe\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Also $\ell(SP) = a(1-e\cos\theta)$.

$$\text{Now } \frac{b^2}{p^2} = \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2(1-e\cos\theta)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{b^2\cos^2\theta + a^2 - a^2\cos^2\theta}{a^2(1-e\cos\theta)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{a^2 - a^2e^2\cos^2\theta}{a^2(1-e\cos\theta)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{1+e\cos\theta}{1-e\cos\theta} = \frac{2a}{\ell(SP)} - 1.$$

Q.74 Let the coordinates of P, P', Q, Q' be

$(a\cos\alpha, b\sin\alpha), (-a\cos\alpha, -b\sin\alpha), (a\cos\alpha, a\sin\alpha)$ & $(-a\cos\alpha, -a\sin\alpha)$ respectively.

Now The quadrilateral formed by tangents at these points to the respective curves will be a parallelogram as tangents at the extremities of diameters are parallel.

Equation of tangent at P : $\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b} = 1 \ldots(i)$

Equation of tangent at P' : $\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=-1$ …(ii)

Equation of tangent at Q : $x\cos\alpha+y\sin\alpha=a$ …(iii)

Equation of tangent at Q' : $x\cos\alpha+y\sin\alpha=-a$ …(iv)

Point of intersection of (i) & (iii) : $\left(\frac{a}{\cos\alpha},0\right)$

Point of intersection of (i) & (iv) : $\left(\frac{a(a+b)}{(b-a)\cos\alpha},-\frac{2ab}{(b-a)\sin\alpha}\right)$

Point of intersection of (ii) & (iv) : $\left(-\frac{a}{\cos\alpha},0\right)$

Required area = $\left\|\begin{matrix}1 & \frac{a}{\cos\alpha} & 0\\ 1 & -\frac{a}{\cos\alpha} & 0\\ 1 & \frac{a(a+b)}{(b-a)\cos\alpha} & -\frac{2ab}{(b-a)\sin\alpha}\end{matrix}\right\|=\frac{4a^2b}{(a-b)\sin\alpha\cos\alpha}$.

Q.75 Normal at P(θ) : $\frac{\sqrt{14}x}{\cos\theta}-\frac{\sqrt{5}y}{\sin\theta}=9$.

As this normal passes through Q(2θ), hence $\frac{14\cos2\theta}{\cos\theta}-\frac{5\sin2\theta}{\sin\theta}=9$

$\Rightarrow 18\cos^2\theta-9\cos\theta-14=0\Rightarrow\cos\theta=-\frac{2}{3}$.

Q.76 Slope of normal at P(θ) : $-\frac{a}{b}\tan\theta$

As normal is inclined to x – axis at 45^o, hence $\frac{a}{b}\tan\theta=1$

$\Rightarrow \sin\theta=\frac{b}{\sqrt{a^2+b^2}},\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$.

Hence coordinates of P are $\left(\frac{a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$.

Now any point at a distance r on this normal will be $\left(\frac{a^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}},\frac{b^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}}\right)$

Substituting these coordinates in the equation of the ellipse gives

$$b^2\left(\frac{a^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}}\right)^2+a^2\left(\frac{b^2}{\sqrt{a^2+b^2}}-\frac{r}{\sqrt{2}}\right)^2=a^2b^2$$

$$\Rightarrow \frac{\left(a^2+b^2\right)r^2}{2}=2\sqrt{2}\frac{a^2b^2r}{\sqrt{a^2+b^2}}\Rightarrow r^2=\frac{32a^4b^4}{\left(a^2+b^2\right)^3}.$$

Q.77 Chord of contact of the tangents drawn from R(h, k) to the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ is

$\frac{hx}{c^2}+\frac{ky}{d^2}=1\ldots$(i)

Any tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1\ldots$(ii)

comparing (i) & (ii) gives $h=\frac{c^2\cos\theta}{a}$ & $k=\frac{d^2\sin\theta}{b}\ldots$(iii)

But the tangents from R are mutually perpendicular,

therefore $h^2+k^2=c^2+d^2\ldots$(iv)

From (iii) we get $\frac{a^2h^2}{c^4}+\frac{b^2k^2}{d^4}=1$

Comparing the above relation with that in (iv) gives

$\frac{a^2}{c^4}=\frac{b^2}{d^4}=\frac{1}{c^2+d^2}\Rightarrow\frac{a^2}{c^2}=\frac{c^2}{c^2+d^2}$ & $\frac{b^2}{d^2}=\frac{d^2}{c^2+d^2}$ or $\frac{a^2}{c^2}+\frac{b^2}{d^2}=1.$

Q.78 Let the tangents be drawn at $P(\alpha)$ & $Q(\beta)$,

then equations of the tangents will be

$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1\ldots$(i)

& $\frac{x\cos\beta}{a}+\frac{y\sin\beta}{b}=1\ldots$(ii)

and point of intersection of these tangents will be $P(x,y)\equiv\left(\frac{a\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}},\frac{b\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\right)$.

The tangents will meet the x – axis at $\left(\frac{a}{\cos\alpha},0\right)$ & $\left(\frac{a}{\cos\beta},0\right)$

Hence as given $\left|\frac{a}{\cos\alpha}-\frac{a}{\cos\beta}\right|=c$.

$$\Rightarrow\left|\frac{4\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}\right|=\frac{c}{a}\Rightarrow\left|\frac{2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}-\sin^2\frac{\alpha-\beta}{2}}\right|=\frac{c}{a}$$

Now for the point P we have $\frac{x}{a}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}, \frac{y}{b}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$.

Using $\sin\frac{\alpha-\beta}{2}=\frac{y}{b}\cos\frac{\alpha+\beta}{2}$, we get $\left|\tan\frac{\alpha+\beta}{2}\right|=\frac{c\left(b^2-y^2\right)}{2aby}$

Also $\frac{x}{a}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}, \frac{y}{b}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\tan^2\frac{\alpha+\beta}{2}$

Hence required locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{c^2\left(b^2-y^2\right)^2}{4a^2b^2y^2}$.

Q.79 Pair of tangents drawn from $\left(\frac{a^2}{\sqrt{a^2-b^2}}, \sqrt{a^2+b^2}\right)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2}{a^2-b^2}+\frac{a^2+b^2}{b^2}-1\right)=\left(\frac{1}{\sqrt{a^2-b^2}}x+\frac{\sqrt{a^2+b^2}}{b^2}y-1\right)^2 \quad \{\text{By } SS_1=T^2\}$$

Let these lines meet the ordinate through (ae, 0) at (ae, k_1) & (ae, k_2)

Substitute (ae, k) in the equation of pair of tangents

$$\left(\frac{a^2e^2}{a^2}+\frac{k^2}{b^2}-1\right)\left(\frac{a^2}{a^2-b^2}+\frac{a^2+b^2}{b^2}-1\right)=\left(\frac{ae}{\sqrt{a^2-b^2}}+\frac{k\sqrt{a^2+b^2}}{b^2}-1\right)^2$$

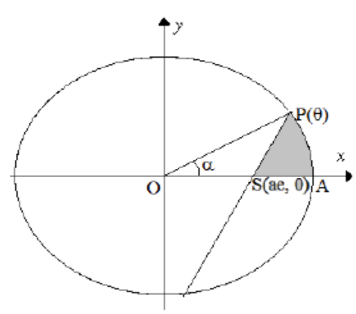
$$\Rightarrow \frac{a^4}{\left(a^2-b^2\right)}\left(e^2+\frac{k^2}{b^2}-1\right)=\frac{k^2\left(a^2+b^2\right)}{b^2}$$

$$\Rightarrow \left(\frac{a^4}{\left(a^2-b^2\right)b^2}-\frac{a^2+b^2}{b^2}\right)k^2=\frac{a^4}{a^2-b^2}-a^2 \Rightarrow k^2=a^2$$

Hence $k_1=a$ & $k_2=-a \Rightarrow k_1-k_2=2a$.

Q.80 Substitute x = r cos α & y = r sin α in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to get

polar equation of the ellipse as $r^2=\frac{a^2b^2}{b^2\cos^2\alpha+a^2\sin^2\alpha}$.



Area of sector OAP = $\frac{1}{2}\int_0^{\alpha}\frac{a^2b^2}{b^2\cos^2\alpha+a^2\sin^2\alpha}d\alpha$

$=\frac{ab}{2}\tan^{-1}\left(\frac{a\tan\alpha}{b}\right)$.

Also $\tan\alpha=\frac{b}{a}\tan\theta$, hence Area of sector OAP = $\frac{ab\theta}{2}$

Area of triangle OPS = $\frac{1}{2} \times ae \times b\sin\theta$.

The required area = area of sector OAP – area of triangle OPS

$= \frac{ab\theta}{2} - \frac{1}{2} \times ae \times b\sin\theta$.