LIMITS

EXERCISE 1(A)

1. $LHL = \lim_{x \to 1^-} x^2 = \lim_{h \to 0^+} (1-h)^2 = \lim_{h \to 0^+} \left(1 + h^2 - 2h\right) = 1$

$RHL = \lim_{x \to 1^+} x = \lim_{h \to 0^+} (1+h) = 1$

$\therefore LHL = RHL =$ a finite quantity.

Hence , $\lim_{x \to 1} f(x) = 1$

2. $LHL = \lim_{h \to 0^+} \frac{|2-h-2|}{(2-h)-2} = \frac{(-h)}{-h} = \frac{h}{-h} = -1$

$RHL = \lim_{h \to 0^+} \frac{|2+h-2|}{(2+h-2)} = \frac{|h|}{h} = \frac{h}{h} = 1$

$\therefore \lim_{x \to 2} \frac{|x-2|}{x-2} = 1$

3. $LHL = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \left(\frac{2}{5-x}\right) = \lim_{h \to 0^+} \frac{2}{5-(3-h)} = \lim_{h \to 0^+} \frac{2}{2+h} = 1$

$RHL = \lim_{x \to 3^+} f(x) = \lim_{h \to 0^+} \left[5-(3+h)\right] = 2$

4. $LHL = \lim_{h \to 0^+} 3(1-h) = 3$, $\qquad RHL = \lim_{h \to 0^+} 5 - 3(1+h) = 2$

5. $LHL = \lim_{h \to 0^+} \frac{|h|}{h} = -1 \qquad RHL = \lim_{h \to 0^+} \frac{|h|}{h} = 1$

6. $\lim_{x \to 1} \left(3x^2 + 4x + 5\right) = 3 + 4 + 5 = 12$

7. $\lim_{x \to 2} \frac{\left(3^{x/2} - 3\right)}{\left(3^{x/2} - 3\right)\left(3^{x/2} + 3\right)} = \lim_{x \to 2} \frac{1}{\left(3^{x/2} + 3\right)} = \frac{1}{6}$

8. $\lim_{x \to a} \frac{x^5 - 4^5}{x - a} = 5a^4$

9. $\lim_{h \to 0} \frac{1}{h}\left[\frac{x-(x+h)}{(x+h)x}\right] = \lim_{h \to 0} \frac{-1}{x(x+4)} = \frac{-1}{x^2}$

10. $\lim_{x \to 0} \frac{\left(\sqrt{1-x^2} - \sqrt{1+x^2}\right)\left(\sqrt{1-x^2} + \sqrt{1+x^2}\right)}{x^2\left(\sqrt{1-x^2} + \sqrt{1+x^2}\right)} = \lim_{x \to 0} \frac{\left(1-x^2\right) - \left(1+x^2\right)}{x^2\left(\sqrt{1-x^2} + \sqrt{1+x^2}\right)}$

$= \lim_{x \to 0} \frac{-2x^2}{2x^2} = -1$

11. $$\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{(x-2)-(4-x)}=\lim_{x\to 3}\frac{1}{2}(2)=1$$

12. $$\lim_{x\to\infty}\frac{x^2}{x^2}\frac{\left(a+\frac{b}{x}+\frac{c}{x^2}\right)}{\left(d+\frac{e}{x}+\frac{f}{x^2}\right)}=\frac{a}{d}$$

13. $$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\frac{\left(\sqrt{x+\sqrt{x+x}}+\sqrt{x}\right)}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}=\lim_{x\to\infty}\frac{\left(x+\sqrt{x+\sqrt{x}}-x\right)}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}\right)}$$

$$=\lim_{x\to\infty}\left(\frac{\sqrt{x}\left(1+\sqrt{\frac{1}{x}}\right)^{1/2}}{\sqrt{x}\left[\left(1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}\right)^{1/2}+1\right]}\right)=\frac{(1)^{1/2}}{(1)^{1/2}+1}=\frac{1}{2}$$

14. $$\lim_{x\to 1}(1+x)^{\frac{1}{x}}=2$$

$$\lim_{x\to\frac{\pi}{6}}\frac{\left(3\sin x-\sqrt{3}\cos x\right)}{6\left(x-\frac{\pi}{6}\right)}$$

$$=\frac{3}{6}\lim_{x\to\frac{\pi}{6}}\frac{\left(\sin x-\frac{1}{\sqrt{3}}\cos x\right)}{\left(x-\frac{\pi}{6}\right)}=\frac{1}{2}\lim$$

$$=\frac{2\sqrt{3}}{6}\lim_{x\to\frac{\pi}{6}}\frac{\left(\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right)}{\left(x-\frac{\pi}{6}\right)}$$

$$\lim_{x\to\frac{\pi}{6}}\left(\frac{1}{\sqrt{3}}\right)\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(x-\frac{\pi}{6}\right)}=\frac{1}{\sqrt{3}}$$

17. $$\lim_{x\to 3}\frac{\left(x^3-27\right)-\left(x^2-9\right)}{(x-3)}=\lim_{x\to 3}\frac{(x-3)}{(x-3)}\left[\left(x^2+3x+9\right)-(x+3)\right]=\lim_{x\to 3}\left(x^2+2x+6\right)=21$$

18. $$\lim_{x\to 2}\frac{(x-2)\left(x^2+2x+4\right)}{(x-2)(x+2)}=2$$

19. $\lim_{x\to 0}\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+2}+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{x\sqrt{1+x}-\sqrt{1-x}}{2x}=1$

$\lim_{x\to a}\frac{\left(\sqrt{a+2z}-\sqrt{3x}\right)\left(\sqrt{a+2x}+\sqrt{3x}\right)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{\sqrt{3a+x}-2\sqrt{x}\left(\sqrt{3a+x}+2\sqrt{x}\right)\left(\sqrt{a+2x}+\sqrt{3}x\right)}=\frac{(a-x)(a+2x)+\sqrt{3x}}{(a-x)\left(\sqrt{3a+x}+2\sqrt{x}\right)}$

20. $\lim_{x\to a}\frac{(a-x)}{3(a-x)}\frac{\left(\sqrt{2a+x}+2\sqrt{x}\right)}{\left(\sqrt{a+2x}+\sqrt{3x}\right)}=\frac{1}{3}\frac{4\sqrt{a}}{2\sqrt{3}\sqrt{a}}=\frac{2}{3\sqrt{3}}$

21. $\lim_{n\to\infty}\frac{\left(n^{49}+n^{98}+.......1^{99}\right)(n-1)}{n^{180}(n-1)}=\left(n^{100}-1\right)$

22. $\lim_{x\to 7}\frac{2-\sqrt{x-3}}{\left(x^2-49\right)}\times\frac{\left(2+\sqrt{x-3}\right)}{\left(2+\sqrt{x-3}\right)}=\lim_{x\to 7}=\frac{(7-x)}{(x-7)(x+7)\left(x+\sqrt{x+3}\right)}=\frac{-1}{6}$

23. $\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}=\lim_{n\to\infty}\frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6n^3}=\frac{1}{6}$

24. $\lim_{x\to 0}\left[\left(\frac{4^x-1}{x}\right)-\left(\frac{9^x-1}{x}\right)\right]\frac{1}{\left(4^x+9^x\right)}=\frac{1}{2}(\log 4-\log 9)=\log\left(\frac{2}{3}\right)$

25. $\lim_{x\to 3}\frac{2}{x-3}+\frac{x-3}{x+4}-\frac{2(2x+1)}{x^2+x-12}=\lim_{x\to 3}\frac{(2x+8-4x-2)}{\left(x^2+x-12\right)}=\lim_{x\to 3}\frac{-2(x-3)}{\left(x^2+x+2\right)}=\frac{-2}{7}$

26. $\lim_{x\to 0}\frac{\sin x+\log(1-x)}{x^2}=\lim_{x\to 0}\frac{x-\frac{x^3}{3!}-x-\frac{x^2}{2!}-\frac{x^3}{3!}}{x^2}=\frac{1}{2}$

27. $\lim_{x\to\infty}\left[\frac{x^3(1-a)+x^2(-b)+x(-a)+(a-b)}{x^2+1}\right]=2$

$\therefore 1-a=0\Rightarrow a=1$

$\Rightarrow\lim_{x\to\infty}\frac{-b+\left(\frac{-a}{x}\right)+\left(\frac{1-b}{x^2}\right)}{1+\frac{a}{x^2}}=2\Rightarrow -b=2$

28. $\lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\left(1+\frac{3}{x}\right)^{10}+........\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\left(\frac{10}{x}\right)^{10}\right]}=100$

29. $\lim_{x\to 0}\frac{\left(xe^x - x\right)}{x^2}+\left(\frac{x-\log\left(1+x\right)}{x^2}\right)=L_1+L_2$

$L_1=\lim_{x\to 0}\frac{e^x-1}{x}=1,\ L_2=\lim_{x\to 0}\frac{x-\log\left(1+x\right)}{x^2}$, let $\log\left(1+x\right)=t$

$\therefore L_2=\lim_{t\to 0}\frac{e^t-1-t}{\left(e^t-1\right)^2}=\lim_{t\to 0}\frac{e^t-t-1}{t^2}=\frac{1}{2}$

30. $\lim_{x\to 0}\left(\frac{\sin^{-1}x-x}{x^3}\right)+\lim_{x\to 0}\left(\frac{x-\tan^{-1}x}{x^3}\right)=L_1+L_2$

Let $\sin^{-1}x=t_1\Rightarrow L_1=\lim_{t\to 0}\frac{t_1-\sin t}{\sin^3 t}=\lim_{t_1\to 0}\frac{t_1-\sin t}{t_1^{\ 3}}=\frac{1}{6}$

Let $\tan^{-1}x=t_2\Rightarrow L_2=\lim_{t_2\to 0}\frac{\tan t_2-t_2}{\tan^3 t_2}=\lim_{t_2\to 0}\frac{\tan t_2-t_2}{t_2^2}=\frac{1}{3}$

31. $RHL=\lim_{h\to 0^+}\frac{\sqrt{1-\sin\left(\frac{\pi}{2}+h\right)}}{2\left(\frac{\pi}{2}-\frac{\pi}{2}+h\right)}=\lim_{h\to 0^+}\frac{\sqrt{1-\cos h}}{2h}=\frac{\sqrt{2}\left|\ell n\frac{h}{2}\right|}{4\left(\frac{h}{2}\right)}=\frac{1}{2\sqrt{2}}$

$LHL=\lim_{h\to 0^+}\frac{\sqrt{2}\left|\sin\frac{h}{2}\right|}{(-2h)}=\frac{-1}{2\sqrt{2}}$

32. $\lim_{x\to 0}\left[\frac{1-\cos 2x}{\left(2x\right)^2}\right]\left(\frac{\sin 5x}{5x}\right)\left(\frac{3x}{\sin 3x}\right)\times\left(\frac{4\times 5}{3}\right)=\frac{10}{3}$

33. $\lim_{x\to 0}\frac{\left(x^2\right)}{\sin\left(x^2\right)}\times x=1\times 0=0$

34. $\lim_{x\to 0}3\left(\frac{\sin 3x}{3x}\right)+\lim_{x\to 0}\frac{\sin x}{x}=4$

35. $\lim_{x\to 0}\frac{\left(1+x\right)^8-\left(1+8x\right)}{x^2}=\lim_{x\to 0}\frac{\left(1+8x+\frac{8.7}{2}x^2+\frac{8.7.6}{3!}x^3+...\right)-\left(1+3x\right)}{x^2}=28$

36. $LHL=\lim_{H\to 0^+}\frac{\sin(-h)}{(-h)}=1\qquad RHL=0$

37. $\lim_{x\to 0}8.\left(\frac{\sin 2x}{2x}\right)\left[\frac{1-\cos 2x}{\left(2x\right)^2}\right]\left(\frac{1}{\cos 2x}\right)=4$

38. $\lim_{x\to 2}f\left(x\right)=\lim_{x\to 2}\frac{\sin\left(e^{x-2}-1\right)}{\left(e^{x-2}-1\right)}\times\frac{\left(e^{x-2}-1\right)}{\left(x-2\right)}\frac{\left(x-2\right)}{\log\left[1+\left(x-2\right)\right]}=1$

39. $\lim_{x\to\frac{\pi}{2}} \frac{a^{\cos x}\left(a^{\cot x-\cos x}-1\right)}{\cot x-\cos x}=\ell n(a)$

40. $\lim_{x\to 0}\frac{f(x)}{x^2}=\lim_{x\to 0}\begin{vmatrix}\sin x & \cos^2 x & \tan x\\ x & 1 & 1/x\\ 2x & 1 & 1\end{vmatrix}=\lim_{x\to 0}\left(\frac{\begin{vmatrix}\sin x & \cos x & \tan x\\ x^2 & x & 1\\ 2x & 1 & 1\end{vmatrix}}{x}\right)$

41. $=\begin{vmatrix}\cos x & \cot x & \tan x\\ 2x & x & 1\\ 2x & 1 & 1\end{vmatrix}+\begin{vmatrix}\sin x & -\sin x & \tan x\\ x^2 & 1 & 1\\ 2x & 0 & 1\end{vmatrix}+\begin{vmatrix}\sin x & \cos x\\ x^2 & x\\ 2x & 1\end{vmatrix}$

42. $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)\times\frac{1}{(1+\tan x)}\times\left(\frac{1-\cos x}{x^2}\right)=1\times 1\times\frac{1}{2}=\frac{1}{2}$

43. $\lim_{x\to 0}\left[2(a-2)-\frac{\tan x}{x}\right]\frac{\sin 2x}{x}=0\Rightarrow 2[2a-4-1]=0\Rightarrow a=\frac{5}{2}$

44. $\lim_{h\to 0}\frac{\log_e\frac{(1+4h)}{(1+2h)^2}}{h^2}=\lim_{h\to 0}\frac{\ell n\left(1+\left(\frac{1+4h}{(1+2h)^2}-1\right)\right)}{\left(\frac{1+4h}{(1+2h)^2}-1\right)}\frac{\left(\frac{1+4h}{(1+2h)^2}-1\right)}{h^2}$

$\lim_{h\to 0}\frac{-4h^2}{h^2}=-4$

45. $\lim_{x\to a}\frac{\ell n(1+(x-a))}{(x-a)}=1$

46. $\lim_{h\to 0}\frac{\ell n(1+h)}{h}\times\log_{10}e=\log_{10}e$

47. $k=\lim_{x\to 0}\frac{\log\left(1+\frac{3+2x}{3-2x}-1\right)}{\left(\frac{4x}{3-2x}\right)}\times\frac{4x}{(3-2)x}=\frac{4}{3}$

48. $\lim_{x\to 0}\frac{(e^x-1)^2}{(x)^2}+\left(\frac{e^x-x-1}{x^2}\right)=1+\frac{1}{2}=\frac{3}{2}$

49. $\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.......\right)-(x+1)}{x^2}=\frac{1}{2}$

50. $\lim\limits_{x\to 0}\dfrac{\left(2^x-1\right)\left(\sqrt{1+x}+1\right)}{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}=\lim\limits_{x\to 0}2\dfrac{\left(2^x-1\right)}{x}=\ell n4$

51. $e\lim\limits_{x\to 0}\left(x+3\right)\left(\dfrac{x+4}{x+1}-1\right)=e^{\lim\limits_{x\to\infty}3}=e^3$

52. $e\lim\limits_{x\to 0}\left(c+dx\right)\left(\dfrac{1}{a+bx}\right)=e^{\frac{d}{b}}$

53. $\lim\limits_{x\to 0}\left(2x\right)^{3x}=1$

54. $\lim\limits_{x\to 0}\left(1+1+\dfrac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2}x^2+......\right)+\left(\dfrac{x}{2}-1\right)\left(1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+.....\right)$

55. $e^{\lim\limits_{m\to\infty}}m\left(\cos\dfrac{x}{m}-1\right)=e^{\lim\limits_{m\to\infty}}\dfrac{\left(\cos\frac{x}{m}-1\right)}{\left(\frac{x^2}{m^2}\right)}\times\left(\dfrac{x^2}{m^2}\right)\times m=1$

56. $e^{\lim\limits_{n\to\infty}\ n\left(n-1\right)}\dfrac{\left(2\right)}{\left(n^2-n-1\right)}=e^2$

57. $\lim\limits_{x\to 0}\left(\dfrac{1-\cos mx}{m^2x^2}\right)\times\left(\dfrac{n^2c^2}{1-\cos nx}\right)\times\dfrac{m^2}{n^2}$

$=\dfrac{m^2}{b^2}$

58. $\lim\limits_{x\to\frac{\pi}{8}}\dfrac{\sin\left(2x-\frac{\pi}{4}\right)}{2\left(x-\frac{\pi}{8}\right)}\left(\sqrt{2}\right)=2\sqrt{2}$

59. $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x-a}=\lim\limits_{x\to a}x^2+xa+a^2=3a^2$

60. $\lim\limits_{h\to 0}\dfrac{2\left\{\left[1+\frac{\sin h}{4}\right]^{1/2}-1\right\}}{h}=li2\dfrac{\left[\frac{1}{8}\sin h+........\right]}{h}=$

61. $\lim\limits_{\alpha\to\beta}\dfrac{\sin\left(\alpha+\beta\right)}{\left(\alpha+\beta\right)}\dfrac{\sin\left(\alpha-\beta\right)}{\left(\alpha-\beta\right)}=\dfrac{\sin\left(2\beta\right)}{\left(2\beta\right)}$

62. $\lim_{x\to 0}\left(\frac{\tan 4x}{x}-2\right)\Big/\left(6-\frac{\sin 3x}{x}\right)=(4-2)/(6-3)=2/3$

63. $\lim_{x\to 1}\frac{\left(5-\sqrt{26-x^2}\right)\left(5+\sqrt{26-x^2}\right)}{\left(5+\sqrt{26-x^2}\right)(x-1)}=\lim_{x\to 1}\frac{\left(x^2-1\right)}{(x-1)}\frac{1}{10}=\frac{1}{5}$

64. $\lim_{x\to -\infty}\frac{|x|\left(1-\frac{1}{x^2}+\sqrt{1-\frac{2}{x^2}}\right)}{x\left(1+\frac{1}{x}\right)}=(-1)\times 2=-2$

65. $\lim_{x\to -\infty}\frac{\sqrt{x^2-1}+\sqrt{x^2-2}}{x+1}=-\lim_{y\to 0^-}\frac{\sqrt{1-y^2}+\sqrt{1-2y^2}}{1+y}=2$

LIMITS
EXERCISE 1(B)

1 If $f(x) = \begin{cases} x^2 + 2, x \geq 1 \\ 2x + 1, x < 1 \end{cases}$, then $\underset{x \to 1}{\text{Lim}}$ f(x) equals -

(A) 1 (B) 2
(C) 3 (D) Does not exist

Sol. $\lim_{x \to 1-0} f(x) \lim_{h \to 0} = [2(1-h)+1] = 3$

$\lim_{x \to 1+0} f(x) = \lim_{h \to 0} [(1+h)^2 + 2] = 3$

$\therefore$ LHL = RHL, so $\lim_{x \to 1} f(x) = 3$. **Ans.[C]**

2 $\lim_{x \to 0} \frac{1 + e^{-1/x}}{1 - e^{-1/x}}$ is equal to -

(A) 1 (B) –1
(C) 0 (D) Does not exist

Sol. LHL = $\lim_{h \to 0} \frac{1 + e^{1/h}}{1 - e^{1/h}}$

$= \lim_{h \to 0} = \frac{e^{-1/h} + 1}{e^{-1/h} - 1} -- 1$

RHL = $\lim_{h \to 0} = \frac{1 + e^{-1/h}}{1 - e^{-1/h}} = \frac{1+0}{1-0} = 1$ **Ans.[D]**

LHL $\neq$ RHL, so given limit does not exist.

3 If $f(x) = \begin{cases} x - 1, x < 0 \\ 1/4, x = 0 \\ x^2, x > 0 \end{cases}$ then $\lim_{x \to 0}$ f(x) equals -

(A) 0 (B) 1
(C) –1 (D) Does not exist

Sol. Here $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$

and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x-1) = -1$

$\because \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\therefore \lim_{x \to 0} f(x)$ does not exist. **Ans.[D]**

4 $\lim_{x \to 3} \frac{x - 3}{|x - 3|}$, is equal to -

(A) 1 (B) –1
(C) 0 (D) Does not exist

Sol. LHL = $\lim_{h \to 0} \frac{(3-h) - 3}{|(3-h) - 3|}$

$= \lim_{h \to 0} \frac{-h}{|-h|} = -1$

$$\text{RHL} = \lim_{h\to 0} \frac{(3+h)-3}{|(3+h)-3|}$$

$$= \lim_{h\to 0} \frac{h}{|h|} = 1$$

LHL $\neq$ RHL, so limit does not exist. **Ans.[D]**

5 $\lim_{x\to\infty} \frac{2x^2+3x}{3x^2+4}$ equals -

(A) 1/2 (B) 2/3

(C) 3/4 (D) 0

Sol. $= \lim_{x\to\infty} \frac{2+(3/x)}{3+(4/x^2)} = \frac{2}{3}$ **Ans.[B]**

6 $\lim_{x\to\infty} \left(\sqrt{x^2+1}-x\right)$ equals -

(A) –1 (B) 0

(C) 1 (D) None of these

Sol. Limit $= \lim_{x\to\infty} x\left[\left(1+\frac{1}{x^2}\right)^{1/2}-1\right]$

$$= \lim_{x\to\infty} x\left[1+\frac{1}{2x^2}-\frac{1}{8x^4}+\ldots-1\right]$$

$$= \lim_{x\to\infty} \left[\frac{1}{2x}-\frac{1}{8x^3}+\ldots\right] = 0.$$ **Ans.[B]**

7 If $f(x) = \sqrt{\frac{x-\sin x}{x+\cos^2 x}}$, then $\lim_{x\to\infty} f(x)$ equals -

(A) 0 (B) ∞

(C) 1 (D) None of these

Sol. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{\frac{\{1-(\sin x/x)\}}{\{1+(\cos^2 x/x)\}}}$

$$= \sqrt{\frac{1-0}{1+0}} = 1.$$ **Ans.[C]**

8 $\lim_{x\to a} \left[\frac{x^2-(a+1)x+a}{x^3-a^3}\right]$ is equal to -

(A) $\frac{a-1}{3a^2}$ (B) a – 1

(C) a (D) 0

Sol. $\lim_{x\to a} \left[\frac{x^2-(a+1)x+a}{x^3-a^3}\right]$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to a} = \frac{2x-a-1}{3x^2} = \frac{a-1}{3a^2}$$

(D.L.Hospital rule) **Ans.[A]**

9 $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}}$ is equal to -

(A) 1/2 (B) 2

(C) 1 (D) 0

Sol. Limit = $\lim_{x\to 0} \frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)}$

$= \lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{2} = 1.$ **Ans.[C]**

10 $\lim_{x\to 0} \frac{\tan 2x - x}{3x - \sin x}$ equals-

(A) 2/3 (B) 1/3

(C) 1/2 (D) 0

Sol. The given limit is in the form , therefore applying L 'Hospital's rule, we get

Limit $= \lim_{x\to 0} \frac{2\sec^2 2x - 1}{3 - \cos x} = \frac{2-1}{3-1} = \frac{1}{2}$ **Ans.[C]**

11 $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$ is equal to -

(A) e^3 (B) $e^{1/3}$ (C) 1 1 (D) e

Sol. Limit = $\lim_{x\to 0} \left(\frac{x + x^3/3 + ...}{x}\right)^{1/x^2}$

$= \lim_{x\to 0} \left(1 + \frac{x^3}{3}\right)^{1/x^2}$

[$\because$ x $\to$ 0, so neglecting higher powers of x]

$= \lim_{x\to 0} \left[\left(1 + \frac{x^2}{3}\right)^{3/x^2}\right]^{1/3} = e^{1/3}$ **Ans.[B]**

12 $\lim_{x\to 0} \frac{\sin x^{o}}{x}$ is equal to -

(A) 1 (B) π (C) x (D) $\pi/180$

Sol. Limit = $\lim_{x\to 0} \frac{\sin(\pi/180)x}{x}$

$= \lim_{x\to 0} \frac{(\pi/180)\cos(\pi/180)x}{1}$

$= \frac{\pi}{180}$ **Ans.[D]**

13 $\lim_{x\to \infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$ equals -

(A) 0 (B) 1 (C) ∞ (D) -1

Sol. Let $y = \lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$

$= \lim_{x\to\infty} (\cot^{-1}x)^{1/x}$

$\therefore \log y = \lim_{x\to\infty} \frac{\log \cot^{-1} x}{x}$ $\left(\frac{\infty}{\infty}\text{form}\right)$

$= \lim_{x\to\infty} - \frac{1}{(1+x^2)\cos^{-1} x}$ (0x ∞ form)

$= - \lim_{x\to\infty} \frac{(1+x^2)^{-1}}{\cot^{-1} x}$ $\left(\frac{0}{0}\text{form}\right)$

$= - \lim_{x\to\infty} \frac{\frac{-2x}{(1+x^2)^2}}{\frac{-1}{1+x^2}}$ $= -2 \lim_{x\to\infty} \frac{x}{1+x^2}$

$= -2 \lim_{x\to\infty} \frac{1}{2x} = 0$ $\therefore y = e^0 = 1.$ **Ans.[B]**

14 If $G(x) = -\sqrt{25-x^2}$,

then $\lim_{x\to 1} \frac{G(x)-G(1)}{x-1}$ equals -

(A) 1/24 (B) 1/5

(C) $-\sqrt{24}$ (D) None of these

Sol. Here $G(1) = -\sqrt{25-x^2} = -\sqrt{24}$

$\therefore$ Given limit

$= \lim_{x\to 1} \frac{-\sqrt{25-x^2}+\sqrt{24}}{x-1}$ $\left(\frac{0}{0}\text{form}\right)$

$= \lim_{x\to 1} \frac{x}{\sqrt{25-x^2}}$ (By L Hospital ruel)

$= \frac{1}{\sqrt{24}}$ **Ans.[D]**

15 If f(9) = 9 and f'(9) = 4, then $\lim_{x\to 9} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ is equal to -

(A) 1 (B) 3 (C) 4 (D) 9

Sol. Given limit is in 0/0 form, so using Hospital rule, we get

$\text{Limit} = \lim_{x\to 9} \frac{\frac{1}{2\sqrt{f(x)}}.f(x)}{\frac{1}{2\sqrt{x}}}$

$= \frac{f(9).\sqrt{9}}{\sqrt{f(9)}} = \frac{4.3}{3} = 4$ **Ans.[C]**

16 By L'hospital's rule

$\lim_{x\to 1} \frac{f(1)\,g(x) - f(1) - g(1)\,f(x) + g(1)}{g(x) - f(x)} = \lim_{x\to 1} \frac{f(1)\,g'(x) - g(1)\,f'(x)}{g'(x) - f'(x)} = k = 4$ **Ans.[A]**

17 By L'hospital's rule

$$\lim_{x\to 0}\left(\frac{\int_0^{2x^2}\sec^2 2t\,dt}{x\sin x}\right)=\lim_{x\to 0}\left(\frac{4x\sec^2 4x^2}{x\cos x+\sin x}\right)=\lim_{x\to 0}\left(\frac{4\sec^2 4x^2}{\cos x+\frac{\sin x}{x}}\right)=2$$ **Ans.[B]**

18 $$\lim_{x\to 0}\frac{\sin^{-1}\left(3x-4x^3\right)}{\ln(1+2x)}=\lim_{x\to 0}\frac{3\sin^{-1}x}{\ln(1+2x)}=\lim_{x\to 0}\frac{3\frac{\sin^{-1}x}{x}}{2\frac{\ln(1+2x)}{2x}}=\frac{3}{2}.$$ **Ans.[C]**

19 $$\lim_{x\to 2}\frac{\left(x^2+5\right)^{\frac{1}{2}}-\left(x^3+1\right)^{\frac{1}{2}}}{x^2-4}=\lim_{x\to 2}\frac{\left(x^2+5\right)-\left(x^3+1\right)}{\left(x^2-4\right)\left(\left(x^2+5\right)^{\frac{1}{2}}+\left(x^3+1\right)^{\frac{1}{2}}\right)}$$

$$=\lim_{x\to 2}\frac{-(x-2)\left(x^2+x+2\right)}{(x-2)(x+2)\left(\left(x^2+5\right)^{\frac{1}{2}}+\left(x^3+1\right)^{\frac{1}{2}}\right)}=\lim_{x\to 2}\frac{-\left(x^2+x+2\right)}{(x+2)\left(\left(x^2+5\right)^{\frac{1}{2}}+\left(x^3+1\right)^{\frac{1}{2}}\right)}=-\frac{1}{3}$$ **Ans.[C]**

20 $$\lim_{x\to 0}\frac{\tan 2x-\sin x}{x}=2\lim_{x\to 0}\frac{\tan 2x}{2x}-\lim_{x\to 0}\frac{\sin x}{x}=1$$ **Ans.[A]**

21 $$\lim_{x\to 0}\frac{\sin 4x-\sin 2x+\sin x}{x\cot x+1}=\lim_{x\to 0}\frac{\sin 4x-\sin 2x+\sin x}{\left(\frac{x}{\tan x}+1\right)}=\frac{0}{2}=0$$

22 $$\lim_{x\to 0}\frac{\ln(1+\sin x)}{e^{\tan 2x}-1}=\lim_{x\to 0}\frac{\ln(1+\sin x)}{\sin 2x}\times\frac{\tan 2x}{e^{\tan 2x}-1}\times\frac{\frac{\sin 2x}{2x}}{\frac{\tan 2x}{2x}}=1$$ **Ans.[A]**

23 $$\lim_{x\to\frac{\pi}{4}}(1+\cos 2x)^{\frac{1}{1-\tan x}}=e^{\lim_{x\to\frac{\pi}{4}}\frac{\cos 2x}{1-\tan x}}$$

$$=e^{\lim_{x\to\frac{\pi}{4}}\frac{1-\tan^2 x}{1+\tan^2 x}\times\frac{1}{1-\tan x}}=e^{\lim_{x\to\frac{\pi}{4}}\frac{1+\tan x}{1+\tan^2 x}}=e.$$ **Ans.[A]**

24 $$\lim_{x\to 0}\frac{a\sin x+b\cos x+ce^x}{x^2}=\lim_{x\to 0}\frac{a\left(x-\frac{x^3}{6}\right)+b\left(1-\frac{x^2}{2}\right)+c\left(1+x+\frac{x^2}{2}\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{(b+c)+(a+c)x+(c-b)\frac{x^2}{2}-a\frac{x^3}{6}}{x^2}$$

$\Rightarrow b+c=0, a+c=0$ & $c-b=4$

$\Rightarrow a=b=-2, c=2.$

25 $$\lim_{x\to\infty}\left(\frac{ax+1}{ax+2}\right)^{2x}=e^{\lim_{x\to\infty}2x\left(\frac{ax+1}{ax+2}-1\right)}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{-2x}{ax+2}\right)} = e^{-\frac{2}{a}} = e^{\frac{1}{2}} \Rightarrow a = -4$$ **Ans.[C]**

26 In RHL $[x] = 0$ hence limit is not defined. **Ans.[D]**

27 $\lim\limits_{x\to 0^-}\left(1+[x]\right)^{\frac{1}{x}}$ not defined as $1+[x]=0$ & $\frac{1}{x}\to -\infty$ **Ans.[D]**

$\lim\limits_{x\to 0^+}\left(1+[x]\right)^{\frac{1}{x}} = 1$ as $[x]=0$

28 $$\lim_{x\to\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right) = \lim_{x\to\infty}\left(\frac{4x}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{4}{\sqrt{1+\frac{8}{x}+\frac{3}{x^2}}+\sqrt{1+\frac{4}{x}+\frac{3}{x^2}}}\right) = 2$$ **Ans.[C]**

29 $$\lim_{x\to 0}\frac{2x\left(3^x-1\right)}{1-\cos 2x} = \lim_{x\to 0}\frac{2x\left(3^x-1\right)}{2\sin^2 x} = \lim_{x\to 0}\frac{\left(\frac{3^x-1}{x}\right)}{\frac{\sin^2 x}{x^2}} = \ln 3$$ **Ans.[A]**

30 $$\lim_{n\to\infty}\frac{1^5+2^5+3^5+........+n^5}{n^6} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^5$$

$$= \int_0^1 x^5 dx = \frac{1}{6}.$$ **Ans.[D]**

31 $$\lim_{n\to\infty}\left[\frac{3}{1+n^3}+\frac{12}{8+n^3}+\frac{27}{27+n^3}+....+n \text{ terms}\right] = 3\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3+1}$$

$$= 3\int_0^1\frac{x^2}{1+x^3}dx = \int_1^2\frac{1}{t}dt \quad \{\text{by substitution } 1+x^3=t\}$$

= ln 2. **Ans.[B]**

32 $$\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{\sin^{-1}2x} = \lim_{x\to 0}\frac{2x}{\sin^{-1}2x}\times\frac{1}{\sqrt{4+x}+\sqrt{4-x}} = \frac{1}{4}.$$ **Ans.[D]**

33 $$\lim_{x\to\infty}\left(\frac{1^x+3^x+5^x+...+(2n-1)^x}{n}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to\infty}(2n-1)\left(\frac{\left(\frac{1}{2n-1}\right)^x+\left(\frac{3}{2n-1}\right)^x+\left(\frac{5}{2n-1}\right)^x+...+\left(\frac{2n-3}{2n-1}\right)^x+1}{n}\right)^{\frac{1}{x}}$$

$$= 2n - 1 \quad \left\{ \text{as all } \frac{1}{2n-1}, \frac{3}{2n-1}, \ldots, \frac{2n-3}{2n-1} < 1 \right\}$$ **Ans.[A]**

34 $$\lim_{x \to 0} \left(\frac{1^x + 2^x + 3^x + \ldots n^x}{n} \right)^{\frac{1}{x}} = e^{\lim_{x \to 0} \frac{1}{x} \left(\frac{1^x + 2^x + 3^x + \ldots n^x}{n} - 1 \right)}$$

$$= e^{\lim_{x \to 0} \frac{1}{n} \left(\frac{1^x - 1 + 2^x - 1 + 3^x - 1 + \ldots n^x - 1}{x} \right)} = e^{\frac{\ln 2 + \ln 3 + \ldots + \ln n}{n}} = (n!)^{\frac{1}{n}}$$ **Ans.[B]**

35 $$\lim_{x \to 0} \frac{8\left(2^x - 3^x\right)\tan x}{1 - \cos 4x} = \lim_{x \to 0} \frac{8\left(\frac{2^x - 1}{x} - \frac{3^x - 1}{x} \right) \frac{\tan x}{x}}{8 \frac{\sin^2 2x}{4x^2}}$$

$$= \ln 2 - \ln 3 = \ln \frac{2}{3}$$ **Ans.[D]**

LIMITS
EXERCISE 1(C)

1. $\lim\limits_{x\to 1}\dfrac{x^{\frac{1}{13}}-x^{\frac{1}{7}}}{x^{\frac{1}{5}}-x^{\frac{1}{3}}}$ $\left(\dfrac{0}{0}\text{form}\right)$

Apply L'Hospital rule

$$\lim_{x\to 1}\frac{\frac{1}{13}x^{-1+\frac{1}{13}}-\frac{1}{7}x^{-1+\frac{1}{7}}}{\frac{1}{5}x^{\frac{1}{5}-1}-\frac{1}{3}x^{\frac{1}{3}-1}}$$

$$\lim_{x\to 1}\frac{\frac{1}{13}-\frac{1}{7}}{\frac{1}{5}-\frac{1}{3}}=\frac{45}{91}$$ **Ans.**

2. $\lim\limits_{x\to 1}\dfrac{\sum_{k=1}^{100}x^k-100}{x-1}$

$\Rightarrow \lim\limits_{x\to 1}\dfrac{x+1}{x-1}+\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}+\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}.....\lim\limits_{x\to 1}\dfrac{x^{100}-1}{x-1}$

Applying L'Hospital rule

$\Rightarrow \lim\limits_{x\to 1}1+\lim\limits_{x\to 1}2x+\lim\limits_{x\to 1}3x^2+\lim\limits_{x\to 1}4x^3+.........\lim\limits_{x\to 1}100x^9$

= 1 + 2 + 3 + 4...........100

$= \dfrac{100(100+1)}{2} = 50\times 101\left(1+2+3...........n=\dfrac{n(n+1)}{2}\right)$

= 5050

3. $\lim\limits_{x\to 1}\left(\dfrac{p}{1-x^p}-\dfrac{q}{1-x^2}\right)$ $(\infty-\infty\text{ form})$

$$\lim_{x\to 1}\left(\frac{p(1-x^q)-q(1-x^p)}{(1-x^p)\,(1-x^q)}\right)$$

$\lim\limits_{x\to 1}\left(\dfrac{p-px^2-q+qx^p}{1-x^q-x^p+x^{p+q}}\right)$ $\left(\dfrac{0}{0}\text{form}\right)$

Apply L'Hopital rule

$$\lim_{x\to 1}\left(\frac{-pqx^{q-1}+qpx^{p-1}}{-qx^{q-1}-px^{p-1}+(p+q)x^{(p+q-1)}}\right) \qquad \left(\frac{0}{0}\text{form}\right)$$

$$\lim_{x\to 1}\left(\frac{-pq(q-1)x^{q-2}+qp(p-1)x^{p-2}}{-q(q-1)x^{q-2}-p(p-1)x^{p-2}+(p+q)(p+q-1)x^{(p+q-2)}}\right)$$

$$\Rightarrow \frac{-pq^2+pq+p^2q-pq}{-q^2+q-p^2+p+p^2+pq-p+qp+q^2-q}$$

$$\Rightarrow \frac{pq(p-q)}{2pq}=\frac{p-q}{2}$$

4. $$\lim_{x\to 3\pi/4}\frac{1+(\tan x)^{\frac{1}{3}}}{1-2\cos^2 x}$$

$$\lim_{x\to 3\pi/4}\frac{1+(\tan x)^{\frac{1}{3}}}{-\cos 2x} \qquad \left(\frac{0}{0}\text{form}\right)$$

Applying L'Hospital rule

$$\lim_{x\to\frac{3\pi}{4}}\frac{\frac{1}{3}(\tan x)^{\frac{-2}{3}}\times\sec^2 x}{+2\sin 2x}$$

$$\Rightarrow \frac{-1}{6}\left(-\sec\frac{\pi}{4}\right)^2$$

$$\frac{-1}{6}\times 2=-\frac{1}{3}$$

5. $$\lim_{x\to 0}\frac{\ell n(1+\sin^2 x)}{\tan[\ell n^2(1+x)]}$$

$$\Rightarrow \lim_{x\to 0}\ell n(1+\sin^2 x)\frac{1}{\tan[\ell n^2(1+x)]}$$

$$\Rightarrow \lim_{x\to 0}\frac{(\sin x/x)^2}{\frac{\tan(\ell n^2(1+x))}{\ell n^2(1+x)}}\frac{1}{\left(\frac{1}{x^2}\right)\ell n^2(1+x)}$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{\frac{\ell n(1+x)}{x}\ \frac{\ell n(1+x)}{x}} = 1$$

6. $$\lim_{x\to\infty} \frac{2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 5x^{\frac{1}{5}}}{(3x-2)^{\frac{1}{2}} + (3x-3)^{\frac{1}{3}}}$$

divide numerator & denominator by $x^{1/2}$

$$\Rightarrow \lim_{x\to\infty} \frac{2 + \frac{3}{x^{\frac{1}{6}}} + \frac{5}{x^{\frac{3}{10}}}}{\left(3-\frac{2}{x}\right)^{\frac{1}{2}} + \frac{1}{x^{6}}\left(2-\frac{3}{x}\right)^{\frac{1}{3}}} \Rightarrow \frac{2+0+0}{\sqrt{3}+0} \Rightarrow \frac{2}{\sqrt{3}}$$

7. $$\lim_{x\to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$$

$$\lim_{x\to 0} \frac{\frac{1}{\cos 4x} - \frac{1}{\cos 2x}}{\frac{1}{\cos 3x} - \frac{1}{\cos x}}$$

$$\lim_{x\to 0} \frac{\cos 2x - \cos 4x}{\cos x - \cos 3x} \times \frac{\cos x.\cos 3x}{\cos 4x.\cos 2x}$$

$$\lim_{x\to 0} \frac{+2\sin 3x.\sin x}{2\sin 2x.\sin x}$$

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 2x} \times \frac{2x}{3x} \times \frac{3x}{2x}$$

$$\lim_{x\to 0} \frac{3x}{2x} = \frac{3}{2}$$

8. Let first term of on infinite G.P. is a & common ratio of infinite G.P. is r

given $a = \lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x} \left(\frac{0}{0}\text{form}\right)$

Apply L'Hospital rule

$$\lim_{x\to 0} \frac{\sec^2 x - \cos x}{3\sin^2 x \cos x}$$

$$\lim_{x\to 0} \frac{1-\cos^3 x}{3\sin^2 x \cos 3x}$$

$$\lim_{x\to 0} \frac{1-\cos 3x}{3\sin^2 x \cos 2x} \times \lim_{x\to 0} \frac{1}{\cos x}$$

$$\lim_{x\to 0}\frac{4}{4}\times\frac{1-\cos^3 x}{3\sin^2 x\cos^2 x}\times 1$$

$$\lim_{x\to 0}\frac{4(1-\cos 3x)}{3(\sin 2x)^2} \qquad \left(\frac{0}{0}\text{form}\right)$$

Apply L'Hospital rule

$$\lim_{x\to 0}\frac{4}{3}\times\frac{(+3\cos 2x\sin x)}{2(\sin 2x)\times 2\cos 2x}$$

$$a = \lim_{x\to 0}\frac{\cos^2 x\sin x}{(2\sin x\cos x)\cos 2x} = \frac{1}{2}$$

$$r = \lim_{x\to 1}\frac{1-\sqrt{x}}{(\cos^{-1} x)^2}$$

$$\lim_{x\to 1}\frac{1\frac{1}{2\sqrt{x}}}{2(\cos^{-1} x)\times -\frac{1}{\sqrt{1-x^2}}}$$

$$\lim_{x\to 1}\frac{1\times\sqrt{1-x^2}}{4\sqrt{x}(\cos^{-1} x)}$$

$$\lim_{x\to 1}\frac{\sqrt{1-x^2}}{4}\ \cos^{-1}x \qquad \text{put } x=\cos t$$

$$r = \lim_{t\to 1}\frac{\sin t}{4t} = \frac{1}{4}$$

so sum of infinite G.P. is $\int\infty = \frac{a}{1-r}$

$$= \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{1}{2}\times\frac{4}{3} = \frac{2}{3}$$

9. $\lim_{x\to\infty}(x-\ell n(\cos nx))$ $\qquad$ $\cosh x = \frac{e^x+e^{-x}}{2}$

$$\lim_{x\to\infty}\left(x-\ell n\left(\frac{e^x+e^{-x}}{2}\right)\right)$$

$$\lim_{x\to\infty}\left(x-\ell n\left(\frac{e^{2x}+1}{2e^x}\right)\right)$$

$$\lim_{x\to\infty}\left[x-[\ell n(e^{2x}+1)-\ell n 2e^{x}]\right]$$

$$\lim_{x\to\infty}[x-\ell n(e^{2x}+1)-\ell n 2-\ell n e^{x}]$$

$$\lim_{x\to\infty}[x-\ell n(e^{2x}+1)-\ell n 2-x]$$

$$\lim_{x\to\infty}[-\ell n^{2}-\ell n(e^{2x}+1)]$$

$$-\ell n 2-\lim_{x\to\infty}\ell n\frac{(e^{2x}+1)}{e^{2x}}.e^{2x}$$

$$-\ell n 2-\lim_{x\to\infty}.\ell n\left(\frac{1+\frac{1}{e^{2x}}}{\frac{1}{e^{2x}}}\right)$$

10. $$\operatorname*{Lim}_{x\to 0}\frac{8}{x^{8}}\left[\left(1-\cos\frac{x^{2}}{2}\right)-\cos\frac{x^{2}}{4}\left(1-\cos\frac{x^{2}}{2}\right)\right]$$

$$\operatorname*{Lim}_{x\to 0}=\frac{8}{x^{8}}\left[\left(1-\cos\frac{x^{2}}{2}\right)\left(1-\cos\frac{x^{2}}{4}\right)\right]$$

$$\operatorname*{Lim}_{x\to 0}=\frac{8}{x^{8}}\left[1-\left(2\cos^{2}\frac{x^{2}}{4}-1\right)\left(1-\cos\frac{x^{2}}{4}\right)\right]$$

$$\operatorname*{Lim}_{x\to 0}=\frac{8}{x^{8}}\left[2-2\cos^{2}\frac{x^{2}}{4}\left(1-\cos\frac{x^{2}}{4}\right)\right]$$

$$\operatorname*{Lim}_{x\to 0}=\frac{8\times 2}{x^{8}}\left[\left(1-\cos^{2}\frac{x^{2}}{4}\right)\left(1-\cos\frac{x^{2}}{4}\right)\right]$$

$$\operatorname*{Lim}_{x\to 0}=\frac{16}{x^{8}}\left[\sin^{2}\frac{x^{2}}{4}\quad 2\sin^{2}\frac{x^{2}}{8}\right]$$

$$\lim_{x\to 0} = \frac{32}{x^8}\left[\sin^2\frac{x^2}{4}\sin^2\frac{x^2}{8}\right]$$

$$\lim_{x\to 0} = \frac{1}{32}\left(\frac{\sin\frac{x^2}{4}}{\left(\frac{x^2}{4}\right)}\right)\left(\frac{\sin\frac{x^2}{8}}{\left(\frac{x^2}{8}\right)}\right)^2 = \frac{1}{32}$$

11. $\lim_{\theta\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$ $\left(\frac{0}{0}\text{form}\right)$

Apply L'Hospital rule

$\lim_{\theta\to\frac{\pi}{4}} \frac{\sin\theta-\cos\theta}{2(4\theta-\pi).4}$ $\left(\frac{0}{0}\text{form}\right)$

Again apply L'Hospital rule

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\cos\theta+\sin\theta}{8.4} = \frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{32} = \frac{\sqrt{2}}{32} = \frac{1}{16\sqrt{2}}$$

12. $\lim_{\theta\to\frac{\pi}{2}} \frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)}$

$\lim_{\theta\to\frac{\pi}{2}} \frac{1-2^{\cos x}}{2^{\cos x}\left(x^2-\frac{\pi}{2}x\right)}$ $\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to\frac{\pi}{2}} \frac{1-2^{\cos x}}{x^2-\frac{\pi}{2x}}$ $\left(\frac{0}{0}\text{form}\right)$

again apply L'Hospital rule

$$\lim_{x\to\frac{\pi}{2}} \frac{-2^{\cos x}\log 2}{2x-\frac{\pi}{2}} \Rightarrow \frac{-2\log 2}{\pi}$$

13. $\lim_{x\to 1}\left[\ell n\left(\frac{1+x}{2}\right)\frac{1}{\sin(x-1)}\right].3.\left[\frac{4^{x-1}-x}{(7+x)^{1/3}-(1+3x)^{1/2}}\right]$

here $(1)^{\infty}$

$$=\frac{x-1}{2\sin(x-1)}.3\left[\frac{\ell n4\times(4^{x-1})-1}{\frac{1}{3}(7+x)^{-2/3}-\frac{3}{2}(1+3x)^{-1/2}}\right]\Rightarrow\frac{1}{2}.3\left[\frac{\ell n4-1}{\frac{1}{3}.\frac{1}{4}-\frac{3}{2}.\frac{1}{2}}\right]$$

$$\Rightarrow\frac{1}{2}.3.\left[\frac{\ell n4-1}{\frac{1-9}{12}}\right]\Rightarrow\boxed{\frac{-9}{4}\ell n\frac{4}{e}}$$

14. If $\ell=\lim_{n\to\infty}\sum_{r=2}^{n}\left((r+1)\sin\frac{\pi}{r+1}-r\sin\frac{\pi}{r}\right)$

$$\ell=\lim_{n\to\infty}\left(3\sin\frac{\pi}{3}-2\sin\frac{\pi}{2}+4\sin\frac{\pi}{4}-3\sin\frac{\pi}{3}+5\frac{\sin\pi}{5}-4\frac{\sin\pi}{4}+n\sin\frac{\pi}{4}-(n-1)\frac{\sin\pi}{n-1}\right.$$
$$\left.+(n+1)\frac{\sin\pi}{n+1}-n\frac{\sin\pi}{n}\right)$$

$$\ell=\lim_{n\to\infty}\left((n+1)\frac{\sin\pi}{n+1}-2\sin\frac{\pi}{2}\right)$$

$$\ell=\lim_{n\to\infty}\left((n+1)\frac{\frac{\sin\pi}{n+1}}{\frac{\pi}{n+1}}\times\frac{\pi}{n+1}-2\right)$$

$$\ell=\lim_{n\to\infty}(\pi-2)=3.14-2$$

$$=1.14$$

so $\{\ell\}=\ell-[\ell]$

$=1.14-1=.14$ or $\pi-3$

15. $\lim_{x\to-\infty}\frac{(3x^4+2x^2)\left(\sin\frac{1}{x}\right)\frac{1}{x}}{(|x|^3+|x|^2+|x|+1)\frac{1}{2}}+\frac{|x|^3+5}{|x|^3+|x|^2+|x|+1}$

$$\lim_{x\to-\infty} \frac{3x^3+2x}{-(x)^3-(x)^2-(x)+1}+\frac{-(x)^3+5}{-(x)^3+(x)^2-(x)+1}$$

$$\lim_{x\to-\infty} \frac{3+\frac{2}{x^2}}{-1-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}}+\frac{-1+\frac{5}{x^3}}{-1-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}$$

$$\lim_{x\to-\infty} \frac{-3}{1}+1=-2$$

16. $$\lim_{x\to3}\frac{(x^3+27)\ell n(x-2)}{(x^2-9)}$$

$$\lim_{x\to3}\frac{(x-3)\ (x^2+9-6x)\ \ell n(1+(x-2))}{(x+3)\ (x-3)}$$

$$\lim_{x\to3} x^2+9-6x\ =18-9=9$$

17. $$\lim_{x\to0}\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}\times\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$\lim_{x\to0}\frac{2\sqrt{2}(27^x-9^x-3^x+1)}{2-1-\cos x}$$

$$\lim_{x\to0}\frac{2\sqrt{2}(27^x-9^x-3^x+1)}{1-\cos}\times x^2$$

$$\lim_{x\to0}\frac{4\sqrt{2}(27^x-9^x-3^x+1)}{x^2}$$

$$\lim_{x\to0}4\sqrt{2}\left(\frac{9^x(3^x-1)-1(3^x-1)}{x^2}\right)$$

$$\lim_{x\to0}4\sqrt{2}\left(\frac{(9^x-1)\ (3^x-1)}{x^2}\right)\quad\Rightarrow\quad\lim_{x\to0}4\sqrt{2}\left(\frac{9^x-1}{x}\right)\times\lim_{x\to0}\frac{3^x-1}{x}\ \Rightarrow\ 8\sqrt{2}(\ell n3)^2$$

$$\Rightarrow\ 4\sqrt{2}(\ell n9)\ (\ell n3)$$

18. $$\operatorname*{Lim}_{x\to0}\left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}\quad(1)^{\infty}$$

$$\boxed{L=e^{\operatorname*{Lim}_{x\to0}\frac{(1+x)^{1/x}-e}{e}.\frac{1}{x}}}$$

Act.: $\lim\limits_{x\to 0}\dfrac{\dfrac{e[e^{\frac{\ell n(1+x)}{x}-1}-1]}{e(\ell n(1+x)-x)}}{x}\cdot\left(\dfrac{\ell n(1+x)-x}{x^2}\right)$

$\lim\limits_{t\to 0}\dfrac{t-e^t+1/t^2}{(e^t-1)^2/t^2}=-\dfrac{1}{2}$

(taking commone)

$= e^{-1/2}$

19. $e^{\lim\limits_{n\to\infty}\left[\frac{\sqrt{n^2+n}-1-n}{n}\right](2\sqrt{n^2+n}-1)}$

$e^{\lim\limits_{x\to\infty}\frac{[(n^2+n)-(1+n)^2](2\sqrt{n^2+n}-1)}{n\left\{\sqrt{n^2+n}+(1+n)\right\}}}$

$e^{\lim\limits_{n\to\infty}\frac{(-n-1)(2\sqrt{n^2+n}-1)}{n\left\{\sqrt{n^2+n}+(1+n)\right\}}}$ [taking n^2 as common]

$e^{\lim\limits_{n\to\infty}\frac{\left(-1-\frac{1}{n}\right)\left(2\sqrt{1+\frac{1}{n}}-\frac{1}{n^2}\right)n^2}{n^2\left\{\sqrt{1+\frac{1}{n}}+\frac{1}{n}+1\right\}}}=e^{-1}$

20. $\lim\limits_{x\to 1}\left[\tan\dfrac{\pi x}{4}\right]^{\tan\frac{\pi x}{2}}$

$e^{\lim\limits_{x\to 1}\left(\tan\frac{\pi x}{4}-1\right)\tan\frac{\pi x}{2}}$

$e^{\lim\limits_{x\to 1}\frac{\sin\left(\frac{\pi x}{4}-\frac{\pi}{4}\right)}{\cos\frac{\pi}{4}.\cos\frac{\pi x}{4}}.\tan\frac{\pi x}{2}}$

$e^{\lim\limits_{x\to 1}\frac{\sin\frac{\pi}{4}(x-1)}{\frac{\pi}{4}(x-1)\cos\frac{\pi}{4}\cos\frac{\pi}{4}x}.\tan\frac{\pi x}{2}.\frac{\pi}{4}(x-1)}$

$e^{\lim\limits_{x\to 1}\frac{\pi}{2}(x-1)\cot\left(\frac{\pi}{2}-\frac{\pi}{2x}\right)}=e^{-1}=(e^{-1})$

21. $\lim\limits_{x\to 0}\left[\tan\left(\dfrac{\pi}{4}+x\right)\right]^{1/x}=e^{\lim\limits_{x\to 0}\frac{1}{x}\left[\tan\left(\frac{\pi}{4}+x\right)-1\right]}$

$=e^{\lim\limits_{x\to 0}\frac{1}{x}\left[\frac{2\tan x}{1-\tan x}\right]}=e^{\lim\limits_{x\to 0}\left[\frac{\tan x}{x}\right]\frac{2}{1-\tan x}}=e^2\cdot$

22. $\lim_{x\to 0}\left(\frac{3^x-1}{x}\right)=\ln 3$

23. $\lim_{x\to 0}\frac{\cos(\sin x)-1}{x^2}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{\sin x}{2}\right)}{x^2}$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\frac{\sin x}{2}\right)}{\frac{\sin^2 x}{4}}\times\frac{\sin^2 x}{4x^2}=\frac{1}{2}$$

24. $\lim_{x\to 0^+}\frac{e^{1/x}}{e^{\frac{1}{x}}+1}=\lim_{x\to 0^+}\frac{1}{e^{-\frac{1}{x}}+1}=1$ & $\lim_{x\to 0^-}\frac{e^{1/x}}{e^{\frac{1}{x}}+1}=0$

LHL $\neq$ RHL

Limit does not exist.

25. $\lim_{x\to 0}\frac{a^{\sin x}-1}{b^{\sin x}-1}=\lim_{x\to 0}\frac{\frac{a^{\sin x}-1}{\sin x}}{\frac{b^{\sin x}-1}{\sin x}}=\frac{\ln a}{\ln b}$

26. $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1$

27. $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+1}=e^{\lim_{x\to\infty}\left(\frac{x+3}{x+1}-1\right)(x+1)}=e^{\lim_{x\to\infty}\left(\frac{2}{x+1}\right)(x+1)}=e^2$

28. $\lim_{x\to 0}(1-ax)^{\frac{1}{x}}=e^{\lim_{x\to 0}\frac{-ax}{x}}=e^{-a}$

29. By L'hospitals rule

$$\lim_{x\to 0}\frac{x^n-\sin x^n}{x-\sin^n x}=\lim_{x\to 0}\frac{nx^{n-1}\left(1-\cos x^n\right)}{1-n\sin^{n-1}x\cos x}$$

Now for any value of n greater than 1, denominator will be nonzero and numerator will be zero.

But for n = 1, limit becomes $\lim_{x\to 0}\frac{1-\cos x}{1-\cos x}=1$.

30. $\lim_{x\to 0}\frac{2x+ax\cos x+b\sin x}{x^3}=\lim_{x\to 0}\frac{2x+ax\left(1-\frac{x^2}{2}\right)+b\left(x-\frac{x^3}{6}\right)}{x^3}$

$=\lim_{x\to 0}\frac{(a+b+2)x-\left(\frac{a}{2}+\frac{b}{6}\right)x^3}{x^3}=2$

$\Rightarrow a+b+2=0\ \ \&\ 3a+b=-12$

$\Rightarrow a=-5, b=3$.

31. $\lim_{h\to 0}\frac{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h(\sqrt{3}\cos h-\sin h)}=\lim_{h\to 0}\frac{\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h\left(\frac{\sqrt{3}}{2}\cos h-\frac{1}{2}\sin h\right)}$

$=\lim_{h\to 0}\frac{\cos\frac{\pi}{6}\sin\left(\frac{\pi}{6}+h\right)-\sin\frac{\pi}{6}\cos\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h\left(\cos\frac{\pi}{6}\cos h-\sin\frac{\pi}{6}\sin h\right)}$

$=\lim_{h\to 0}\frac{\sin h}{\sqrt{3}h\cos\left(\frac{\pi}{6}+h\right)}=\frac{2}{3}$.

32. $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}=\lim_{x\to 0}\frac{x\left(1-\frac{x^2}{2}\right)-\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)}{x^2}$

$=\lim_{x\to 0}\frac{\frac{x^2}{2}-\frac{2x^3}{3}}{x^2}=\frac{1}{2}$.

33. $\lim_{x\to\frac{\pi}{4}}\left(\frac{1-\tan x}{1-\sqrt{2}\sin x}\right)=\lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x}\left(\frac{\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x}{\frac{1}{\sqrt{2}}-\sin x}\right)$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x}\left(\frac{\sin\left(\frac{\pi}{4}-x\right)}{\sin\frac{\pi}{4}-\sin x}\right)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x}\left(\frac{2\sin\left(\frac{\pi}{8}-\frac{x}{2}\right)\cos\left(\frac{\pi}{8}-\frac{x}{2}\right)}{2\cos\left(\frac{\pi}{8}+\frac{x}{2}\right)\sin\left(\frac{\pi}{8}-\frac{x}{2}\right)}\right)$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{1}{\cos x}\left(\frac{\cos\left(\frac{\pi}{8}-\frac{x}{2}\right)}{\cos\left(\frac{\pi}{8}+\frac{x}{2}\right)}\right) = 2$$

34. $\lim_{x\to 0}\frac{3^x-1}{\sqrt{4+x}-2} = \lim_{x\to 0}\frac{3^x-1}{x}\left(\sqrt{4+x}+2\right) = 4\ln 3$

35. $\lim_{x\to 0}\left(\frac{\sin 2(1+x)+\sin 2(1-x)-2\sin 2}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{2\sin 2\cos 2x-2\sin 2}{x\sin x}\right)$

$$= -2\sin 2\lim_{x\to 0}\left(\frac{2\sin^2 x}{x\sin x}\right) = -4\sin 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = -4\sin 2$$

36. By L'hospitals rule

$$\lim_{x\to\frac{\pi}{4}}\frac{\int_{\pi/4}^{x} t^2\,dt}{\cos 2x} = \lim_{x\to\frac{\pi}{4}}\frac{x^2}{-2\sin 2x} = -\frac{\pi^2}{32}$$

37. By L'hospitals rule

$$\lim_{h\to 0}\frac{(x+h)\sec(x+h)-x\sec x}{\sin h} = \lim_{h\to 0}\frac{(x+h)\sec(x+h)\tan(x+h)+\sec(x+h)}{\cos h}$$

$$= x\sec x\tan x+\sec x = \sec x\left(x\tan x+1\right)$$

Alternately

$$\lim_{h\to 0}\frac{(x+h)\sec(x+h)-x\sec x}{\sin h} = \lim_{h\to 0}\frac{(x+h)\sec(x+h)-(x+h)\sec x-x\sec x+(x+h)\sec x}{\sin h}$$

$$= \lim_{h \to 0} \frac{(x+h)\left(\sec(x+h) - \sec x\right)}{\sin h} - \lim_{h \to 0} \frac{\left(x - (x+h)\right)\sec x}{\sin h}$$

$$= \lim_{h \to 0} \frac{(x+h)\left(\cos x - \cos\left(x+h\right)\right)}{\sin h \cos x \cos\left(x+h\right)} + \lim_{h \to 0} \frac{h \sec x}{\sin h}$$

$$= \lim_{h \to 0} \frac{2(x+h)\sin\left(x + \frac{h}{2}\right)\sin\frac{h}{2}}{2\sin\frac{h}{2}\cos\frac{h}{2}\cos x \cos\left(x+h\right)} + \lim_{h \to 0} \frac{h \sec x}{\sin h}$$

$$= \frac{x \sin x}{\cos^2 x} + \sec x = x \sec x \tan x + \sec x \, .$$

38. $$\lim_{x \to \frac{\pi}{6}} \frac{\sin\left(\frac{\pi}{3} - 2x\right)}{2\cos 2x - 1} = \lim_{x \to \frac{\pi}{6}} \frac{\sin\left(\frac{\pi}{3} - 2x\right)}{4\cos^2 x - 3}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\sin\left(\frac{\pi}{3} - 2x\right)\cos x}{\cos 3x}$$

$$= \lim_{y \to 0} \frac{\sin 2y \cos\left(\frac{\pi}{6} - y\right)}{\sin 3y} \quad \left\{x = \frac{\pi}{6} - y\right\}$$

$$= \lim_{y \to 0} \frac{2\frac{\sin 2y}{2y}\cos\left(\frac{\pi}{6} - y\right)}{3\frac{\sin 3y}{3y}} = \frac{2}{3} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$$

39. $$\lim_{x \to 0} \frac{e^{x^4} - \cos x^2}{x^4} = \lim_{x \to 0} \frac{\left(1 + x^4 + \frac{x^8}{2}\right) - \left(1 - \frac{x^4}{2} + \frac{x^8}{4!}\right)}{x^4} = \frac{3}{2} \, .$$

40. $$\lim_{x \to 0} \frac{x\cos^2 x - \sin^2 x}{x + \sin x} = \lim_{x \to 0} \frac{\cos^2 x - x\frac{\sin^2 x}{x^2}}{1 + \frac{\sin x}{x}} = \frac{1}{2}$$

LIMITS
EXERCISE 2(A)

1. (A)(D)

put $\theta = -1$; $\frac{1-1-2}{2} \le f(-1) \le \frac{1-2-1}{2}$

$-1 \le f(-1) \le -1 \quad \Rightarrow \quad f(-1) = -1$

$\lim\limits_{\theta \to -1} \frac{\theta^2+\theta-2}{\theta+3} = -1 = \lim\limits_{\theta \to -1} \frac{\theta^2+2\theta-1}{\theta+3}$

using sequeeze play theorem

$$\lim_{\theta \to -1} \frac{f(\theta)}{\theta^2} = -1; \ \lim_{\theta \to -1} f(\theta) = -1.$$

2. (A)(B)(C)(D)

$$l_1 = \lim_{x \to \infty} \sqrt{\frac{1-\frac{\cos^2 x}{x}}{1+\frac{\sin x}{x}}} = 1$$

$$l_2 = \lim_{h \to 0^+} 2\int_0^1 \frac{h\,dx}{h^2+x^2} = \lim_{h \to 0^+} \left[2\frac{h}{h}\tan^{-1}\frac{x}{h}\right]_{0^+}^{1} = \pi \quad \textbf{Ans.}$$

Note: $\frac{22}{7} > \pi \qquad \left[\frac{22}{7} = 3.1428571 \text{ and } \pi \approx 3.1415929\right].$

3. (B)(C)

(A)

$\lim\limits_{x \to 3^+}\left(\left[[x]\right]-[2x+1]\right) = 3-7 = -4$

$\lim\limits_{x \to 3^-}\left(\left[[x]\right]-[2x+1]\right) = 2-6 = -4$

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^-} f(x)$

Hence limit exists.

(B)

$\lim\limits_{x \to 1^+}\left([x]-x\right) = 1-1 = 0$

$\lim\limits_{x \to 1^-}\left([x]-x\right) = 0-1 = -1$

$\lim\limits_{x \to 1^+} f(x) \ne \lim\limits_{x \to 1^-} f(x)$

Limit does not exist

(C)

$\lim\limits_{x \to 0^+}\left(\{x\}^2-\{-x\}^2\right) = 0-1 = -1$

$\lim\limits_{x \to 0^-}\left(\{x\}^2-\{-x\}^2\right) = 1-0 = 1$

$\lim\limits_{x \to 1^+} f(x) \ne \lim\limits_{x \to 1^-} f(x)$

Limit does not exist

(D)

$\lim\limits_{x \to 0^+} \frac{\tan(\operatorname{sgn}(x))}{\operatorname{sgn}(x)} = \frac{\tan 1}{1} = \tan 1$

$\lim\limits_{x \to 0^-} \frac{\tan(\operatorname{sgn}(x))}{\operatorname{sgn}(x)} = \frac{\tan(-1)}{-1} = \tan 1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x)$

Hence limit exists.

4. (A)(C)(D)

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{\tan^2\{x\}}{x^2-[x]^2} = \lim\limits_{x \to 0^+} \frac{\tan^2 x}{x^2} = 1$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{\{x\}\cot\{x\}} = \lim_{x\to 0^+} \sqrt{\frac{x+1}{\tan(x+1)}} = \sqrt{\cot 1}$$

5. (B)(D)

$$(A)\ \lim_{x\to\infty} x^{\frac{1}{4}} \sin\frac{1}{\sqrt{x}} = \lim_{y\to 0^+} \frac{\sin\sqrt{y}}{y^4} = \infty$$

$$(B)\ \lim_{x\to\frac{\pi}{2}} (1-\sin x)\tan x = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right)\sin x = \lim_{x\to\frac{\pi}{2}} \left(\frac{\cos^2 x}{\cos x}\right)\frac{\sin x}{1+\sin x} = 0$$

$$(C)\ \lim_{x\to\infty} \left(\frac{2x^2+3}{x^2+x-5}\right)\operatorname{sgn}(x) = \lim_{x\to\infty} \left(\frac{2+\frac{3}{x^2}}{1+\frac{1}{x}-\frac{5}{x^2}}\right) = 2$$

$$(D)\ \lim_{x\to 3^+} \frac{[x]^2-9}{x^2-9} = 0$$

6. (A)(B)

$$f(x) = \lim_{n\to\infty} \frac{2x^{2n}\sin\frac{1}{x} + x}{1+x^{2n}}$$

$$\text{for } x<1,\ f(x) = \lim_{n\to\infty} \frac{2x^{2n}\sin\frac{1}{x}+x}{1+x^{2n}} = x \qquad \left\{\lim_{n\to\infty} x^{2n} = 0 \text{ if } x<1\right\}$$

$$\text{for } x=1,\ f(x) = 2\sin\frac{1}{x} + x \qquad \left\{\lim_{n\to\infty} x^{2n} = 1 \text{ if } x=1\right\}$$

$$\text{for } x>1,\ f(x) = \lim_{n\to\infty} \frac{2\sin\frac{1}{x} + \frac{1}{x^{2n-1}}}{\frac{1}{x^{2n}}+1} = 2\sin\frac{1}{x} \qquad \left\{\lim_{n\to\infty} \frac{1}{x^{2n}} = 0 \text{ if } x>1\right\}$$

Now

$$(A)\ \lim_{x\to\infty} xf(x) = \lim_{x\to\infty} 2x\sin\frac{1}{x} = 2\lim_{y\to 0}\frac{\sin y}{y} = 2$$

$$(B)\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2\sin\frac{1}{x} = 2\sin 1 \ \&\ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$, hence $\lim_{x\to 1} f(x)$ does not exist

$$(C)\ \lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$$

$$(D)\ \lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} x = -\infty$$

7. (A)(B)(D)

$$\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2} - \sqrt{x^4+2x^3-cx^2+3x-d}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x^4+ax^3+3x^2+bx+2-\left(x^4+2x^3-cx^2+3x-d\right)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{(a-2)x^3+(c+3)x^2+(b-3)x+d+2}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}\right)$$

$$= \lim_{y\to 0}\left(\frac{\left((a-2)+(c+3)y+(b-3)y^2+(d+2)y^3\right)}{y\sqrt{1+ay+3y^2+by^3+y^4}+\sqrt{1+2y-cy^2+3y^3-dy^4}}\right)$$

Clearly a = 2. Now

$$\text{Lim} = \lim_{y\to 0}\left(\frac{(c+3)+(b-3)y+(d+2)y^2}{\sqrt{1+ay+3y^2+by^3+y^4}+\sqrt{1+2y-cy^2+3y^3-dy^4}}\right)=\frac{c+3}{2}$$

Hence $\frac{c+3}{2}=4$ or c = 5.

Also $b, d \in R$.

8. Options need correction

$$\lim_{x\to 0}\frac{a+b\sin x-\cos x+ce^x}{x^3}=\lim_{x\to 0}\frac{a+b\left(x-\frac{x^3}{3!}+...\right)-\left(1-\frac{x^2}{2}+...\right)+c\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+...\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{(a-1+c)+(b+c)x+\left(c+\frac{1}{2}\right)x^2+\frac{(c-b)}{6}x^3}{x^3}$$

Clearly $a-1+c=0, b+c=0 \;\&\; c+\frac{1}{2}=0$

$$\Rightarrow c=-\frac{1}{2}, b=\frac{1}{2}, a=\frac{3}{2}.$$

9. (C)

$$\lim_{x\to 0^+}\frac{a^{[x]+x}}{[x]+x}=\lim_{x\to 0^+}\frac{a^x}{x}=\infty \quad \& \quad \lim_{x\to 0^-}\frac{a^{[x]+x}}{[x]+x}=\lim_{x\to 0^-}\frac{a^{x-1}}{x-1}=-\frac{1}{a}$$

$$\lim_{x\to 0^+}f(x)\neq\lim_{x\to 0^-}f(x).$$

Limit does not exist.

10. (A)(B)

(A) $\lim_{x\to\infty}\left(\frac{x}{2+x}\right)^{2x}=e^{\lim_{x\to\infty}2x\left(\frac{x}{2+x}-1\right)}$ $\{1^\infty \text{ form}\}$

$$= e^{-\lim_{x\to\infty}\left(\frac{4x}{2+x}\right)} = e^{-\lim_{x\to\infty}\left(\frac{4}{\frac{2}{x}+1}\right)} = e^{-4}$$

(B) $\lim_{x\to 1}\left(\frac{x}{2+x}\right)^{2x} = \left(\frac{1}{2+1}\right)^{2} = \frac{1}{9}$

11. (A)(B)(C)

$$\lim_{x\to 0^+}\frac{ae^{1/x}+be^{-1/x}}{ce^{1/x}+de^{-1/x}} = \lim_{x\to 0^+}\frac{a+be^{-2/x}}{c+de^{-2/x}} = \frac{a}{c}$$

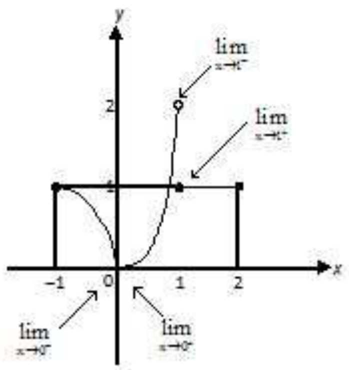
$$\lim_{x\to 0^-}\frac{ae^{1/x}+be^{-1/x}}{ce^{1/x}+de^{-1/x}} = \lim_{x\to 0^+}\frac{ae^{2/x}+b}{ce^{2/x}+d} = \frac{b}{d}$$

Hence a = 2c, b = 2d

Now $bx^2+(a-2c)x-2d=0 \Rightarrow x^2=1$.

12. (B)(C)(D)

Clear from the figure.



13. (A)(B)(D)

Conceptual question.

14. Options need correction

$$\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}} = e^{\lim_{x\to a}\tan\frac{\pi x}{2a}\left(2-\frac{a}{x}-1\right)} \quad \{1^\infty \text{ form}\}$$

$$= e^{\lim_{y\to 0}\tan\frac{\pi(a+y)}{2a}\left(1-\frac{a}{a+y}\right)} = e^{\lim_{y\to 0}\tan\left(\frac{\pi}{2}+\frac{\pi y}{2a}\right)\left(\frac{y}{a+y}\right)}$$

$$= e^{-\lim_{y\to 0}\cot\frac{\pi y}{2a}\left(\frac{y}{a+y}\right)} = e^{-\lim_{y\to 0}\frac{\frac{\pi y}{2a}}{\tan\frac{\pi y}{2a}}\frac{2}{\pi(a+y)}}$$

$$= e^{-\frac{2}{\pi a}} = e^{-\frac{2}{\pi}} \Rightarrow a = 1$$

15. (B)(C)

(A) $\lim_{x\to\infty}\frac{2x^2+1}{x^2+2} = \lim_{y\to 0^+}\frac{2+y^2}{1+2y^2}$

$$=\frac{1}{2}\lim_{y\to 0^+}\left(\frac{3}{1+2y^2}+1\right)=2^-$$

$$\Rightarrow\left[\lim_{x\to\infty}\frac{2x^2+1}{x^2+2}\right]=1$$

(B) $\lim_{x\to 0}\frac{\sin 2x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2^{-1}$

$$\Rightarrow\left[\lim_{x\to 0}\frac{\sin 2x}{x}\right]=1.$$

(C) $\lim_{x\to 0}\frac{\tan 3x}{x}=3\lim_{x\to 0}\frac{\tan 3x}{3x}=3^+$

$$\Rightarrow\left[\lim_{x\to 0}\frac{\tan 3x}{x}\right]=3.$$

(D) $\lim_{x\to -\infty}\frac{\sqrt{4x^2+1}}{x+2}=-\lim_{y\to 0^-}\frac{\sqrt{4+y^2}}{1+2y}=-\left(2^+\right)=-2^-$

$$\Rightarrow\left[\lim_{x\to -\infty}\frac{\sqrt{4x^2+1}}{x+2}\right]=-3$$

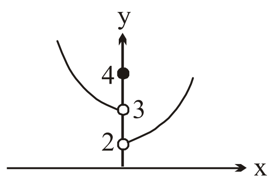
PASSAGE I

16. (C)

$x\to 0$, $x^3-x^2=x^2(x-1)\to 0^-$

$x\to 0$, $2x^4-x^5=x^4(2-x)\to 0^+$

$2(3)=\lambda\ (2) \Rightarrow \lambda=3$



17. (B)

$$\lim_{x\to 0^+}\frac{f(-x)x^2}{\left(\frac{1-\cos x}{[f(x)]}\right)-\left[\frac{1-\cos x}{[f(x)]}\right]}=\frac{3x^2}{\frac{1-\cos x}{2}-0}$$

$=6\times 2=12$

18. (B)

$x\to 0^-$ $\left(\frac{x^3-\sin^3 x}{x^4}\right)=\left(\frac{x-\sin x}{x^3}\right)\left(\frac{x^2+\sin^2 x+x\sin x}{x^2}\right)x=\frac{1}{6}(3)\,x\to 0^- \Rightarrow$

$f(0^-)=3$

$x\to 0^-$ $\frac{\sin x^3}{x}=\frac{\sin x^3}{x^3}x^2\to 0^+ \quad\Rightarrow\quad \left[\frac{\sin x^3}{x}\right]=0 \quad\Rightarrow\quad f(0)=4$

$\therefore\quad 3f\left(\frac{x^3-\sin^3 x}{x^4}\right)>9$

$\Rightarrow\quad [9^+]-f(0)=9-4=5$

19. (A)

$$\left|\sum_{k=1}^{n}\left(3^{k}\left\{f\left(x+ky\right)-f\left(x-ky\right)\right\}\right)\right|\le 1....(i)$$

$$\Rightarrow\left|\sum_{k=1}^{n-1}\left(3^{k}\left\{f\left(x+ky\right)-f\left(x-ky\right)\right\}\right)\right|\le 1...(ii)$$

$$\Rightarrow\left|3^{n}\left\{f\left(x+ny\right)-f\left(x-ny\right)\right\}\right|\le 2$$

$$\Rightarrow\left|f\left(x+ny\right)-f\left(x-ny\right)\right|\le\frac{2}{3^{n}}$$

$$\Rightarrow\lim_{n\to\infty}\left|f\left(x+ny\right)-f\left(x-ny\right)\right|\le 0$$

$\Rightarrow f\left(x\right)$ is a constant function .

20. (D)

$$\lim_{x\to 0}\frac{axe^{x}-b\ln\left(1+x\right)+cxe^{-x}}{x^{2}\sin x}=2$$

$$\Rightarrow\lim_{x\to 0}\frac{ax\left(1+x+\frac{x^{2}}{2}\right)-b\left(x-\frac{x^{2}}{2}+\frac{x^{3}}{3}\right)+cx\left(1-x+\frac{x^{2}}{2}\right)}{x^{2}\sin x}=2$$

$$\Rightarrow\lim_{x\to 0}\frac{\left(a-b+c\right)x+\left(a+\frac{b}{2}-c\right)x^{2}+\left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)x^{3}}{x^{3}\frac{\sin x}{x}}=2$$

$$\Rightarrow a-b+c=0,\ a+\frac{b}{2}-c=0\ \&\ \frac{a}{2}-\frac{b}{3}+\frac{c}{2}=2$$

Hence $a=3, b=12, c=9$

Now $g(x) = 3x^3 - 15x^2 + 9x$.

$g\left(x\right)=f\left(x\right)\Rightarrow h\left(x\right)=3x^{3}-15x^{2}+9x+23=0$

$\Rightarrow h'\left(x\right)=9x^{2}-15x+9>0$ for all x .

Hence g(x) = f(x) has only one real root

21. (A)

$$\lim_{x\to 1}\frac{g\left(x\right)+3}{\ln\left(2-x\right)}=\lim_{x\to 1}\frac{3x^{3}-15x^{2}+9x+3}{\ln\left(2-x\right)}$$

By L'hospital's rule

$$\lim_{x\to 1}\frac{3x^{3}-15x^{2}+9x+3}{\ln\left(2-x\right)}=\lim_{x\to 1}\frac{9x^{2}-30x+9}{-\frac{1}{2-x}}=12$$

22. (A)

$$\lim_{x\to 0}\frac{\ln(f(x))}{\ln(g(x))}=\lim_{x\to 0}\frac{\lim_{n\to\infty}\ln\left(\cos\sqrt{\frac{x}{n}}\right)^n}{\lim_{n\to\infty}\ln\left(1+x\left(1+e^{1/n}\right)\right)^n}=\lim_{x\to 0}\lim_{n\to\infty}\frac{n\ln\left(\cos\sqrt{\frac{x}{n}}\right)}{n\ln\left(1+x\left(1+e^{1/n}\right)\right)}$$

$$=\lim_{x\to 0}\lim_{m\to 0}\frac{\ln\left(\cos\sqrt{mx}\right)}{\ln\left(1+x\left(1+e^m\right)\right)}=\lim_{x\to 0}\lim_{m\to 0}\frac{\frac{\ln\left(1-\left(1-\cos\sqrt{mx}\right)\right)}{1-\cos\sqrt{mx}}}{\frac{\ln\left(1+x\left(1+e^m\right)\right)}{x\left(1+e^m\right)}}\frac{1-\cos\sqrt{mx}}{x\left(1+e^m\right)}$$

$$=\lim_{x\to 0}\lim_{m\to 0}\frac{\frac{\ln\left(1-\left(1-\cos\sqrt{mx}\right)\right)}{1-\cos\sqrt{mx}}}{\frac{\ln\left(1+x\left(1+e^m\right)\right)}{x\left(1+e^m\right)}}\frac{2\sin^2\frac{\sqrt{mx}}{2}}{x\left(1+e^m\right)}$$

$$=\lim_{x\to 0}\lim_{m\to 0}\frac{\frac{\ln\left(1-\left(1-\cos\sqrt{mx}\right)\right)}{1-\cos\sqrt{mx}}}{\frac{\ln\left(1+x\left(1+e^m\right)\right)}{x\left(1+e^m\right)}}\frac{2\left(\frac{\sin\frac{\sqrt{mx}}{2}}{2\frac{\sqrt{mx}}{2}}\right)^2}{\left(\frac{1+e^m}{m}\right)}=\frac{1}{2}.$$

23. (C)

$$f(x)=e^{\lim_{n\to\infty}n\left(\cos\sqrt{\frac{x}{n}}-1\right)}=e^{\lim_{m\to 0}\frac{1}{m}\left(\cos\sqrt{mx}-1\right)}$$

$$f(x)=e^{\lim_{m\to 0}\frac{-2\sin^2\frac{\sqrt{mx}}{2}}{m}}=e^{-2\lim_{m\to 0}\left(\frac{\sin\frac{\sqrt{mx}}{2}}{\frac{\sqrt{mx}}{2}}\right)^2\frac{x}{4}}\Rightarrow f(x)=e^{-\frac{x}{2}}$$

$$g(x)=\lim_{n\to\infty}\left(1+x\left(1+e^{1/n}\right)\right)^n\quad\{1^\infty\ \text{ form}\}$$

$$g(x)=e^{\lim_{n\to\infty}n\left(x\left(1+e^{1/n}\right)\right)}=e^{\lim_{m\to 0}\left(x\left(\frac{1+e^m}{m}\right)\right)}\Rightarrow g(x)=e^x$$

$$f^{-1}(x)=-2\ln x,\ g^{-1}(x)=\ln x$$

$$\Rightarrow h(x)=\tan^{-1}\left(\ln(-2\ln x)\right)$$

Now $-2\ln x>0\Rightarrow \ln x<0\Rightarrow x\in(0,1)$

24. (D)

As range of $\ln(-2\ln x)$ is R, hence $-\frac{\pi}{2} < h(x) < \frac{\pi}{2}$.

25. (A)→(R); (B)→(S); (C)→(P); (D)→(Q)

(A) $l = \lim_{x\to 0} \frac{e^{x^2} - 1 + x - e^x + 1}{2\frac{\sin^2 x}{x^2}\cdot x^2} = \frac{1}{2}\left[\lim_{x\to 0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to 0}\frac{x - e^x + 1}{x^2}\right] =$

$\frac{1}{2}\left[1 - \lim_{x\to 0}\frac{e^x - x - 1}{x^2}\right]$

$= \frac{1}{2}\left[1 - \frac{1}{2}\right] = \frac{1}{4} \quad \Rightarrow \quad \frac{1}{l} = 4$

(B) $l = \lim_{x\to 0}\left(\frac{3+x}{3-x}\right)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{3+x}{3-x}-1\right)} = e^{\lim_{x\to 0}\frac{2x}{x(3-x)}} = e^{2/3} \Rightarrow 2 + 3 = 5$

(C) $\lim_{x\to 0}\frac{(\tan^3 x - x^3) - (\tan x^3 - x^3)}{x^5} = \lim_{x\to 0}\frac{\tan^3 x - x^3}{x^5} - \underbrace{\lim_{x\to 0}\frac{\tan x^3 - x^3}{x^5}}_{\text{zero (by expansion)}}$

$= \lim_{x\to 0}\frac{(\tan x - x)}{x^3}\cdot\frac{(\tan^2 x + x\tan x + x^2)}{x^2} = \frac{1}{3}\times 3 = 1$

(D) rationalising gives

$\lim_{x\to 0}\frac{(x + 2\sin x)\left[\sqrt{(x^2 + 2\sin x + 1)} + \sqrt{\sin^2 x - x + 1}\right]}{(x^2 + 2\sin x + 1) - (\sin^2 x - x + 1)}$

$2\cdot\lim_{x\to 0}\frac{x + \sin 2x}{x^2 - \sin^2 x + 2\sin x + x} = 2\cdot\lim_{x\to 0}\frac{1 + \frac{\sin 2x}{x}}{x - \frac{\sin^2 x}{x} + 2 + 1} = 2\left(\frac{1+2}{3}\right) = 2.$

26. (A)→(Q), (B)→(R), (C)→(P), (D)→(P)

(A) $\lim_{n\to\infty}\cos^2\left(\pi\left(\sqrt[3]{n^3 + n^2 + 2n}\right)\right) = \lim_{n\to\infty}\cos^2\left(n\pi - \left(n^3 + n^2 + 2n\right)^{1/3}\pi\right)$

$= \lim_{n\to\infty}\cos^2\pi\left(\frac{n^2 + 2n}{n^2 + n\left(n^3 + n^2 + 2n\right)^{1/3} + \left(n^3 + n^2 + 2n\right)^{2/3}}\right)$

$= \lim_{n\to\infty}\cos^2\pi\left(\frac{1 + \frac{2}{n}}{1 + \left(1 + \frac{1}{n} + \frac{2}{n^2}\right)^{1/3} + \left(1 + \frac{1}{n} + \frac{2}{n^2}\right)^{2/3}}\right) = \cos^2\frac{\pi}{3} = \frac{1}{4}.$

(B) $\lim_{n\to\infty} \dfrac{\sin\left(2\pi\sqrt{1+n^2}\right)}{1/n} \rightarrow \left\{\dfrac{0}{0} \text{ form as } \sin\left(2\pi\sqrt{1+n^2}\right) \rightarrow 0\right\}$

Also $\lim_{n\to\infty}\left(n-\sqrt{1+n^2}\right) = \lim_{n\to\infty}\left(\dfrac{-1}{n+\sqrt{1+n^2}}\right) = 0$

$$\therefore \lim_{n\to\infty} \frac{\sin\left(2\pi\sqrt{1+n^2}\right)}{1/n} = -\lim_{n\to\infty} \frac{\sin\left(2n\pi - 2\pi\sqrt{1+n^2}\right)}{2n\pi - 2\pi\sqrt{1+n^2}} 2n\pi\left(n-\sqrt{1+n^2}\right)$$

$$= \lim_{n\to\infty} \frac{\sin\left(2n\pi - 2\pi\sqrt{1+n^2}\right)}{2n\pi - 2\pi\sqrt{1+n^2}} \frac{2n\pi}{n+\sqrt{1+n^2}}$$

$$= \lim_{n\to\infty} \frac{\sin\left(2n\pi - 2\pi\sqrt{1+n^2}\right)}{2n\pi - 2\pi\sqrt{1+n^2}} \frac{2\pi}{1+\sqrt{\frac{1}{n^2}+1}} = \pi$$

(C) $\operatorname{Lim}_{n\to\infty} (-1)^n \sin\left(\pi\sqrt{n^2+0.5n+1}\right)$

$$= -\operatorname{Lim}_{n\to\infty} \pi\left(n-\sqrt{n^2+0.5n+1}\right)\frac{\sin\left(n\pi - \pi\sqrt{n^2+0.5n+1}\right)}{\pi n - \pi\sqrt{n^2+0.5n+1}}$$

$$= \operatorname{Lim}_{n\to\infty} \left(\frac{0.5n+1}{n+\sqrt{n^2+0.5n+1}}\right)\frac{\sin\left(\pi - \pi\sqrt{n^2+0.5n+1}\right)}{\pi - \pi\sqrt{n^2+0.5n+1}}$$

$$= \operatorname{Lim}_{n\to\infty} \left(\frac{\frac{1}{2}+\frac{1}{n}}{1+\sqrt{1+\frac{1}{2n}+\frac{1}{n^2}}}\right)\frac{\sin\left(\pi - \pi\sqrt{n^2+0.5n+1}\right)}{\pi - \pi\sqrt{n^2+0.5n+1}} = \frac{1}{2}.$$

(D) $\lim_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{x+a}{x-a}-1\right)} = e^{\lim_{x\to\infty}\left(\frac{2ax}{x-a}\right)} = e^{2a} \Rightarrow a = \dfrac{1}{2}$

27. (A) $\rightarrow$ (S), (B) $\rightarrow$ (P), (C) $\rightarrow$ (Q), (D) $\rightarrow$ (R)

(A) $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}}\right) = \lim_{x\to\infty}\left(\dfrac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\right)$

$$= \lim_{x\to\infty}\left(\frac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}}\right) = 1$$

(B) $\lim_{x\to 0}\dfrac{\sin 2x-2\tan x}{\ln(1+x^3)}=\lim_{x\to 0}\dfrac{2\sin x\cos x-2\dfrac{\sin x}{\cos x}}{\ln(1+x^3)}$

$=-\lim_{x\to 0}\dfrac{2\sin^3 x}{\cos x\ln(1+x^3)}=-2\lim_{x\to 0}\dfrac{\dfrac{\sin^3 x}{x^3}}{\cos x\dfrac{\ln\left(1+x^3\right)}{x^3}}=-2$.

(C) $\lim_{x\to 0^+}\left(\ln\sin^3 x-\ln\left(x^4+ex^3\right)\right)=\lim_{x\to 0^+}\left(3\ln\sin x-3\ln x-\ln\left(x+e\right)\right)$

$=\lim_{x\to 0^+}\left(3\ln\dfrac{\sin x}{x}-\ln\left(x+e\right)\right)=-1$

(D) $\tan\left(2\pi|\sin\theta|\right)=\cot\left(2\pi|\cos\theta|\right)\Rightarrow\tan\left(2\pi|\sin\theta|\right)=\tan\left(\dfrac{\pi}{2}-2\pi|\cos\theta|\right)$

$\Rightarrow 4|\sin\theta|=2n+1-4|\cos\theta|$

$\Rightarrow|\sin\theta|+|\cos\theta|=\dfrac{2n+1}{4}$.

Now range of $|\sin\theta|+|\cos\theta|$ is $\left[1,\sqrt{2}\right]$ hence $\dfrac{2n+1}{4}=\dfrac{5}{4}$

Further $f\left(x\right)=\left(\dfrac{5}{4}\right)^x$

$\Rightarrow\lim_{x\to\infty}\left[\dfrac{2}{f\left(x\right)}\right]=\lim_{x\to\infty}\left[2\left(\dfrac{4}{5}\right)^x\right]=0$.

28. (A)$\to$(S); (B)$\to$(R); (C)$\to$(P); (D)$\to$(Q); (E)$\to$(P)

(A) $\lim_{x\to 1}\dfrac{\ln x}{x^4-1}=\lim_{x\to 1}\dfrac{\ln x}{x-1}\dfrac{1}{\left(x^2+1\right)\left(x+1\right)}=\dfrac{1}{4}$

(B) $\lim_{x\to 0}\dfrac{3e^x-x^3-3x-3}{\tan^2 x}=\lim_{x\to 0}\dfrac{3\left(1+x+\dfrac{x^2}{2}\right)-x^3-3x-3}{\tan^2 x}$

$=\lim_{x\to 0}\dfrac{\dfrac{3x^2}{2}-x^3}{\tan^2 x}=\lim_{x\to 0}\dfrac{\dfrac{3}{2}-x}{\dfrac{\tan^2 x}{x^2}}=\dfrac{3}{2}$

(C) $\lim_{x\to\infty}\dfrac{\pi-2\tan^{-1}x}{\ln\left(1+\dfrac{1}{x}\right)}=2\lim_{x\to\infty}\dfrac{\cot^{-1}x}{\ln\left(1+\dfrac{1}{x}\right)}$

$$= 2\lim_{x\to\infty} \frac{\dfrac{\tan^{-1}\frac{1}{x}}{\frac{1}{x}}}{\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = 2$$

(D) $$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x(\cos x - \cos 2x)} = -\lim_{x\to 0} \frac{2\sin x(1-\cos x)}{x(2\cos^2 x - \cos x - 1)}$$

$$= -\lim_{x\to 0} \frac{2\sin x(1-\cos x)}{x(2\cos x + 1)(\cos x - 1)} = 2$$

(E) $$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) - 2x}{x - \left(x - \frac{x^3}{6}\right)}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3}}{\frac{x^3}{6}} = 2.$$

29. (A)$\to$(S); (B)$\to$(R); (C)$\to$(Q); (D)$\to$(P)

(A) $$\lim_{x\to\infty} \left(\frac{x}{1+x}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{x}{1+x}-1\right)} = e^{\lim_{x\to\infty}\left(\frac{-x}{1+x}\right)} = \frac{1}{e}$$

(B) $$\lim_{x\to\infty} \left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} x\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)} = e^{\lim_{x\to\infty}\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}} + \frac{\cos\frac{1}{x}-1}{\frac{1}{x}}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}} - \left(\frac{2\sin^2\frac{1}{2x}}{\frac{1}{4x^2}}\right)\frac{1}{4x}\right)} = e$$

(C) $$\lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{\lim_{x\to 0}\cot^2 x(\cos x - 1)} = e^{-\lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{\tan^2 x}\right)}$$

$$= e^{-\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}\frac{x^2}{\tan^2 x}\right)} = e^{-\frac{1}{2}}$$

(D) $$\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\left(\tan\left(\frac{\pi}{4}+x\right)-1\right)} = e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{1+\tan x}{1-\tan x}-1\right)} = e^{\lim_{x\to 0}\frac{2}{1-\tan x}\left(\frac{\tan x}{x}\right)} = e^2.$$

LIMITS
EXERCISE 2(B)

1. Since the function is conti

$VF|_{x=0} = RHL|_{x=0} = LHL|_{x=0}$

$f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\left(\frac{\sin 3x + A\sin 2x + B\sin x}{x^5}\right)$$

$$= \lim_{x\to 0}\left(\frac{3\cos 3x + 2A\cos 2x + B\cos x}{5x^4}\right); \left(\frac{3+2A+B}{0}\right) \text{ form}$$

$3 + 2A + B = 0$ ...(1)

$$= \lim_{x\to 0}\left(\frac{-9\sin 3x - 4A\sin 2x - B\sin x}{20x^3}\right); \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\left(\frac{-27\cos 3x - 8A\cos 2x - B\cos x}{60x^2}\right); \left(\frac{-27-8A-B}{0}\right) \text{ form}$$

$27 + 8A + B = 0$ ...(2)

$$= \lim_{x\to 0}\left(\frac{81\sin 3x + 16A\sin 2x + B\sin x}{120x}\right); \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\left(\frac{243\cos 3x + 32A\cos 2x + B\cos x}{120}\right)$$

$$f(0) = \frac{243 + 32A + B}{120} \quad \text{...(3)}$$

using (1) & (2)

$$\boxed{A = -4, B = 5}$$

then

$f(0) = 1$

2. $\therefore$ $$100\, l = \lim_{x\to 0}\frac{x^m - (\sin x)^m}{x^2 \cdot (\sin x)^m} = \lim_{x\to 0}\frac{x^m\left[1 - \left(\frac{\sin x}{x}\right)^m\right]}{(\sin x)^m x^2} = \lim_{x\to 0}\frac{1 - \left(\frac{\sin x}{x}\right)^m}{x^2}$$

$$\left(\text{as } \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^m = 1\right)$$

Using L'Hospital's Rule

$$100l = \lim_{x\to 0} \frac{-D\left(\frac{\sin x}{x}\right)^m}{2x}$$

now let $y = \left(\frac{\sin x}{x}\right)^m$; $\frac{dy}{dx} = m\left(\frac{\sin x}{x}\right)^{m-1}\left[\frac{x\cos x - \sin x}{x^2}\right]$

$$\therefore \quad 100l = -\lim_{x\to 0} \frac{6000\cdot\cos x\cdot(x-\tan x)}{2x^3} = 1000 \qquad \left(\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{m-1} = 1\right)$$

using expansion of $\tan x$ we get $\boxed{l = 10}$

3. Using L'Hospital's Rule, $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{3x^2}$; $\frac{\frac{x}{\sqrt{x^2+1}}}{6x} = \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{6x} = \frac{1}{6} = \frac{p}{q}$

$\therefore \left|p^2 - q^2\right| = 35$.

Alternate :

$$\text{Let } x = \tan\theta, \text{ then } L = \lim_{\theta\to 0}\frac{\tan\theta + \ln(\sec\theta - \tan\theta)}{\tan^3\theta}$$

$$L = \lim_{\theta\to 0}\frac{\tan\theta + \ln(1-\sin\theta) - \frac{1}{2}\ln(1-\sin^2\theta)}{\tan^3\theta}$$

$$L = \lim_{\theta\to 0}\frac{\sin\theta - \left(\sin\theta + \frac{\sin^2\theta}{2} + \frac{\sin^3\theta}{3} - - -\right)\cos\theta + \frac{1}{2}\left(\sin^2\theta + \frac{\sin^4\theta}{2} + - - -\right)\cos\theta}{\tan^3\theta\cos\theta}$$

$$L = \lim_{\theta\to 0}\left(\frac{\sin\theta(1-\cos\theta)}{\tan^3\theta\cos\theta} - \frac{\sin^3\theta}{3\tan^3\theta\cos\theta}\right) \text{ ....rest of the terms have degree} > 3$$

$$L = \lim_{\theta\to 0}\left(\left(\frac{\sin\theta}{\tan\theta\cos\theta}\right)\left(\frac{\sin^2\frac{\theta}{2}}{\frac{\tan^2\theta}{4}}\right)\frac{1}{2} - \frac{\sin^3\theta}{3\tan^3\theta\cos\theta}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

4. Let radius of circle is r &

$\angle AOB = 2O$

$\angle ACB = \pi - 2\theta$

($\because$ points A, C, B, O are concyclic)

$\therefore \angle AOP = \angle BOP = \theta$

$\angle$ ACP = $\angle$BCP = $\frac{\pi}{2} - \theta$

In ΔAOP,

$\sin\theta = \frac{AP}{OA} = \frac{AP}{r}$

$\Rightarrow AP = r \sin\theta$

$\cos\theta = \frac{OP}{OA} = \frac{OP}{r}$

$\Rightarrow OP = r \cos\theta$

$AB = 2AP = 2r \sin\theta$

in ΔAOC,

$\sin\left(\frac{\pi}{2} - \theta\right) = \frac{OA}{OC}$

$\Rightarrow \cos\theta = \frac{r}{OC}$

$= \frac{r}{\cos\theta}\ r \cos\theta$

$= r\left(\frac{1}{\cos\theta} - \cos\theta\right)$

$= \frac{r}{\cos\theta}\ (1 - \cos^2\theta)$

$\therefore\ PC = \frac{r}{\cos\theta}\sin^2\theta$

$\therefore\ ar(\Delta ABC) = \frac{1}{2}\ AB.\ PC$

$= \frac{1}{2}\ 2r \sin\theta\ .\ \frac{r}{\cos\theta}.\sin^2\theta$

$= r^2 \tan\theta\ .\ \sin^2\theta$

$\because\ OC = \frac{r}{\cos\theta}\ \ OR = r$

RC = OC – OR

$$= \frac{r}{\cos\theta} - r$$

$$= \frac{r}{\cos\theta}(1-\cos\theta)$$

In ΔDRC,

$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{DR}{RC}$$

$$\Rightarrow \cot\theta = \frac{DR}{RC} \Rightarrow DR = RC\cos\theta$$

$$= \frac{r}{\cos\theta}\ (1-\cos\theta).\ \cot\theta$$

$$= \frac{r}{\cos\theta}(1-\cos\theta).\frac{\cos\theta}{\sin\theta} = \frac{r(1-\cos\theta)}{\sin\theta}$$

DR + RE = DE $\Rightarrow$ DE = 2DR

$$\Rightarrow OC = \frac{r}{\cos\theta}$$

PC = OC – OP $\Rightarrow$ DE = 2DR

$$= \frac{2r}{\sin\theta}(1-\cos\theta)$$

$$\text{are }(\Delta DEC) = \frac{1}{2}\times DE\times RC$$

$$= \frac{1}{2}\frac{2r(1-\cos\theta)}{\sin\theta}.\frac{r(1-\cos\theta)}{\cos\theta}$$

$$= \frac{r^2(1-\cos\theta)(1-\cos\theta)}{\sin\theta.\cos\theta}$$

$$= \frac{r^2.2\sin^2\frac{\theta}{2}\cdot 2\sin^2\frac{\theta}{2}}{\sin\theta.\cos\theta}$$

$$= \frac{4r^2\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}{\sin\theta.\cos\theta} = \frac{4r^2\sin^4\frac{\theta}{2}}{\sin\theta\cos\theta}$$

AB = 2r sinθ

If AB $\to 0$ i.e. $2\sin\theta \to 0 \Rightarrow \theta \to 0$

$$\therefore \lim_{AB\to 0} \frac{ar(\Delta ABC)}{ar(\Delta CDE)}$$

$$= \lim_{\theta\to 0} \frac{r^2 \tan\theta.\sin^2\theta}{\dfrac{4r^2 \sin\theta/2}{\sin\theta.\cos\theta}}$$

$$= \lim_{\theta\to 0} \frac{\tan\theta.\sin^3\theta.\cos\theta}{4\sin^4\theta/2}$$

$$= \lim_{\theta\to 0} \frac{\dfrac{\tan\theta.\sin^3\theta.\cos\theta}{\theta^4}}{4\dfrac{\sin^4\theta/2}{\theta^4}}$$

$$= \lim_{\theta\to 0} \frac{\left(\dfrac{\tan\theta}{\theta}\right)\cdot\left(\dfrac{\sin\theta}{\theta}\right)^3\cdot\cos\theta}{4\left(\dfrac{\sin\theta/2}{\theta/2}\right)^4\cdot\dfrac{1}{2^4}}$$

$$= \frac{1.1.1}{4.1.1/16} = 4$$

5. $$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$$

$$\lim_{x\to 0} \frac{a\left[1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+.....\right]-b\left[1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-.....\right]+c\left[1-\dfrac{x}{1!}+\dfrac{x^2}{2!}-.....\right]}{x^2\left[\dfrac{\sin x}{x}\right]}$$

(i) $a - b + c = 0$ = constant

coeff. of x = (ii) $a - c = 0 \Rightarrow a = c$

coeff. of x^2 = (iii) $\frac{a}{2}+\frac{b}{2}+\frac{c}{2} = 2 \Rightarrow a + b + c = 4$

$2(a + c) = 4 \Rightarrow a - c = 2$

$\boxed{a = 1 = c}$ $\boxed{b = 2}$

6. $$L = \left(1-\frac{4}{3^2}\right)\left(1-\frac{4}{4^2}\right)\left(1-\frac{4}{5^2}\right)...........$$

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{2}{6}\right) \qquad \left(1+\frac{2}{3}\right)\left(1+\frac{2}{4}\right)-\left(1+\frac{2}{5}\right)$$

$\frac{1}{3}\times\frac{2}{4}\times\frac{3}{5}\times\frac{4}{6}$ .... $\left(\frac{5}{3}\right)\left(\frac{6}{4}\right)\left(\frac{7}{5}\right)$

$= \frac{1}{3}\times\frac{2}{4} = \left(\frac{1}{6}\right)$

$$M = \frac{2^3-1}{2^3+1}.\frac{3^3-1}{3^3+1}.\frac{4^3-1}{4^3+1}........$$

$$\frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \left(\frac{2-1}{2+1}.\frac{3-1}{3+1}.\frac{4-1}{4+1}.\frac{5-1}{5+1}.....\right)$$

$$\left(\frac{2^2+2+1}{2^2-2+1}.\frac{3^2+3+1}{3^2-3+1}\right).....$$

$$\Rightarrow \frac{102}{(n+1)_n}.\frac{n^2+n+1}{3} = \frac{2}{3}$$

7. $\frac{1}{1+n^2} \le \frac{1}{1+n^2} \le \frac{1}{1+n^2}$

$$\frac{2}{n+n^2} \le \frac{2}{2+n^2} \le \frac{2}{1+n^2}$$

$$\frac{3}{n+n^2} \le \frac{3}{n+n^2} \le \frac{3}{1+n^2}$$

$$\frac{n(n+1)}{2(n+n^2)} \le Sn \le \frac{n(n+1)}{2(1+n^2)}$$

(a) $\frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2}} \le \frac{1}{\sqrt{n^2}}$

$$\frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2+1}} \le \frac{1}{\sqrt{n^2}}$$

$$\frac{2n}{\sqrt{n^2+2n}} \le Sn \le \frac{2n}{\sqrt{n^2}}$$

t = 2 .

8. $l = \lim_{x\to 0^+} x^{(x^x-1)}$ (0^0 form)

$\ln l = \lim_{x\to 0} (x^x-1).\ln x = \text{Limit}_{x\to 0}\frac{(e^{x\ln x}-1)}{x\ln x}\ \text{Limit}_{x\to 0}\ x\ln x\ .\ln x$

$= \text{Limit}_{x\to 0}\ x(\ln x)^2$ (as $x \to 0$ $x\ln x \to 0$)

$$= \underset{x\to 0}{\text{Limit}} \frac{(\ln x)^2}{1/x} = \underset{x\to 0}{\text{Limit}} -\frac{2\ln x}{x}.x^2 \quad \text{(use Lopital's rule)}$$

$$= \underset{x\to 0}{\text{Limit}} -2\ln x.x = 0 \quad \Rightarrow \quad l = e^0 = 1$$

9. Let $U_n = \underset{x\to 0}{\text{Lim}} \frac{1-\cos 3x\cdot\cos 3^2x\cdot\cos 3^3x.........\cos 3^n x}{x^2}$

and $V_n = \underset{x\to 0}{\text{Lim}} \frac{1-\cos\frac{x}{3}\cdot\cos\frac{x}{3^2}\cdot\cos\frac{x}{3^3}.........\cos\frac{x}{3^n}}{x^2}$

$$U_n = \underset{x\to 0}{\text{Lim}} \frac{-D(\cos 3x\cdot\cos 3^2x\cdot\cos 3^3x.........\cos 3^n x)}{2x}$$

now let $y = \cos 3x \cdot \cos 3^2x \cdot \cos^3 x \ .... \cos 3^n x$

$ln\, y = ln \cos 3x + ln \cos 3^2x + ... + ln \cos 3^n x$

$$\frac{1}{y}\frac{dy}{dx} = -[3\tan 3x + 3^2\tan 3^2x + ... + 3^n\tan 3^n x]$$

$$\frac{dy}{dx} = -\prod_{r=1}^{n}\cos 3^r x[3\tan 3x + 3^2\tan 3^2x + ... + 3^n\tan 3^n x]$$

$$\therefore \quad U_n = \underset{x\to 0}{\text{Lim}} \frac{3\tan 3x + 3^2\tan 3^2x + ... + 3^n\tan 3^n x}{2x} = \frac{3^2 + (3^2)^2 + (3^3)^2 + ... + (3^n)^2}{2}$$

$$U_n = \frac{3^2[3^{2n}-1]}{(3^2-1)\cdot 2} \quad ....(1)$$

|||ly replacing 3^r by $\frac{1}{3^r}$ we get

$$V_n = \frac{\frac{1}{3^2}\left[1-\frac{1}{3^{2n}}\right]}{\left(1-\frac{1}{3^2}\right)\cdot 2} = \frac{(3^{2n}-1)}{3^{2n}(3^2-1)\cdot 2} \quad ....(2)$$

$$\therefore \quad \frac{U_n}{V_n} = 3^{2n+2} = 3^{10} \text{ (given)}$$

$$\therefore \quad 2n + 2 = 10 \quad \Rightarrow \quad \boxed{n = 4}$$

10. $\lim_{x\to 0}\frac{1}{x^3}\left[\frac{1}{\sqrt{1+x}} - \frac{1+ax}{1+bx}\right]$

$$= \lim_{x\to 0}\frac{1}{x^3}\left[\frac{1+bx-(1+ax)\sqrt{1+x}}{\sqrt{1+x}\ \ (1+bx)}\right]$$

$$= \lim_{x\to 0}\frac{1+bx-(1+ax)\ (1+x)^{1/2}}{x^3(1+x)^{1/2}(1+bx)}$$

We know that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)\ (n-2)}{3!}x^3+....$$

where, $n \in Q$ & $|x|<1$

$$= \lim_{x\to 0}\frac{1+bx-(1+ax)\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+.......\right]}{x^3(1+x)^{1/2}\ \ (1+bx)}$$

$$\lim_{x\to 0}\frac{1+bx-1-\frac{1}{2}x-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3-...-ax\left(1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+......\right)}{x^3(1+x)^{1/2}(1+bx)}$$

$$=\lim_{x\to 0}\frac{bx-\frac{1}{2}x-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3-...-ax\left(1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+.....\right)}{x^3(1+x)^{1/2}(1+bx)}$$

$$=\lim_{x\to 0}\frac{b-\frac{1}{2}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-x\right)x^2}{3!}-.....-a\left(1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+...\right)}{x^2(+x)^{1/2}(1+bx)}$$

$$= \lim_{x\to 0}\frac{\left(b-a-\frac{1}{2}\right)-\left(\frac{a}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\right)x-\left(\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}+a.\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\right)x^2-(\ )x^3-(\ )x^4-.......}{x^2.1.1}$$

Limit exists finitaly if

(i) $b-a-\frac{1}{2}=0$

(ii) $\frac{a}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}=0$

$\Rightarrow \frac{a}{2}-\frac{\frac{1}{2}.\frac{1}{2}}{2}=0$

$\Rightarrow a-\frac{1}{4}=0$

$\Rightarrow\ a=\frac{1}{4}$

$\because b-a=\frac{1}{2}$

$\Rightarrow b=a+\frac{1}{2}$

$=\frac{3}{4}$

$a=\frac{1}{4}$ & $b=\frac{3}{4}$

$$=\lim_{x\to 0}\frac{\left(-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}a-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\right)x^2-(\ )x^3-(\ )x^4-......}{x^2}$$

$$=\lim_{x\to 0}\left[\left(\frac{a}{8}-\frac{\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{2.3}\right)-(\ )x-(\ )x^2-......\right]$$

$$=\frac{1}{32}-\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{1}{2}=-\frac{1}{16}+\frac{1}{32}=-\frac{1}{32}$$

$$\Rightarrow\ \lim_{x\to 0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)=-\frac{1}{32}$$

$$\therefore\ \ell=-\frac{1}{32}$$

$$\therefore \frac{1}{a}-\frac{2}{\ell}+\frac{3}{b}$$

$$=\frac{1}{\frac{1}{4}}-\frac{2}{\frac{-1}{32}}+\frac{3}{\frac{3}{4}}$$

$= 4 + 2.32 + 4$

$= 4 + 64 + 4$

$= 72$

11. Clearly from $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ & $x^2 + (a+b)x + 36 = 0$, the roots are (3 & 4), (3 & 5) and (3 & 12) respectively.
Hence the common root is 3.

12. $$\lim_{x\to\frac{\pi}{3}} \frac{3\tan x - \tan^3 x}{3\cos\left(x+\frac{\pi}{6}\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\left(3\cos^2 x - \sin^2 x\right)\sin x}{3\cos\left(x+\frac{\pi}{6}\right)\cos^3 x}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\left(3-4\sin^2 x\right)\sin x}{3\cos\left(x+\frac{\pi}{6}\right)\cos^3 x} = \lim_{x\to\frac{\pi}{3}} \frac{\sin 3x}{3\cos\left(x+\frac{\pi}{6}\right)\cos^3 x}$$

Now let $x = y + \frac{\pi}{3}$, then $L = \lim_{y\to 0} \frac{\sin 3y}{3\sin y\cos^3\left(y+\frac{\pi}{3}\right)}$

$$L = \lim_{y\to 0} \frac{\frac{\sin 3y}{3y}}{\frac{\sin y}{y}\cos^3\left(y+\frac{\pi}{3}\right)} = 8.$$

13. $$\left[\frac{4}{x^3 - x^{-1}} - \frac{1-3x+x^2}{1-x^3}\right]^{-1} + \frac{3.\left(x^4-1\right)}{x^3 - x^{-1}}$$

$$= \lim_{x\to 1}\left[\left[\frac{4x}{x^3-1} + \frac{1-3x+x^2}{x^3-1}\right]^{-1} + \frac{3x\left(x^4-1\right)}{x^4-1}\right]$$

$$= \lim_{x\to 1}\left[\left[\frac{1+x+x^2}{x^3-1}\right]^{-1} + 3x\right] = \lim_{x\to 1}\left[x-1+3x\right] = 3.$$

14. $$\lim_{x\to 0}\frac{x\left(1-m\cos x\right)+n\sin x}{x^3}=\lim_{x\to 0}\frac{x\left(1-m\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-...\right)\right)+n\left(x-\frac{x^3}{6}+\frac{x^5}{5!}-...\right)}{x^3}=1$$

Now $n-m+1=0$ & $\frac{m}{2}-\frac{n}{6}=1$

$\Rightarrow m=\frac{5}{2}$ & $n=\frac{3}{2}$

Hence m + n = 4.

15. $$x+\cfrac{x^{1/3}}{x+\cfrac{x^{1/3}}{x+....\infty\text{ terms}}}=y\Rightarrow x+\frac{x^{1/3}}{y}$$

$$\Rightarrow y^2-xy-x^{1/3}=0\Rightarrow y=\frac{x+\sqrt{x^2+4x^{1/3}}}{2}$$

$$\Rightarrow \lim_{x\to\infty}\cfrac{x}{x+\cfrac{x^{1/3}}{x+\cfrac{x^{1/3}}{x+....\infty\text{ terms}}}}=\lim_{x\to\infty}\frac{2x}{x+\sqrt{x^2+4x^{1/3}}}=2$$

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LIMITS

EXERCISE - 3

1. $$f(x)=\begin{cases}\dfrac{x}{\sin x} & , \quad x>0\\ 2-x & ; \quad x\le 0\end{cases}$$

$$g(x)=\begin{cases}x+3 & , \quad x<0\\ x^2-2x-2 & ; \quad 1\le x<2\\ x-5 & ; \quad x\ge 2\end{cases}$$

$$g(f(x))=\begin{cases}f(x)+3 & , \quad f(x)<1\\ f^2(x)-2f(x)-2 & ; \quad 1\le f(x)<2\\ f(x)-5 & ; \quad f(x)\ge 2\end{cases}$$

$\boxed{b=t}$

$$=\begin{cases}\dfrac{x}{\sin x}+3 & ; \quad x>0\cap \dfrac{x}{\sin x}<1\\ 2-x+3 & ; \quad x\le 0\cap 2x<1\\ \left(\dfrac{x}{\sin x}\right)^2-2\left(\dfrac{x}{\sin x}\right)-2 & ; \quad x>0\cap 1\le \dfrac{x}{\sin x}<2\\ (2-x)^2-2(2-x)-2 & ; \quad x\le 0\cap 1\le 2-x<2\\ \dfrac{x}{\sin x}-5 & ; \quad x>0\cap \dfrac{x}{\sin x}\ge 2\\ 2-x-5 & ; \quad x\le 0\cap 2-x\ge 2\end{cases}$$

$$=\begin{cases}\phi\\ \phi\\ \left(\dfrac{x}{\sin x}\right)^2-2\left(\dfrac{x}{\sin x}\right)-2 & ; \quad x>0\cap 1\le \dfrac{x}{\sin x}<2\\ \phi\\ \dfrac{x}{\sin x}-5 & ; \quad x>0\cap \dfrac{x}{\sin x}\ge 2\\ 2-x-5 & ; \quad x\le 0\end{cases}$$

$$\Rightarrow g(f(x)) = \begin{cases} -x-3 & ; \quad x \le 0 \\ \left(\frac{x}{\sin x}\right)^2 - 2\left(\frac{x}{\sin x}\right) - 2 & ; \quad 1 \le \frac{x}{\sin x} < 2 \cap x > 0 \\ \frac{x}{\sin x} - 5 & ; \quad x > 0 \cap \frac{x}{\sin x} \ge 2 \end{cases}$$

$\because$ gf(0) = – 0 – 3

= – 3

$$\text{LHL} = \lim_{x \to 0^-} g(f(x))$$

$$= \lim_{x \to 0^-} -(x + 3)$$

$$= -0 - 3 = -3$$

$$\text{RHL} = \lim_{x \to 0^+} g(f(x))$$

$$= \lim_{x \to 0^+} \left[\left(\frac{x}{\sin x}\right)^2 - 2\left(\frac{x}{\sin x}\right) - 2\right]$$

$$= \lim_{x \to 0^+} \left[\frac{1}{\left(\frac{\sin x}{x}\right)^2} - \frac{2}{\left(\frac{\sin x}{x}\right)} - 2\right]$$

$$= \lim_{x \to 0^+} \frac{1}{\left(\frac{\sin x}{x}\right)^2} - \lim_{x \to 0^+} \frac{2}{\left(\frac{\sin - x}{x}\right)} - 2$$

$$= 1 - 2 - 2$$

$$= -3$$

$\because$ LHL = RHL = – 3

$$\therefore \lim_{x \to 0} g(f(x)) = -3$$

2. Given $p_n = a^{p_{n-1}} - 1$; $p_{n-1} = a^{p_{n-2}} - 1$

Let $p_1 = a^x - 1$

$$\lim_{x \to 0} \frac{p_n}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{a^{p_{n-1}} - 1}{x} \times \frac{p_{n-1}}{p_{n-1}}$$

$$\Rightarrow \lim_{x\to 0} \ell na \times \frac{p_{n-1}}{x}$$

$$\Rightarrow \lim_{x\to 0} \ell na \times \frac{a\,p_{n-1}-1}{x} \times \frac{p_{n-2}}{p_{n-2}}$$

$$\Rightarrow \lim_{x\to 0} (\ell na)^2 \times \frac{p_{n-2}}{x}$$

$$\Rightarrow \lim_{x\to 0} (\ell na)^{n-1} \times \frac{p_1}{x}$$

$$\Rightarrow (\ell na)^{n-1} \times \lim_{x\to 0} \frac{a^x - 1}{x}$$

$$\Rightarrow (\ell na)^n$$

3. $x^3 - (2x + 1)x^2 + (2x - 1)\,x + 1 = 0$ ...(1)

roots of equation a_n, b_n, c_n $\quad a_n < b_n < c_n$

$x = 1$ is a root of equation (1)

so $(x - 1)(x^2 - 2nx - 1) = 0$

$$x = \frac{2n \pm \sqrt{4x^2 + 4.1}}{2.1}$$

$$x = n \pm \sqrt{n^2 + 1}$$

$$x = n \pm \sqrt{n^2 + 1}$$

$$x = n \pm \sqrt{n^2 + 1}$$

$$\begin{array}{l} (x-1)\overline{\big)\,x^3 - (2x+1)x^2 + (2n-1)x + 1} \\ \quad\quad x^3 - x^2 \\ \quad\quad - \;\; + \;\; 1 \\ \hline \quad\quad -2nx^2 + (2n-1)x + 1 \\ \quad\quad -2nx^2 + 2nx \\ \quad\quad + \quad\quad - \\ \hline \quad\quad\quad -x + 1 \\ \quad\quad\quad -x + 1 \\ \hline \end{array}$$

so the three roots of cubic equation.

$$n + \sqrt{n^2 + 1},\, n - \sqrt{n^2 + 1},\, 1$$

so $\lim_{n\to\infty} n\, a_n \Rightarrow \lim_{n\to\infty} n(n-\sqrt{n^2+1})\times\frac{n+\sqrt{n^2+1}}{n+\sqrt{n^2+1}}$

$$\Rightarrow \lim_{n\to\infty}\frac{n(n^2-n^2-1)}{n+\sqrt{n^2+1}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{-n}{n\left(1+\sqrt{1+\frac{1}{n^2}}\right)}=-\frac{1}{2}.$$

4. $a_n = 2^2\ [1^2 + 2^2 + 3^2 + ...n^2]$

$$a_n = 2^2\left[\frac{n(n+1)\ (2n+1)}{6}\right]$$

$$a_n + b_n = \frac{2n(2n+1)\ (4n+1)}{6}$$

$$b_n = \frac{2n(2n+1)\ (4n+1)}{6} - \frac{4(n(n+1))\ (2n+1)}{6}$$

$$b_n = \frac{n(2n+1)}{6}[2(4n+1)-4(n+1)]$$

$$b_n = \frac{n(2n+1)}{6}\ [4n-2] = \frac{n(2n+1)(2n-1)}{3}$$

$$\lim_{n\to\infty}\frac{\sqrt{a_n}-\sqrt{b_n}}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{\sqrt{\frac{2^n n(n+1)\ (2n+1)}{6}}}{\sqrt{n}}-\frac{\sqrt{\frac{n(n+1)\ (2n+1)}{2}}}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{3}}\left(\sqrt{4n^2+6n+2}-\sqrt{4n^2-1}\right) \text{ by ratio radization}$$

$$\lim_{n\to\infty} = \frac{\frac{1}{\sqrt{3}}(6n+2+1)}{\sqrt{4n^2+6n+2}+\sqrt{4n^2-1}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{3}}\frac{6+\frac{3}{n}}{\sqrt{4+\frac{6}{n}+\frac{2}{n^2}}+\sqrt{\frac{4n^2-1}{n^2}}}$$

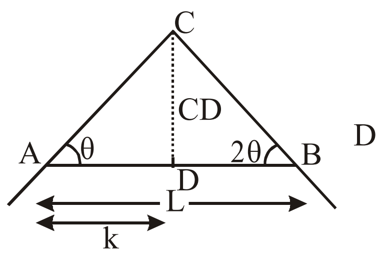
$$=\frac{1}{\sqrt{3}}\cdot\frac{6}{2+2}$$

$$=\frac{\sqrt{3}}{2}$$

5. $\tan\theta = \frac{CD}{x}$

$\tan 2\theta = \frac{CD}{L-x}$

$x\tan\theta = (L-x)\ \frac{2\tan\theta}{1-\tan^2\theta}$



$x = \frac{2(L-x)}{1-\tan^2\theta}$

$x(1-\tan^2\theta) = 2L - 2x$

$x(2 + 1 - \tan^2\theta) = 2L$

$x = \frac{2L}{3-\tan^2\theta}$, $\tan\theta \to 0$

$\theta \to 0$ gives $\boxed{x = \frac{2L}{3}}$

6. Put $\frac{x}{1+x} = y \Rightarrow y \to 1$ as $x \to \infty$

$1 - \frac{1}{1+x} = y \Rightarrow 1 + x = \frac{1}{1-y} \Rightarrow x = \frac{1}{1-y} - 1 \Rightarrow x = \frac{y}{1-y}$

$$e^{\lim_{y\to 1}\left[\cos 2\pi y^a - 1\right]\left(\frac{y}{1-y}\right)^2} = e^{\lim_{y\to 1}(-2\sin^2\pi y^a)\left(\frac{y^2}{(1-y)^2}\right)}$$

$$e^{\lim_{y\to 1}-2\left[\frac{\sin^2\pi(1-y^a)}{\pi^2(1-y^a)^2}\cdot\frac{\pi^2(1-y^a)^2}{(1-y)^2}\cdot y^2\right]}$$

$e^{-2\pi^2\left(\frac{1-y^a}{1-y}\right)^2 . y^2 = e^{-2\pi^2 a^2}}$ $\left[\because \lim_{x\to 1}\frac{x^n-1}{x-1}=n\right]$

Alt.: Let limit = e^L

Let $= \frac{x}{1+x} = t$

$$L = \lim_{t\to 1}\left[\frac{\cos(2\pi - 2\pi t^a)-1}{(2\pi(1-t^a))^2}\right]\frac{t^2}{(1-t)^2}\frac{4\pi^2(1-t^a)^2}{a^2}$$

$$= \left(\frac{-1}{2}\right)4\pi^2(a^2) \quad e^L = e^{-2\pi^2 a^2}$$

7. $\lim_{x\to 0}\left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$

$$\lim_{x\to 0}\left(\frac{x-1+1-\frac{x^2}{2!}+\frac{x^4}{4!}+.....}{x}\right)^{1/x}$$

$$\lim_{x\to 0}\left(1-\frac{x}{2}+\frac{x^3}{4!}.........\right)^{1/x} \Rightarrow e^{\lim_{x\to 0}\left(\frac{-x}{2}+\frac{x^2}{4!}+.........\right)\frac{1}{x}} = e^{-1/2}$$

Alt: $e^{\lim_{x\to 0}\left(\frac{x-1+\cos x - x}{x}\right)\frac{1}{x}} \Rightarrow e^{-1/2}$

$$e^{\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{\frac{4x^2}{4}}} = e^{-1/2}$$

8. $\lim_{x\to\infty}\left(\frac{a_1^{\left(\frac{1}{x}\right)}+a_2^{\left(\frac{1}{x}\right)}+a_3^{\left(\frac{1}{x}\right)}+.....+a_n^{\left(\frac{1}{x}\right)}}{n}\right)^{nx}$

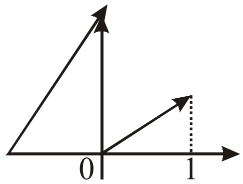
$$e^{\lim_{x\to\infty}\frac{a_1^{1/x}-1}{1/x}+\frac{a_2^{1/x}-1}{1/x}+....+\frac{a_n^{1/x}-1}{1/x}}$$

$$e^{(\ell n a_1)+(\ell n a_2)+...+(\ell n a_n)} = e^{\ell n(a_1.a_2......a_n)}$$

$= a_1.a_2.......a_n.$

9. $\displaystyle \lim_{x\to 0^+} \frac{\sin^{-1}(1-\{x\})\cos^{-1}\left(1-\{x\}\right)}{\sqrt{2}\{x\}.(1-\{x\})}$

$\{x\} \to 0$ for $x \to 0^+$



$\displaystyle \lim_{x\to 0^+} \frac{\sin^{-1}(1-\{x\})\sin^{-1}\sqrt{1-(1-\{x\})^2}}{\sqrt{2}\{x\}(1-\{x\})}$ $[1 - 1 - \{x\}^2 + 2\{x\}\}$

$\displaystyle \lim_{x\to 0^+} \frac{\sin^{-1}(1-\{x\})\sin^{-1}\sqrt{\{x\}(2-\{x\})}}{\sqrt{2}\{x\}(1-\{x\})} = \frac{\pi}{2}$

$\displaystyle \lim_{x\to 0^-} \frac{\sin^{-1}(1-\{x\})(\cos^{-1}(1-\{x\}))}{\sqrt{2}\{x\}(1-\{x\})}[\{x\}\to 1]$

$\displaystyle = \frac{\pi}{2\sqrt{2}}$

10. $\displaystyle \lim_{x\to a} \frac{1}{(a^2-x^2)^2}\left[\frac{a^2+x^2}{ax}+\cos\frac{\pi}{2}(a+x)-\cos\left(\frac{\pi}{2}(a-x)\right)\right]$

$\displaystyle \lim_{x\to a} \frac{1}{(a^2-x^2)^2}\left[\frac{(a-x)^2}{ax}+\cos\frac{\pi}{2}(a+x)+(1+1)\frac{2ax}{ax}-\cos\frac{\pi}{2}(a-x)\right]$

$\displaystyle \lim_{x\to a} \frac{1}{(a+x)^2ax}+\frac{\left\{\cos\frac{\pi}{2}(a+x)\right\}ax}{(a^2-x^2)^2ax}+2ax$

$\displaystyle \lim_{x\to a} \frac{1}{(a+x)^2ax}+\frac{2\cos^2\frac{\pi}{4}(a+x)}{(a^2-x^2)^2}+\frac{2\sin^2\frac{\pi}{4}(a-x)}{(a^2-x^2)^2}$

$\displaystyle \frac{1}{4a^4}+\lim_{x\to a}\frac{4\sin^2\frac{\pi}{4}(a-x)}{(a^2-x^2)^2}$ $\displaystyle \left[\cos\left(\frac{a\pi}{4}+\frac{\pi}{4}x\right)=\sin\left(\frac{a\pi}{2}-\left(\frac{(a+x)\pi}{4}\right)\right)\right]+$

$$\lim_{x\to a}\frac{4\sin^2\frac{\pi}{4}(a-x)\frac{\pi^2}{16}}{\frac{(a-x)^2\pi^2}{16}(a+x)^2}$$

$$=\frac{1}{4a^4}+\frac{\pi^2}{16a^2}=\frac{a^2\pi^2+4}{16a^4}.$$

11. $L=\lim_{x\to 1}\frac{(1-x)(1-x^2)\,(1-x^3)....(1-x^{2n})}{[(1-x)\,(1-x^2)\,(1-x^3).....(1-x^n)]^2}$

$$\frac{1-x^{n+1}}{1-x}.\frac{1-x^{n+2}}{1-x^2}.\frac{1-x^{n+3}}{1-x^3}+.......\frac{(1-x^{2n})}{(1-x^n)}$$

divide by (1 – x) in N^r and D^r both→

$$\frac{(n+1)\,(n+2).......2n}{1.2.3....n}=\frac{2n!}{n!n!}={}^{2n}C_n$$

(a) (b) $\frac{2.6.10.....(4n-2)}{n!}=\frac{2^n[1.3.5.....(2n-1)]}{n!}$

$$=\frac{2^n[1.3.5.....2n-1]\,[2.4.6......2n]}{n!.2^n.n!}=\frac{2n!}{n!n!}$$

(c) ${}^{2n-1}C_n+{}^{2n-1}C_{n+1}$

(d) ${}^{2n}C_n$

12. $\lim_{x\to 1}\frac{1-x+\ell nx}{1+\cos\pi x}$ $\qquad\qquad \lim_{h\to 0}\frac{h+\ell n(1-\ell h)/\pi^2h^2}{\frac{1-\cos\pi\ell h}{\pi^2\ell h^2}}$

put x = 1 – h

$$\lim_{\ell h\to 0}\frac{h+\ell n(1-\ell h)}{1+\cos\pi(1-\ell h)}$$

$$\lim_{\ell h\to 0}\frac{h+\ell h(1-\ell h)}{2\sin^2\frac{\pi\,\ell h}{2}}$$

expanding log $(1-\ell h)$ →

$$\lim_{\ell h\to 0}\frac{\ell h+\left[-h-\frac{h^2}{2}-\frac{h^3}{3}......\right]}{2\sin^2\left(\frac{\pi\;\ell h}{2}\right)}=\lim_{\ell h\to 0}\frac{-\frac{h^2}{2}\frac{\pi^2}{2}}{\frac{\pi^2}{2}.2\sin^2\left(\frac{\pi h}{2}\right)}=-\frac{1}{\pi^2}$$

13. $\lim_{y\to 0}\left[\lim_{x\to\infty}\frac{\exp.\left(x\ell n\left(1+\frac{ay}{x}\right)\right)-\exp.\left(x\ell n\left(1+\frac{by}{x}\right)\right)}{y}\right]$

$$\lim_{y\to 0}\left[\lim_{x\to\infty}\frac{\left(1+\frac{ay}{x}\right)^{x}-\left(1+\frac{by}{x}\right)^{x}}{y}\right]$$

$$\lim_{y\to 0}\frac{e^{ay}-e^{by}}{y}=\lim_{y\to 0}\frac{[e^{(a-b)y}-1]}{y}.e^{by}$$

$$=a-b$$

14. $x_1=\sqrt{2+2\cos\frac{\pi}{6}}=2\cos\frac{\theta}{2}$ if $\theta=\frac{\pi}{6}$

$x_2=2\cos\frac{\theta}{4}$ .... $x_n=2\cos\frac{\theta}{2^n}$

$$\lim_{n\to\infty}2^{(n+1)}\sqrt{2-2\cos\frac{\theta}{2^n}}=\lim_{n\to\infty}2^{n+1}2\sin\frac{\theta}{2^{n+1}}$$

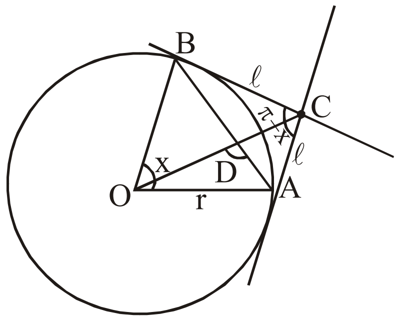
$$=\lim_{n\to\infty}\frac{2\left(\sin\frac{\theta}{2}n+1\right)\theta}{\frac{\theta}{2^{n+1}}}=2\theta=2\times\frac{\pi}{6}=\frac{\pi}{3}$$

15. $\lim_{x\to 0}\frac{\ell n(1+x)^{1+x}}{x^2}-\frac{1}{x}$

$$\frac{\lim_{x\to 0}\frac{(1+x)}{x}\left[\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}......\right]-1}{x}$$

$$\lim_{x\to 0}\frac{(1+x)\left[1-\frac{x}{2}+\frac{x^2}{3}.....\right]-1}{x}=\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}.....\right)+x\left(1-\frac{x}{2}.....\right)}{x}=-\frac{1}{2}+1=\frac{1}{2}$$

16. **Act:** ar (ΔABC) = $\frac{1}{2}\ell^2 \sin(\pi = x)$



$$\text{ar}(\Delta ABC) = \frac{1}{2}r^2 x - \frac{1}{2}r^2 \sin x$$

$$\ell = r \tan \frac{x}{2}$$

$$\frac{T(x)}{S(x)} = \frac{\frac{1}{2}\tan^2\frac{x}{2}\sin x}{\frac{1}{2}(x - \sin x)}$$

(a) $\text{ar}(\Delta ABC) = T(x) = \frac{1}{2}.AB.CD = \frac{1}{2}.2r \sin \frac{x}{2}.\left(r\sec\frac{x}{2} - r\cos\frac{x}{2}\right)$

$$= r^2 \sin \frac{x}{2}\frac{\sin^2\frac{x}{2}}{\cos\frac{x}{2}} = r^2\frac{\sin^3\frac{x}{2}}{\cos\frac{x}{2}} = \frac{1}{2}\tan^2\frac{x}{2}\sin x \quad (\because r = 1)$$

(b) $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x = \frac{1}{2}(x - \sin x)$

(area of ar c = $\frac{1}{2}r^2\theta$| area of $\Delta = \frac{1}{2}$b C sin A)

(c) $\lim\limits_{x\to 0}\frac{\frac{1}{2}\tan^2\frac{x}{2}.\sin x}{\frac{1}{2}(x - \sin x)}$ $\lim\limits_{x\to 0}\frac{\tan^2\frac{x}{2}.\sin x}{x - \left(x - \frac{x^3}{3!} + \ldots\ldots\right)}$

$$\Rightarrow \lim_{x\to 0}\frac{\tan^2\frac{x}{2}}{4\frac{x^2}{4}}.\frac{\sin x}{x\frac{1}{3!}} = \frac{6}{4} = \frac{3}{2}$$

Alternate-

$$\frac{\dfrac{\frac{1}{2}\tan^2\frac{x}{2}.\sin x}{4 \quad \frac{x^2}{4} \quad x}}{\frac{1}{2}\frac{x - \sin x}{x^3}} = \frac{\frac{1}{4}}{\frac{1}{6}} = \frac{3}{2}$$

17. $\sin^3 x = \dfrac{3\sin x - 4\sin^3 x}{4}$ $\qquad\qquad$ $\sin 3x = 3\sin x - 4\sin^3 x$

$$\frac{1}{4}\sum_{n=1}^{n}\left(3\sin\frac{x}{3}-\sin x\right)+3\left(3\sin\frac{x}{3^2}-\sin^3\frac{x}{3^2}\right)+3^2\left(3\sin\frac{x}{3^3}-\sin^3\frac{x}{3^3}\right)$$

$$\frac{1}{4}\sum_{n=1}^{n}3^{n-1}\sin.\frac{x}{3^{n-1}}-\sin x$$

$$\frac{1}{4}\left[\lim_{n\to\infty}\frac{x\sin\left(\frac{x}{3^n}\right)}{\frac{x}{3^n}}-\sin x\right]$$

$$\frac{1}{4}[x-\sin x]$$

$$= g(x) = x - 4\left[\frac{1}{4}(x-\sin x)\right]$$

$g(x) = \sin x$.

18. $f(n,0)=\prod_{r=1}^{n}\left(1-\tan^2\frac{\theta}{2^r}\right)$ $\qquad\qquad$ $\left|\left(\text{use}\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}\right)\right.$

$$f(n,\theta)=\left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)......\left(1-\tan^2\frac{\theta}{2^n}\right)$$

$$=\frac{2\tan\frac{\theta}{2}}{\tan\theta}.\frac{2\tan\frac{\theta}{2^2}}{\tan\frac{\theta}{2}}.\frac{2\tan\frac{\theta}{2^3}}{\tan\frac{\theta}{2^2}}........$$

$$f(n,\theta)=\frac{2^n\tan\frac{\theta}{2^n}}{\tan\theta}$$

$$\lim_{n\to\infty}\frac{\theta\tan\frac{\theta}{2^n}}{\tan\theta\frac{\theta}{2^n}}=\frac{\theta}{\tan\theta}.$$

19. $$\lim_{x\to 0}\frac{\sqrt{\dfrac{\cos 2x+(1+3x)^{1/3}}{2}}-\sqrt[3]{\cos^3 x-\ln(1+x)}}{x}$$

By L'Hospital's Rule

$$\lim_{x\to 0}\left\{\frac{\dfrac{-2\sin 2x+\left(1+3x\right)^{-2/3}}{2}}{2\sqrt{\dfrac{\cos 2x+(1+3x)^{1/3}}{2}}}-\frac{-3\cos^2 x\sin x-\dfrac{1}{1+x}}{3\left(\cos^3 x-\ln(1+x)\right)^{-2/3}}\right\}=\frac{1}{4}+\frac{1}{3}=\frac{7}{12}$$

Hence $a+b=19$.

20. $$e^t=1+t+\frac{t^2}{2!}+\frac{t^3}{3!}.......$$

put $\frac{\pi}{x}=t$, $$e^{-t}=1-t+\frac{t^2}{2!}-\frac{3}{3!}+......$$

$$\lim_{t\to 0}\left(\frac{e^t+e^{-t}}{2\cos t}\right)^{\pi^2/t^2}$$

$$e^{\lim_{t\to 0}\left(\frac{e^t+e^{-t-2+2}-2\cos t}{2\cos t}\right)\left(\frac{\pi^2}{t^2}\right)}$$

$$e^{\lim_{t\to 0}\left[\frac{2+2t^2/2!+2\frac{t^4}{4!}+......-2\cos t}{2\cos t}\right]\left(\frac{\pi^2}{t^2}\right)}$$

$$e^{\lim_{t\to 0}\frac{2\pi^2}{2!}+......+\frac{4\sin^2 t/2}{4t^2/4}.\pi^2\Big/2\cos t}$$

$$e^{\frac{\pi^2+\pi^2}{2}-e^{\pi^2}}$$

Alternate

$$e^{\lim_{t\to 0}\left(\frac{e^t+e^{-t}-2+2(1-\cos t)}{2\cos t}\right)\frac{\pi^2}{t^2}}$$

$$=e^{\frac{\pi^2}{2}\lim_{t\to 0}\frac{e^t+e^{-t}-2}{t^2}+\frac{4\sin^2 t/2}{t^2}}$$

$$=e^{\pi^2/2(1+1)}=e^{\pi^2}$$

21.

when $P\to A$
$T\to A$
$\pi-2\theta\to 0$

$$\frac{AQ}{\sin(\pi-\theta)}=\frac{\ell}{\sin\left(\frac{\pi}{2}-\theta\right)}$$

$AQ = \frac{\ell \sin\theta}{\cos\theta}$ ...(1)

$AP = \ell = 2r \cos\left(2\theta - \frac{\pi}{2}\right) = 2r \sin 2\theta$ when $P \to A$

$AQ = \frac{2r\sin 2\theta.\sin\theta}{\cos\theta} = 4r \sin^2\theta$ $\theta \to \frac{\pi}{2}$

Alt :- gemetrically:-

$AQ = AS + SQ$

in $\Delta TAQ \Rightarrow \angle Q = \frac{0}{2}$

in $\Delta PSQ \to$

$\angle Q = \angle P = \frac{Q}{2}$

$SQ = PS$

as $P \to A \Rightarrow SP = SA = 2r = SQ \Rightarrow AQ = AS + SQ = 2r + 2r = 4r$

22. $L = \lim_{x\to 0}\left[\frac{1}{\ell n(1+x)} - \frac{1}{\ell n(x+\sqrt{1+x^2})}\right]$

put $x = -x$ ($\because x \to 0$, we can put $x = -x$)

$L = \lim_{x\to 0}\left[\frac{1}{\ell n(1-x)} - \frac{1}{\ell n(\sqrt{1+x^2-x})}\right]$

Adding both:

$2L = \lim_{x\to 0}\frac{1}{\ell n(1+x)} + \frac{1}{\ell n(1-x)} - \frac{1}{\ell n(x+\sqrt{1+x^2})} - \frac{1}{\ell n(\sqrt{(1+x^2-x)})}$

$2L = \lim_{x\to 0}\frac{\ell n(1-x^2)}{\ell n(1+x)\ell n(1-x)} - \left\{\frac{\ell n(1+x^2-x^2)}{\ell n(x+\sqrt{1+x^2})\ell n(\sqrt{1+x^2}-x)}\right\} = 0$ ($\because N_r$ is absolute zero)

$= \lim_{x\to 0}\frac{\ell n(1-x^2)}{-x^2}\Big/\frac{\ell n(1+x)}{x}.\frac{\ell n(1-x)}{(-x)} = +1$

$L = +\frac{1}{2}$

$$\frac{+\frac{1}{2}+153}{+\frac{1}{2}} = \frac{306+1}{+1} = 307$$

23 Let $U_n = \lim_{x\to 0} \frac{1-\cos 3x \cdot \cos 3^2 x \cdot \cos 3^3 x ....\cos 3^n x}{x^2}$

and $V_n = \lim_{x\to 0} \frac{1-\cos\frac{x}{3} \cdot \cos\frac{x}{3^2} \cdot \cos\frac{x}{3^3} ....\cos\frac{x}{3^n}}{x^2}$

$$U_n = \lim_{x\to 0} \frac{-D(\cos 3x \cdot \cos 3^2 x \cdot \cos 3^3 x ....\cos 3^n x)}{2x}$$

now let $y = \cos 3x \cdot \cos 3^2x \cdot \cos^3 x .... \cos 3^n x$

$\ln y = \ln \cos 3x + \ln \cos 3^2 x + ... + \ln \cos 3^n x$

$$\frac{1}{y}\frac{dy}{dx} = -[3 \tan 3x + 3^2 \tan 3^2 x + ... + 3^n \tan 3^n x]$$

$$\frac{dy}{dx} = -\prod_{r=1}^{n} \cos 3^r x [3\tan 3x + 3^2 \tan 3^2 x + ... + 3^n \tan 3^n x]$$

$\therefore$ $U_n = \lim_{x\to 0} \frac{3\tan 3x + 3^2 \tan 3^2 x + ... + 3^n \tan 3^n x}{2x} = \frac{3^2 + (3^2)^2 + (3^3)^2 + ... + (3^n)^2}{2}$

$$U_n = \frac{3^2[3^{2n}-1]}{(3^2-1)\cdot 2} \quad ....(1)$$

|||ly replacing 3^r by $\frac{1}{3^r}$ we get

$$V_n = \frac{\frac{1}{3^2}\left[1-\frac{1}{3^{2n}}\right]}{\left(1-\frac{1}{3^2}\right)\cdot 2} = \frac{(3^{2n}-1)}{3^{2n}(3^2-1)\cdot 2} \quad ....(2)$$

$\therefore$ $\frac{U_n}{V_n} = 3^{2n+2} = 3^{10}$ (given)

$\therefore$ $2n + 2 = 10 \Rightarrow \boxed{n = 4}$

24. (a) $\lim_{x\to 0^+}\left(p_1 a_1^x + p_2 a_2^x + ... + p_n a_n^x\right)^{1/x} \to 1^\infty$ form as $\lim_{x\to 0^+} a_i^{\,x} = 1$ & $\sum_{i=1}^{n} p_i = 1$

$$\Rightarrow \lim = e^{\lim_{x\to 0^+}\left(\frac{p_1a_1^x+p_2a_2^x+.......+p_na_n^x-1}{x}\right)} \quad \text{by } \lim_{x\to 0}\left(f(x)\right)^{g(x)} = e^{\lim_{x\to 0} g(x)(f(x)-1)}$$

Now $1 = p_1 + p_2 + p_3 + ...p_n$

$$\Rightarrow \lim = e^{\lim_{x\to 0^+}\left(\frac{p_1a_1^x+p_2a_2^x+.......+p_na_n^x-(p_1+p_2+p_3+...p_n)}{x}\right)}$$

$$\Rightarrow \lim = e^{\lim_{x\to 0^+}\left(p_1\left(\frac{a_1^x-1}{x}\right)+p_2\left(\frac{a_2^x-1}{x}\right)+.......+p_n\left(\frac{a_n^x-1}{x}\right)\right)}$$

$$\Rightarrow \lim = e^{p_1\ln a_1+p_2\ln a_2+.......+p_n\ln a_n} = a_1^{p_1}a_2^{p_2}a_3^{p_3}...a_n^{p_n} \quad .$$

(b) $\lim_{x\to\infty} F(x) = L_2 = \lim_{x\to\infty}\left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)^{1/x}$ (∞^0 form)

[only when a_1 a_2 etc. > 1]

$\therefore \quad \ln L_2 = \lim_{x\to\infty}\frac{\ln\left(p_1a_1^x + p_2a_2^x + ... + p_na_n^x\right)}{x}$

using L'Hospital's Rule

$$L_2 = \lim_{x\to\infty}\frac{\left(p_1\ln a_1\, a_1^x + p_2\ln a_2\, a_2^x + ... + p_n\ln a_n\, a_n^x\right)}{p_1a_1^x + p_2a_2^x + ... + p_na_n^x} \qquad ....(1)$$

dividing by a_1^x and taking limit, we get

$\lim_{x\to\infty}, \left(\frac{a_2}{a_1}\right)^x, \left(\frac{a_3}{a_2}\right)^x$, etc all vanishes as $x \to \infty$

$= \frac{p_1\ln a_1}{p_1} = \ln a_1$

hence $\ln L_2 = \ln a_1 \quad \Rightarrow \quad L_2 = a_1$ **Ans.**

(c) $\lim_{x\to-\infty} F(x) = L_3$ (say)

$\therefore \quad \ln L_3 = \lim_{x\to-\infty}\frac{\left(p_1\ln a_1\, a_1^x + p_2\ln a_2\, a_2^x + ... + p_n\ln a_n\, a_n^x\right)}{p_1a_1^x + p_2a_2^x + ... + p_na_n^x}$

dividing by $(a_n)^x$ and taking $\lim_{x\to-\infty}, \left(\frac{a_1}{a_n}\right)^x, \left(\frac{a_2}{a_n}\right)^x$ etc vanishes

$\therefore \quad \ln L_3 = \frac{p_n\ln a_n}{p_n} \quad \Rightarrow \quad L_3 = a_n$

25. $\sum_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right) = \sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{\frac{3}{4}+r^2}\right) = \sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{1+r^2-\frac{1}{4}}\right)$