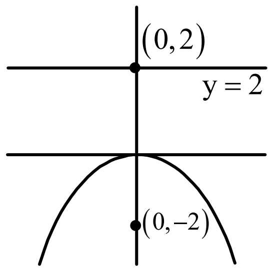
**PARABOLA BOOKLET SOLUTION**
**EXERCISE – 1 (A)**

1. (B)



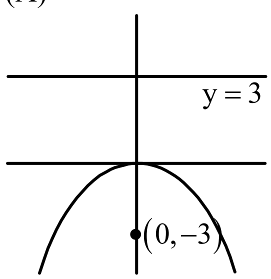

$x^2 = -8y$

$x^2 = -4Ay$

$\Rightarrow A = 2$

Directrix is $y = 2$

2. (A)




$x^2 = -12y$

3. (C)

Vertex $(0,0)$

Directrix $3x - 4y + 2 = 0$

$D = \frac{|2|}{\sqrt{3^2 + 4^2}} = \frac{2}{5} = a$

$L.R = 4a = 4.2/5 = 8/5$

4. (B)

Focus of $y^2 = 8x$ is $(-2,0)$

Focal chord $2x + y + d = 0$

$\Rightarrow 2(-2) + 0 + \lambda = 0$

$\Rightarrow \lambda = 4$

5. (A)

Focal distance is $x + a$

$\therefore x, +3$

6. (B)

Vertex is $(a,b)$

$L.R = \ell = 4A \Rightarrow A = e/4$

$(x-a)^2 = \ell(y-b)$

7. (A)

$x^2 = -y, \qquad x^2 = -4AV$

$\Rightarrow 4A = 1 = \ell$

8. (C)

$D = 2a$ between focus & Directrix

$x^2 = -8y$

$\Rightarrow 2a = 4$

9. (B)

$F(3,0) \qquad \ell = 4A = 8 \Rightarrow A = 2$

$\Rightarrow$ vertex $(1,0)$

10. (B)

$F(2,1)$

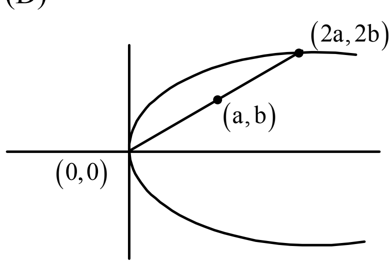
Let the L. R. be $2x-3y+c=0$

Now, $2(2)-3(1)+c=0$

$\Rightarrow c = -1$

$LR \equiv 2x-3y-1=0$

11. (D)



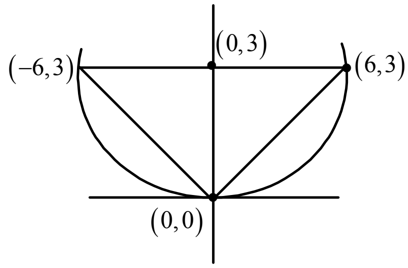

$y^2 = 4x$

$(2b)^2 = 4.2a$

$4b^2 = 8a$

$\Rightarrow b^2 = 2a$

12. (C)



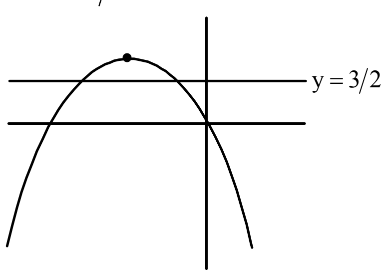

$$A = \frac{1}{2} \times 12 \times 3$$
$$A = 18 \text{ sq.units}$$

13. (A)

$$9x^2 - 6x + 36y + 9 = 0$$
$$(3x)^2 - 2.3x.1 + 1 + 36y + 8 = 0$$
$$(3x-1)^2 = -36y - 8$$
$$9(x - 1/3)^2 = -36\left(y + \frac{2}{9}\right)$$
Vertex $(1/3, -2/9)$

14. (C)

$$x^2 + 4x + 2y = 0$$
$$(x+2)^2 = -2y + 4$$
$$(x+2)^2 = -2(y-2)$$
Now, $4a = 2$
$\Rightarrow a = 1/2$




$LR \equiv y = 3/2$

15. (D)

$$y^2 = x - 2y + 2 = 0$$
$$(y-1)^2 = x - 1$$
Vertex $(1,1), a = 1/4$
Focus $\Rightarrow (5/4, 1)$

16. (D)

$y^2 + 2y + x = 0$

$(y+1)^2 = -(x-1)$

Vertex $(1,-1)$

17. (C)

$x^2 - 4x - 3y + 10 = 0$

$(x-2)^2 = 3(y-2)$

Axis is $x - 2 = 0$

18. (D)

$x^2 + 4x + 2y - 7 = 0$

$(x+2)^2 = -2(y - 11/2)$

Vertex $(-2, 11/2)$

19. (B)

$y^2 = 4y - 2x - 8 = 0$

$(y-2)^2 = 2(x+6)$

$LR = 2$

20. (A)

$y^2 + 4x + 2y - 8 = 0$

$(y+1)^2 = -4(x - 9/4)$

It is focus $(5/4, -1)$

21. (C)

$x = ay^2 + by + c$

$$x = a\left[(y + b/2a)^2 - \frac{b^2}{4a^2} + c/a\right]$$

$$\frac{1}{a}\left(x + \frac{b^2 - 4ac}{4a}\right) = (y + b/2a)^2$$

$L.L.R = 1/a$

22. (C)

For parabola $4^2 = ab$

Option (C) $(4) \neq 2 \times 1$

$2x^2 + y^2 - 4xy = 8$ is not a parabola

23. (B)

Option (B)

$$\frac{x}{y}+\frac{2y}{x}=0$$

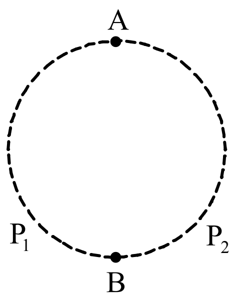
$$x^2=-24y$$

24. (C)

Theory based

25. (C)

Given ends $A \& B$ two parabolas are possible.



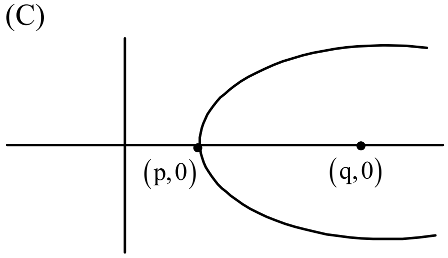

26. $v(0,a)$

$f(0,0)$

$$(x-0)^2=-4a(y-a)$$

$$x^2=4a(a-y)$$

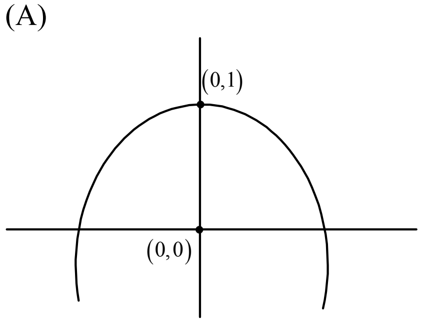
27. (C)




$$y^2=4(q-p)(x-p)$$

$$\Rightarrow y^2=-4(p-q)(x-p)$$

28. (A)



$(x-0)^2 = -4\times1(y-1)$

$x^2 + 4y - 4 = 0$

29. (B)

$v(2,0) \qquad f(5,0)$

$y^2 = 4\times3(x-2)$

$y^2 = 12x - 24$

30. (D)

$x-2 = t^2, y = 2t$

$x-2 = (y/2)^2$

$\Rightarrow y^2 = 4(x-2)$

31. (B)

$x = \frac{t}{4}, y = \frac{t^2}{4}$

$\Rightarrow y = \frac{(4x)^2}{4} = 4x^2$ a parabola

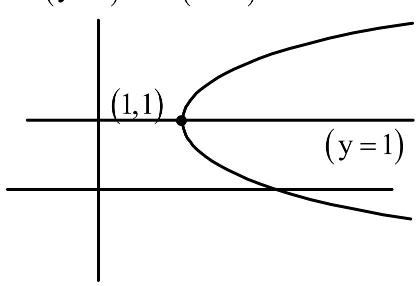
32. (A)

$x = t^2 + 1, y = 2t + 1$

$x = \left(\frac{y-1}{2}\right)^2 + 1$

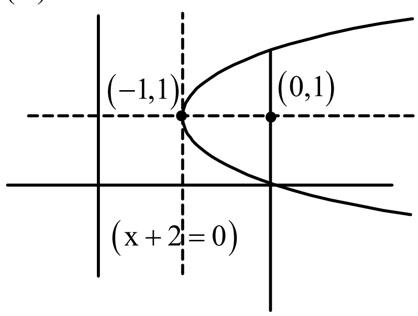
$4x = (y-1)^2 + 4$

$\Rightarrow (y-1)^2 = 4(x-1)$




Directrix is $x = 0$

33. (D)




Vertex $(-1,1)$

$f(0,1)$

$(y-1)^2 = 4\times 1(x+1)$

Now, $x+1 = 1\times t^2 \Rightarrow x = t^2 - 1$

$y-1 = 2\times 1\times t \Rightarrow y = 2t+1$

Any point $(t^2-1, 2t+1)$

34. (A)

$(a/m^2, 2/m)$ doesn't satisfy $y^2 = 4ax$

35. (A)

$y^2 - 12x - 2y - 11 = 0$

$(y-1)^2 = 12(x+1)$

$y-1 = 2\times 3\times t \quad x+1 = 3.t^2$

$y = 1+6t \qquad x = 3t^2 - 1$

36. (B)

$y^2 = 6x$

Vertex $(0,0)$

Negative end of LR $(3/2, -3)$

$\frac{y-0}{x-0} = \frac{3-0}{3/2-0} \Rightarrow \frac{3}{2}y = -3x$

$\Rightarrow y + 2x = 0$

37. (A)

$f(8,0)$

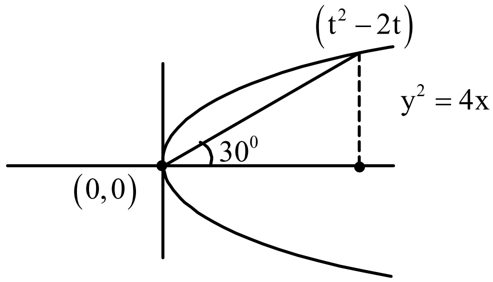
One end $(2,-8)$

$\frac{-2a}{t} = -8 = \frac{-2\times 8}{t} \Rightarrow t = 2$

$(at^2, 2at) \equiv (8\times 2^2, 2\times 8\times 2)$

$\equiv (32,32)$

38. (C)




$\tan 30^0 = \frac{2t}{t^2}$

$\frac{1}{\sqrt{3}} = \frac{2}{t} \Rightarrow$

$t = 2\sqrt{3}$

Length $= \sqrt{\left(\left(2\sqrt{3}\right)^2\right)^2 + \left(2\times 2\sqrt{3}\right)^2}$

$= \sqrt{12^2 + 4^2 \times 3}$

$= 8\sqrt{3}$

39. (B)

$y^2 = 4x$

$(-x)^2 = 4x$

$x^2 = 4x \Rightarrow x = 0 \text{ or } x = 4$

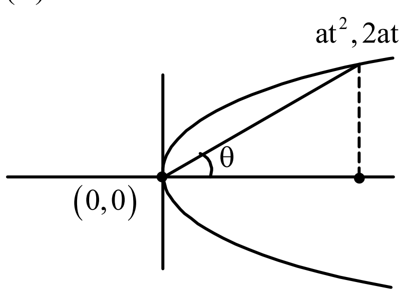
$y^2 = 0$ or $y^2 = 4\times 4$

$4 = 0$ or $y = \pm 4$

$A \equiv (4,4)$ $B \equiv (4,-4)$

$AB = 8$

40. (B)




$\tan\theta = \frac{2at}{at^2}$

$t = 2\cot\theta$

Length $= \sqrt{a^2t^4 + 4a^2t^2}$

$= at\sqrt{t^2+4}$

$= a\times 2\cot\theta\sqrt{4\cot^2\theta+4}$

$= a\times 2\cot\theta\times 2\times \cos\sec\theta$

$= 4a\cos\theta.\cos ec^2\theta$

41. (C)

Put $y=0$

$x^2 = 7x+12=0$

$\Rightarrow x = 4 \text{ or } 3$

Intercept is 1 unit

42. (B)

Put $x=0$

$2y^2+6y-8=0$

$y^2+3y-4=0$

$(y+3/2)^2-9/4-4=0$

$(y+3/2)^2=\frac{25}{4}$

$y+3/2=\pm 5/2$

$y=1,-4$

Intercept length $=5$

43. (A)

$x^2+3x=5-x$

$x^2+4x+4=9$

$(x+2)^2=9$

$x+2=\pm 3$

$x=1,-5$

$y=5-x$

$=5-1$ or $5-(-5)$

$y=4,10$

Points are $(1,4)$.& $(-5,10)$

Length $=6\sqrt{2}$

44. (C)

$2a=2\times 2=\frac{2\times sp\times sa}{sp+sq}$

$2=\frac{6\times s\theta}{6+s\theta}$

$12+2s\theta=65\theta$

$4s\theta = 12$

$5\theta = 3$

45. (B)

$2a = \dfrac{sp \times s\theta}{sp + s\theta}$ $\{A = 2a \text{ in this case}\}$

$2a = \dfrac{2 \times 4}{6} = 4/3$

$a = 2/3$

46. (B)

$y^2 = K(1-y)$

$y^2 + Ky0k = 0$

$D = 0 \Rightarrow K^2 = (4 \times -K) = 0$

$K^2 + 4K = 0$

$K = 0, -4$

47. (C)

$K - x = x - x^2$

$x^2 - 2x + K = 0$

$D = 0 \Rightarrow 4 - 4K = 0$

$\Rightarrow K = 1$

48. (D)

$y^2 = 4a(x+a)$

$y = mx + c$

$y = m(x+a) + c - am$

Now, $c - am = \dfrac{a}{m}$

$\Rightarrow c = am + \dfrac{a}{m}$

49. (A)

$(1-2x)^2 = 4x$

$1 + 4x^2 - 4x = 4x$

$4x^2 - 8x + 1 = 0$

$\because D > 0,$ 2 real points

50. (C)

$y^2 = 4ax$

$\ell x + my + n = 0$

$$y=-\frac{\ell x}{m}-\frac{n}{m}$$

$$c=\frac{a}{M}\Rightarrow\frac{-n}{m}=\frac{a}{-\ell/m}$$

$$\Rightarrow\frac{\ell n}{m^2}=a$$

51. (B)

$$x^2+4\left(\frac{x-k}{2}\right)=0$$

$$x^2+2x-2k=0$$

$$D=0\Rightarrow 4+8K=0$$

$$\Rightarrow K=-1/2$$

52. (D)

$$\left(my+\frac{a}{m}\right)^2=4ay$$

$$m^2y^2+\frac{a^2}{m^2}+2ay=4ay$$

$$(my-a/m)^2=0$$

$$y=\frac{a}{m^2}$$

$$\Rightarrow x=\frac{2a}{m}$$

$$\left(\frac{2a}{m},\frac{a}{m^2}\right)$$

53. (A)

$$y^2=\frac{1}{4}x$$

$$y=\sqrt{3}x+C$$

Hence $c=a/m\Rightarrow c=\frac{1/16}{\sqrt{3}}=\frac{1}{16\sqrt{3}}$

$$(x.y.)\equiv\left(\frac{1}{48},\frac{1}{8\sqrt{3}}\right)$$

54. (A)

$$y.2=4.\frac{(x+1)}{2}$$

$$y=x+1$$

55. (B)

$1-x=x-x^2$

$x^2-2x+1=0$

$\left(x-1\right)^2=0$

$x=1\Rightarrow y=0$

56. (A)

$4y^2+6x=8y+7$

$\left(2y\right)^2-2\times\left(2y\right)\times2+4=-6x+11$

$\left(2y-2\right)^2=-6\left(x-\frac{11}{6}\right)$

$\left(y-1\right)^2=\frac{-3}{2}\left(x-\frac{11}{6}\right)$

Tangent at vertex $x=\frac{11}{6}$

57. (A)

$4x^2=4x=-y+2$

$\left(2x\right)^2-2.2x\times1+1=-y+3$

$\left(2x-1\right)^2=-\left(y-3\right)$

$\left({}^{x-11}2\right)^2==\frac{1}{4}\left(y-3\right)$

58. (D)

$y^2=x$

$y=x+c \qquad \left(\tan 45^0=1\right)$

$c=\frac{1/4}{1}=\frac{1}{4}$

$\left(x+1/4\right)^2=x$

Point $\left(1/4,1/2\right)$

59. (C)

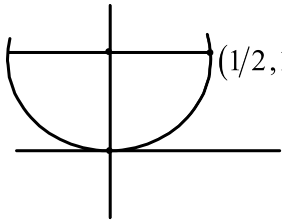
$\left(2x+2\right)^2=16x$

$\left(x+1\right)^2=4x$

$\left(x-1\right)^2=0$

$x=1\Rightarrow y=4$

60. (D)




$x^2 = y$

$x^2 = 4\times\left(\frac{1}{4}y\right)$

Tangent at $(1/2, 1/4)$ is

$x\times 1/2 = \frac{(y+1/4)}{2}$

$4x - 4y = 1$

61. (D)

Tangent of $y^2 = 4x$ is

$y = mx + 1/m$

Also for $x^2 = 32y$

$x^2 = 32(mx + 1/m)$

$x^2 - 32mx - \frac{32}{m} = 0$

For tangency $D = 0$

$\Rightarrow (-32m)^2 + \frac{32}{m}\times 4 = 0$

$\Rightarrow m = -1/2$

$\therefore y = \frac{-1}{2}x - 2$

$\Rightarrow x + 2y + 4 = 0$

62. ()

$$\Delta = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ a\left(t_2^2 t_1^2\right) & 2a\left(t_2 - t_1\right) & 0 \\ a\left(t_3^2 - t_1^2\right) & 2a\left(t_3 - t_1\right) & 0 \end{vmatrix}$$

$$= \frac{1}{2}\times a^2\left(t_2 - t_1\right)\left(t_3 - t_1\right)$$

$$\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ t_1+t_2 & 2 & 0 \\ t_1+t_3 & 2 & 0 \end{vmatrix}$$

$$\Delta = \frac{1}{2}a^2\left(t_2-t_1\right)\left(t_3-t_1\right)\times 2\left(t_2-t_3\right)$$

Required area is $1/2\,\Delta$

$$\therefore \text{Area } = \frac{a^2}{2}\left(t_1-t_2\right)\left(t_2-t_3\right)\left(t_3-t_1\right)$$

63. ()

$t, y = x + at_1^2$ & $t_2 y = x + at_2^2$

Solving $x = at_1 t_2$

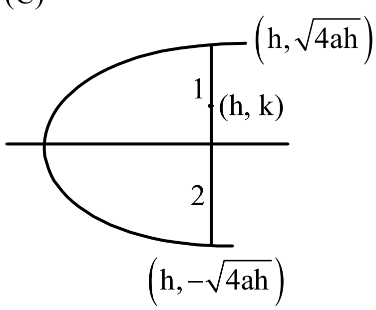
$\therefore$ abscissa is G. M. between ponts.

**EXERCISE 1(B)**

1. (A)
$at_1^2 = p$
$2ah = q$
At conditions of $\theta$ be $\left(\frac{a}{h^2}, \frac{-2a}{h}\right)$
Or $\left(\frac{a^2}{p}, \frac{-4a^2}{q}\right)$

2. (A)
$a + at^2 = 4$
Here a = 2 $\Rightarrow t^2 = 1$
$\Rightarrow t = \pm 1$
Coordinate $(2, \pm 4)$

3. (C)




By section formula
$$\frac{2\sqrt{4ah} - \sqrt{4ah}}{3} = k$$
$$\frac{4ax}{J} = y^2$$

4. (B)
Check option (B)
$$x^2 - 2 = -2\left(2\cos^2\frac{t}{2} - 1\right)$$
$$x^2 - 2 = -4\cos^2\frac{t}{2} + 2$$
$$x^2 - 4 = -y$$

5. (C)
$x_1 = at_1^2$ $\quad t_1 t_2 = -1$
$x_2 = at_2^2$
$x_1 x_2 = a^2$
Square be G.M. of $x_1 x_2 = a^2$

6. (A)

y = mx +c is tangent to

$y^2 = 4ax \Rightarrow c = \frac{a}{m}$

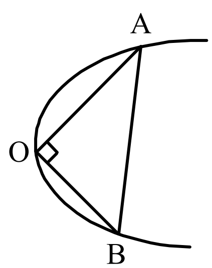
& of $x^2 = 4by \Rightarrow c = -bm^2$

$$\Rightarrow m^3 = -\frac{a}{b}$$

$$\text{Or } m = \frac{-a^{1/3}}{b^{1/3}}$$

Equation of tangent is $a^{1/3}x + b^{1/3}y + (ab)^{2/3} = 0$

7. (A)




Homogenize line with $y^2 = 4ax$

$y^2 - 4ax\left(\frac{y+mx}{c}\right) = 0$

Since $\angle AOB = 90^0$

$\Rightarrow$ coefficient $x^2$ + coefficient $y^2 = 0$

c + 4ax = 0

8. (A)

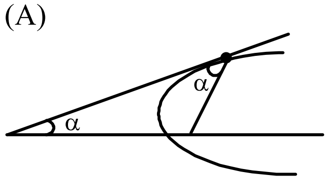
We write equation as

$\left(\frac{2u^2\cos^2\alpha}{g}\right)y = \left(\frac{u^2\sin 2\alpha}{g}\right)x - x^2$

When we form a perfect square

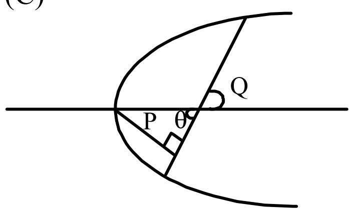
We get $x = \frac{2u^2\cos^2\alpha}{g}$

9. (A)



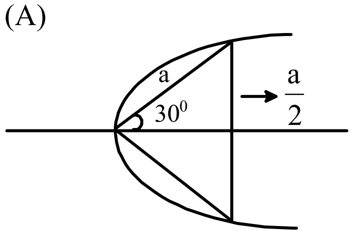

Properly these two angles are same.

10. (C)




$P = a\sin\theta$

11. (A)




The length of double ordinate will be a

$\text{Area} = \frac{1}{2} \times a\cos 90^0 \times a$

$= \frac{\sqrt{3}}{4} a^2$

Here $P^2\left(\frac{\sqrt{3}a}{2}, a/2\right)$ lies on parabola

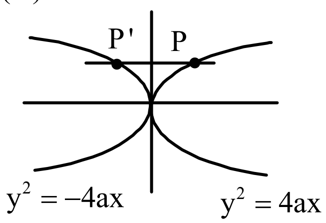
$\frac{a^2}{4} = \frac{4\sqrt{3}\,a}{2}$ $\quad \because A = 48\sqrt{3}$

$a = 8\sqrt{3}$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 12. | (B) | 13. | (D) | 14. | (A) | 15. | (C) | | |
| 16. | (B) | 17. | (A) | 18. | (C) | 19. | (C) | 20. | (D) |
| 21. | (A) | 22. | (A) | 23. | (C) | 24. | (D) | 25. | (C) |
| 26. | (D) | 27. | (A) | 28. | (C) | 29. | (C) | 30. | (C) |
| 31. | (A) | 32. | (D) | 33. | (D) | 34. | (C) | 35. | (C) |
| 36. | (C) | 37. | (D) | 38. | (B) | 39. | (B) | 40. | (C) |
| 41. | (C) | 42. | (B) | 43. | (C) | 44. | (A) | 45. | (B) |
| 46. | (A) | 47. | (D) | 48. | (B) | | | | |

# (2-A) PARABOLA BOOKLET SOLUTION

1. (A)




$P \equiv (at^2, 2at)$. $P' \equiv (-at^2, 2at)$

Let unit point be $\theta(h, k)$

$h = \frac{at^2 - bt^2}{2} \Rightarrow t^2 = \left(\frac{2h}{a-b}\right)$ .......... (I)

$K = \frac{2at - 2bt}{2} \Rightarrow t = \left(\frac{K}{a-b}\right)$ .......... (II)

Eliminating t from (I) and (II)

$\left(\frac{K}{a-b}\right)^2 = \frac{2h}{(a-b)}$

$\Rightarrow K^2 = 2h(a-b)$

$\therefore$ Locus is $y^2 = 2(a-b)x$ . Which is a parabola

2. ()

$y = m(x-a)$ .......... (I)

$y^2 = 4ax$ ........... (II)

Solving (I) and (II)

$[m(x-a)]^2 = 4ax$

$\Rightarrow m^2(x-a)^2 = 4ax$

$\Rightarrow m^2x^2 - 2am^2x + m^2a^2 = 4ax$

$\Rightarrow m^2x^2 - 2ax(m^2+2) + m^2a^2 = 0$

For 2 real solutions $D > 0$ & $m^2 \neq 0$

$\Rightarrow 4a^2(m^2+2)^2 - 4m^2(m^2a^2) > 0$

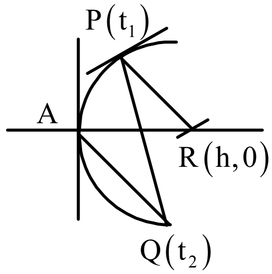
$\Rightarrow 4a^2\left((m^2+2)^2 - m^4\right) > 0$

$\Rightarrow 4a^2\left((2m^2+2)(2)\right) > 0$

$\Rightarrow m \in R$

$\therefore m \in R \sim \{0\}$

3. (C)




$\because P\theta$ is a normal chord

$\Rightarrow t_2 = -t_1 \frac{-2}{t_1}$ .......... (I)

Slope of $AQ = \frac{2}{t_2}$ = slope of PR.

Equation of PR.

$$y - 2ati = \frac{2}{t_2}\left(x - at_1^2\right)$$

For R.

$$0 - 2at_1 = \frac{2}{t_2}\left(h - at_1^{\,2}\right)$$

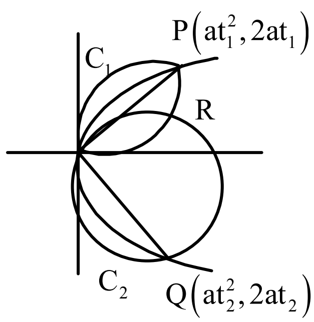
$$\Rightarrow \quad h = at1^2 - at_1t_2$$

$$= at_1^2 - at_1\left(-t_1 - \frac{2}{t_1}\right)$$

$$= at_1^{\,2} + at_1^{\,2} + 2a \qquad = 2a\left(1 + t^2\right)$$

$$= 2 \times \text{focal distance of P.}$$

4. ()




$$\tan\theta_1 = \frac{2}{t_1}$$

$$\tan\theta_2 = \frac{2}{t_2}$$

$C_1 : x^2 + y^2 - at_1^{\,2}x - 2at_1y = 0$

$C_2 : x^2 + y^2 - at_2^{\,2}x - 2at_2y = 0$

Equation of OR

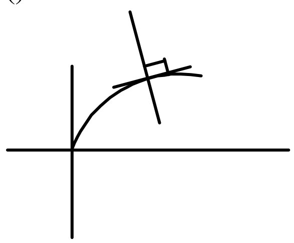
$C_1 - C_2 = 0 \Rightarrow a\left(t_2^2 - t_1^2\right)x + 2a\left(t_2 - t_1\right)y = 0$

$\Rightarrow y = -\frac{1}{2}(t_1 + t_2)x$

$\therefore \tan\phi = -\frac{1}{2}(t_1 + t_2) = -\frac{1}{2}(\cot\theta_1 + \cot\theta_2)$

$\Rightarrow \cot\theta_1 + \cot\theta_2 = -2\tan\phi$

5. ()



$y = \sqrt{x} \Rightarrow y^2 = x$ ....... (I)

Equation of normal to (I) in parametic form.

$$y + tx = \frac{t}{2} + \frac{t^3}{4}$$

$\because$ This passes through $(3,6)$

$\Rightarrow 6 + 3t = \frac{t}{2} + \frac{t^3}{4} \Rightarrow t^3 - 10t - 24 = 0$

$\Rightarrow (t-4)(t^2 + 4t + 6) = 0$

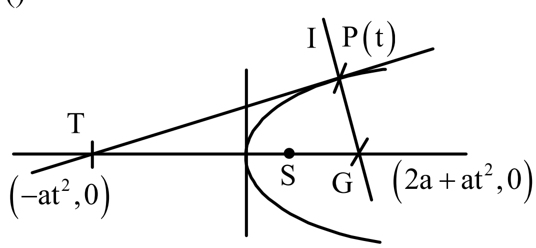
$\Rightarrow t = 4$

$\therefore$ Normal is

$y + 4x = 2 + 16$

$\Rightarrow 4x + y - 18 = 0$

6. ()




Centre of circle passing through P, T, G is $S(a,0)$

$\therefore$ Slope of tangent to circle at

$P = \frac{-1}{\text{slope of PS.}}$

$= \frac{-1}{\left(\frac{2at}{at^2 - a}\right)} = \frac{1-t^2}{2t} = m_1$

Slope of tangent to parabola at P $= \frac{1}{t} = m_2$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{t} - \frac{1}{2t} + \frac{t}{2}}{1 + \frac{1 - t^2}{2t}}\right| = t$$

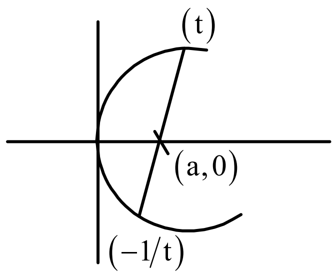
$$\therefore \theta = \tan^{-1}(t)$$

7. ()

Clearly the point of intersection of the tangents will always lie in the line. Hence locus will be the given line its effect.

8. ()

Length of focal chord $= a\left(t + \frac{1}{t}\right)^2$




Also $\tan\alpha = \frac{2}{t - \frac{1}{t}}$

$\Rightarrow t\frac{-1}{t} = 2\cot\alpha$

$\Rightarrow \left(t + \frac{1}{t}\right)^2 = \left(t\frac{1}{t}\right)^2 + 4$

$= 4\cot^2\alpha + 4$

$= 4\cos ec^2\alpha$

$\therefore$ Length of focal chord $= 4\cos ec^2\alpha$

$\because \alpha \in (0, \pi/4)$

$\therefore$ Minimum length $= 4\cos\sec^2\frac{\pi}{4} = 8a$

9. $y = k\sqrt{x} \Rightarrow y^2 = k^2 x$

Let $k^2 = 4a \Rightarrow y^2 = 4ax$

For ABCD

$at_1^2 = 2at_1 \Rightarrow t_1 = 2$

For EFGC

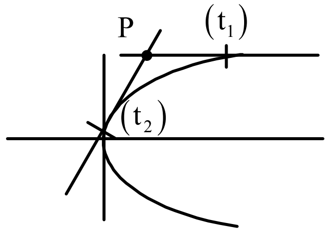
$at_2^2 - ya = 2at_2$

$\Rightarrow at_2^2 - 2at_2 - 4a = 0$

$\Rightarrow t_2 = 1 + \sqrt{5}$

$$\therefore \frac{FG}{BC} = \frac{2at_2}{2at_1} = \frac{2a\left(1+\sqrt{5}\right)}{2a\left(2\right)} = \frac{\sqrt{5}+1}{2}$$

10. ()




Let $P \equiv (h,k)$ and pair of tangents drawn touches parabola at $A(t_1)$ and $B(t_2)$

$\tan\theta_1 = \frac{1}{t_1} \qquad \tan\theta_2 = \frac{1}{t_2}$

Also, $h = t_1 t_2 \qquad k = (t_1 + t_2)$

Given, $\theta_1 + \theta_2 = \frac{\pi}{4}$

$\Rightarrow \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = 1$

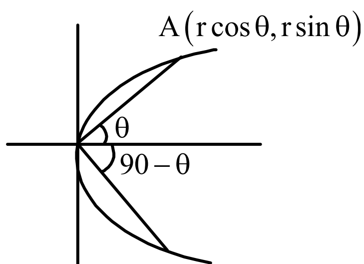
$\Rightarrow \frac{\frac{1}{t_1} + \frac{1}{t_2}}{1 - \frac{1}{t_1 t_2}} = 1$

$\Rightarrow \frac{t_1 + t_2}{t_1 t_2 - 1} = 1$

$\Rightarrow \frac{k}{h-1} = 1 \Rightarrow h - k = 1$

$\therefore$ Locus is $x - y = 1$

11. ()




$\because$ A and B lies on the parabola

For A

$r^2 \sin^2\theta = r\cos\theta$

$\Rightarrow r = \frac{\cos\theta}{\sin^2\theta} = |VA|$

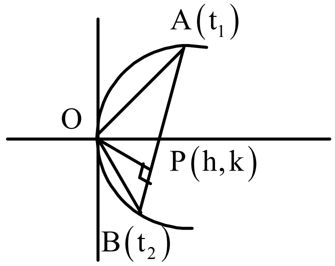
For B

$r^2 \cos^2\theta = r\sin\theta$

$\Rightarrow r = \frac{\sin\theta}{\cos^2\theta} = |VB|$

$\therefore \frac{|VA|}{|VB|} = \frac{\cos^3\theta}{\sin^3\theta} = \cot^3\theta$

12. ()




$\because$ AB subtends $90^0$ at vertex

$\Rightarrow t_1t_2 = -4$ ------ (I)

Equation of AB

$2x - y(t_1 + t_2) + 2at_1t_2 = 0$

$\Rightarrow 2x - y(t_1 + t_2) - 8a = 0$ ……….. (II)

Equation of OP

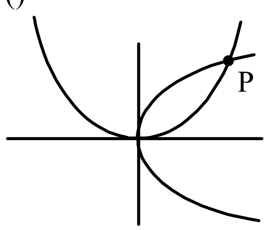
$y = -\frac{(t_1 + t_2)}{2}x$ ……… (III)

To find locus of P eliminate $t_1$ and $t_2$ from (II) And (III)

Required locus is

$x^2 + y^2 - 4ax = 0$

13. ()




Let $P \equiv (at^2, 2at)$

$\because$ P lies on $x^2 = 4by$ as well.

$\Rightarrow a^2t^4 = 4b.2at$

$\Rightarrow t = 2\left(\frac{b}{a}\right)^{1/3}$

Slope of tangent at P to on $y^2 = 4ax$

$= \frac{2a}{y} = \frac{2a}{2at} = \frac{1}{t}.(m_1)$

Slope of tangent at P on $x^2 = 4by$

$= \frac{x}{2b} = \frac{at^2}{2b} = \frac{a}{2b}4.\frac{b^{2/3}}{a^{2/3}}$

$= 2\left(\frac{a}{b}\right)^{1/3} = \frac{4}{t}(m_2)$

$\therefore$ Angle of enter section of parabola

= angle b/w their tangents

$$\therefore \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{4}{t} - \frac{1}{t}}{1 + \frac{4}{t^2}}\right|$$

$$= \left|\frac{3}{t + \frac{4}{t}}\right| = \left|\frac{3/2}{\left(\frac{a}{b}\right)^{1/3} + \left(\frac{b}{a}\right)^{1/3}}\right| = \sqrt{3}. \Rightarrow \theta = \pi/3$$

14. ()

Let point be $P(h,k)$

Equation of polar $xh + yk = r^2$

$$\Rightarrow y = \left(\frac{-h}{k}\right)x + \frac{r^2}{k}$$

For this to be tangent to $y^2 = 4ax$

$$\frac{r^2}{k} = \frac{a}{\left(\frac{-b}{k}\right)} \Rightarrow k^2 = -\frac{r^2}{a}h$$

$\therefore$ Locus is $y^2 = \frac{-r^2}{a}x$

15. ()

$x\cos\alpha + y\sin\alpha = P \Rightarrow y = (-\cot\alpha)x + P\,\text{cosec}\alpha$ ………….. (I)

Tangent to $y^2 = 4a(x+a)$ will be

$$y = m_1(x+a) + \frac{a}{m}$$

$$\Rightarrow y = mx + am + \frac{a}{m} \quad \text{……….. (II)}$$

Comparing (i) and (ii)

$m = -\cot\alpha$ and

$$P\,\text{cosec}\alpha = am + \frac{a}{m}$$

$$\Rightarrow P\,\text{cosec}\alpha = -a\cot\alpha - \frac{a}{\cot\alpha}$$

$$\Rightarrow P\,\text{cosec}\alpha = -\frac{a\,\text{cosec}^2\alpha}{\cot\alpha}$$

$$\Rightarrow P\cos\alpha + a = 0$$

16. ()

Normal to $y^2 = 4c(x-d)$ can be

$$y = m(x-d) - 2cm - cm^3$$

$\Rightarrow y = mx - md - 2cm - cm^3 \ldots\ldots$ (I)

Normal to $y^2 = 4ax$ can be

$y = mx - 2am - am^3 \quad \ldots\ldots\ldots$ (II)

For common normal comparing (I) and (II)

$-md - 2cm - cm^3 = -2am - am^3$

$\Rightarrow d + 2c + cm^2 = 2a + am^2$

$\Rightarrow (a-c)m^2 = (2c + d - 2d)$

$\because a > c > 0 \Rightarrow 2c + d - 2a > 0$

$\Rightarrow 2a < 2c + d$

17. ()

Let normal be $y + tx = 2at + at^3$

$\Rightarrow -t = \tan\phi \Rightarrow t = -\tan\phi$

Normal cells curve again at $t_2$

$$\Rightarrow t_2 = -t - \frac{2}{t} = \tan\phi + 2\cot\phi$$

Slope of tangent at $t_2 = \dfrac{1}{t_2}$

Angle of intersection of normal at $t_2$

$$\tan\theta = \left|\frac{\frac{1}{t_2} + t}{1 - \frac{t}{t_2}}\right| = \left|\frac{1 + tt_2}{t_2 - t_1}\right| = \left|\frac{-1 + t^2}{t_2 - t_1}\right|$$

$$= \left|\frac{1 + \tan^2\phi}{\tan\phi + 2\cot\phi + \tan\phi}\right| = \left|\frac{\tan\phi}{2}\right|$$

$\therefore \theta = \tan^{-1}\left(\dfrac{1}{2}\tan\phi\right)$

18. ()

Equation of focal chord

$$2x - y\left(t - \frac{1}{t}\right) - 2a = 0 \text{ and its length } = a\left(t + \frac{1}{t}\right)^2$$

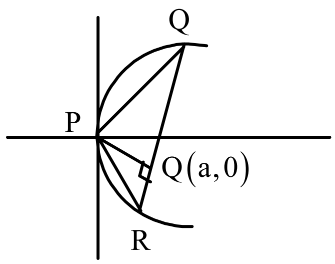
Distance of vertex focus this chord

$$P = \left|\frac{2a}{\sqrt{4 + (t - 1/t)^2}}\right| = \left|\frac{2a}{\sqrt{(t + 1/t)^2}}\right|$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 = \frac{4a^2}{P^2}$$

$\therefore$ length of chord $= a\left(t + \dfrac{1}{t}\right)^2 = \dfrac{4a^3}{P^2}$

19. ()




Area of $\Delta = A = \frac{1}{2} \times a\left(t+\frac{1}{t}\right)^2 \times \frac{2a}{(t+1/t)}$

$$\left[\because QR = a\left(t+\frac{1}{t}\right)^2, PQ = \frac{2a}{\left(t+\frac{1}{t}\right)}\right]$$

$\Rightarrow A = a^2\left(t+\frac{1}{t}\right) \Rightarrow t+\frac{1}{t} = \frac{A}{a^2}$

Difference between ordinates $= 2at - \left(\frac{2a}{-t}\right) = 2a\left(t+\frac{1}{t}\right)$

$$= 2a\frac{A}{a^2} = \frac{24}{a}$$

20. ()
Same question as in Q. 3

21. ()
Slope of normal at $P(t_1)$ and $Q(t_2)$ is

$-t_1$ and $-t_2$ respectively

$\because$ Normal are perpendicular to each other

$\Rightarrow t_1 t_2 = -1$

Equation of chord PQ

$2x - y(t_1+t_2) + 2at_1t_2 = 0$

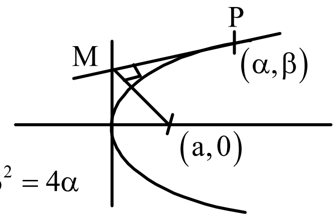
$\Rightarrow 2(x-a) - (t_1+t_2)(y) = 0$

Which passes through intersection of lines

$x - a = 0$ and $y = 0$

$\therefore$ Fixed Pt is $(a, 0)$

22. ()




Equation of tangent at P

$y\beta = 2a(x-\alpha)$

Coordinates of $M \equiv \left(0, \frac{2a\alpha}{\beta}\right)$

Area of $\Delta = \frac{1}{2}\begin{vmatrix} \alpha & \beta & 1 \\ 0 & \frac{2\alpha}{\beta} & 1 \\ 1 & 0 & 1 \end{vmatrix}$

$= \frac{1}{2}\left(\frac{2\alpha^2}{\beta} + \left(\beta - \frac{2\alpha}{\beta}\right)\right) = \frac{1}{2}\left(\frac{\beta^3}{8} + \frac{\beta}{2}\right)$

Maximum area $= (\text{for } \beta = 2) = \frac{1}{2}\left(\frac{8}{8} + \frac{2}{2}\right) = 1$

23. (A)

$y^2 - 2y - 4x + 5 = 0$

$\Rightarrow (y-1)^2 = 4(x-1)$

Equation of normal

$y - 1 = m(x-1) - 2m - m^3$

Let point on axis of parabola be $(x,1)$

$\Rightarrow 0 = m(x-1) - 2m - m^3 \Rightarrow 0 = x - 3 - m^2$

$\Rightarrow m^2 = x - 3 \geq 0 \Rightarrow x \geq 3$

24. ()

Slope of line $= \sqrt{3} = \tan\theta \Rightarrow \theta = 60^0$

Coordinate of a point at distance r from

$(\sqrt{3}, 0)$. On the line $y - \sqrt{3x} + 3 = 0$

$\equiv (\sqrt{3} + r\cos 60^0, r\sin 60^0)$

$\equiv \left(\sqrt{3} + \frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$

Putting this on parabola

$y^2 = x + 2$

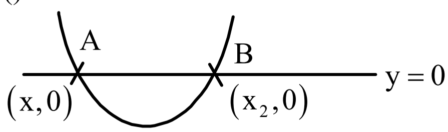
$\Rightarrow \left(\frac{\sqrt{3}r}{2}\right)^2 = \left(\sqrt{3} + \frac{r}{2}\right) + 2$

$$\Rightarrow \frac{3}{4}r^2 - \frac{r}{2} - \left(\sqrt{3}+2\right) = 0$$

$$PA.PB = |\text{Product of roots}| = \frac{\sqrt{3}+2}{3/4}$$

$$= \frac{4\left(2+\sqrt{3}\right)}{3}$$

25. ()




Equation of circle passing through A and B can be given by

$(x - x_1)(x - x_2) + y^2 + \lambda y = 0$

Where $x_1$ and $x_2$ are roots of $ax^2 + bx + c = 0$

$\therefore$ Equation will be

$$x^2 + y^2 + \frac{b}{a}x + \lambda y + \frac{c}{a} = 0$$

Length of tangent from origin $= \sqrt{\frac{c}{a}}$

26. $\because$ $\left(ap^2, 2ap\right), \left(aq^2, 2aq\right), \left(ar^2, 2ar\right)$ are

Co – normal point

$\therefore p + q + r = 0$

$\Rightarrow x = 1$ is a root of $px^2 + qx + r = 0$

Also $x = 1$ satisfies

$a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

$\therefore$ common root is $x = 1$

27. ()

$$y = \frac{a^3}{3}\left(x^2 + \frac{3}{2a}x\right)$$

$$\Rightarrow \frac{3y}{a^3} = \left(x + \frac{3}{4a}\right)^2 - \frac{9}{16a^2}$$

$$\Rightarrow \left(\frac{3y}{a^3} + \frac{9}{16a^2}\right) = \left(x + \frac{3}{4a}\right)^2$$

Vertex $h = \frac{-3}{4a}$ $k = \frac{-3a}{16}$

$hk = \frac{9}{64}$ $\therefore$ locus is $xy = \frac{9}{64}$

28. ()

Equation of tangent at $(1,2)$

$$2y = 2(x+1) \Rightarrow x - y + 1 = 0$$

Image of a variable point $(t^2, 2t)$ in the tangent

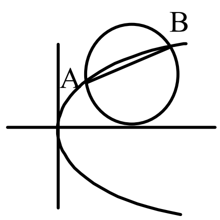
$$\frac{x - t^2}{1} = \frac{y - 2t}{-1} = \frac{-\cancel{2}(t^2 - 2t + 1)}{\cancel{2}}$$

$\Rightarrow x = 2t - 1 \quad \& \quad y = t^2 + 1$

$\Rightarrow t = \left(\frac{x+1}{2}\right) \ \& \ t^2 = (y-1)$

$\left(\frac{x+1}{2}\right)^2 = (y-1) \Rightarrow (x+1)^2 = 4(y-1)$ is the required image

29. ()




$\because$ circle touches $x$ – axis

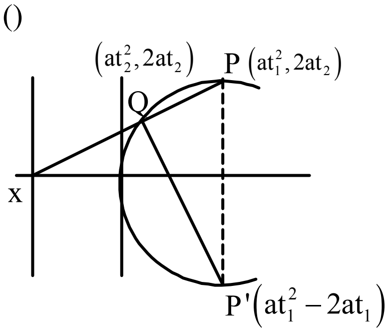
$y$ coordinate of center = radius

$$\Rightarrow \frac{2at_1 + 2at_2}{2} = r$$

$$\Rightarrow t_1 + t_2 = \frac{r}{a}$$

Slope of $AB = \frac{2}{t_1 + t_2} = \frac{2a}{r}$

30. ()




Equation of PQ

$2x - y(t_1 + t_2) + 2at_1t_2 = 0$

$\because$ It passes through $(-a, 0)$

$\Rightarrow -2a + 0 + 2at_1t_2 = 0$

$\Rightarrow t_1t_2 = 1 \quad$ ………. (I)

$\Rightarrow t_2 = \frac{1}{t_1}$

Equation of P'Q $\Rightarrow 2x - y(t_2 - t_1) - 2at_1t_2 = 0$

$\Rightarrow 2x - y(t_2 - t_1) - 2a = 0$

$\Rightarrow 2(x - a) - (t_2 - t_1)y = 0$

Which passes through $(a, 0)$ i.e. focus

31. Let mid point of $PQ$ be $(h, k)$

Equation of chord PQ

$xh + yk = h^2 + k^2$

$\Rightarrow y = \left(\frac{-h}{k}\right)x + \frac{h^2 + k^2}{k}$

For this to be tangent to parabola

$c = a/m$

$\Rightarrow \frac{h^2 + k^2}{k} = \frac{a}{(-h/k)} \Rightarrow h(h^2 + k^2) + ak^2 = 0$

$\therefore$ locus is

$x(x^2 + y^2) + ay^2 = 0$

32. ()

Let mid point be $P(h, k)$

Equation of chord $\Rightarrow xh - 2(y + k) = h^2 - 4k$

$\Rightarrow x = \left(\frac{2}{h}\right)y + \frac{h^2 - 2k}{h}$ …… (I)

Equation of normal to $x^2 = 4y$

$x = my - 2m - m^3$ ………. (II)

Comparing (I) and (II)

$\frac{h^2 - 2k}{h} = -2\left(\frac{2}{h}\right) - \left(\frac{2}{h}\right)^3$

$\Rightarrow h^2 - 2k = -4 - \frac{8}{h^2} \Rightarrow 2k = h^2 + \frac{8}{h^2} + 4$

$\therefore$ Locus is $2y = x^2 + \frac{8}{x^2} + 4$

33. ()

Equation of tangent to parabola $y^2 = 4ax$ is

$yt = x + at^2$ ………… (I)

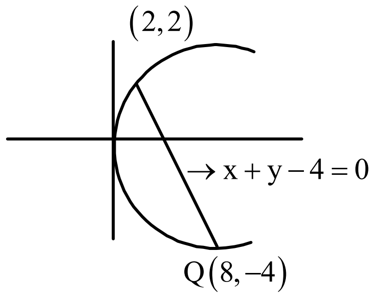
For (I) to be normal to circle $x^2 + y^2 - 2ax - 2by + c = 0$ it should pass through its centre $(a, b)$

$\therefore bt = a + at^2 \Rightarrow at^2 - bt + a = 0$

For 2 distinct tangent $D > 0$

$\Rightarrow b^2 - 4a^2 > 0 \Rightarrow b^2 > 4a^2$

34. ()




Point $\left(P^2, P-2\right)$ should lie inside the parabola as well as on the same side of the chord PQ as the origin

$(P-2)^2 - 2P^2 < 0 \Rightarrow -P^2 - 4P + 4 < 0$

$\Rightarrow P^2 + 4P - 4 > 0$

$\Rightarrow P \in \left(-\infty, -2-2\sqrt{2}\right) \cup \left(-2+2\sqrt{2}, \infty\right)$ ……….. (I)

Also, $P^2 + P - 2 - 4 < 0 \Rightarrow P^2 + P - 6 < 0$

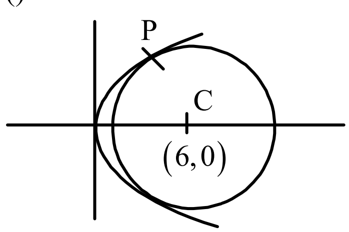
$\Rightarrow (P+3)(P-2) < 0$

$\Rightarrow -3 < P < 2$ …………… (II)

$(I) \cap (II)$

$\Rightarrow -2 + 2\sqrt{2} < P < 2$

35. ()




For having a common tangent circle should touch the parabola

$\therefore$ It centre well lie on normal to the parabola at $P\left(t^2 - 2t\right)$

Substituting $(6,0)$ in equation of normal at P, we get $t = 0, 2 \,\&\, -2$

Least distance $CP = \sqrt{20}$

$\therefore$ For no common tangent

$CP >$ radius

$\Rightarrow r < \sqrt{20}$

36. ()

Equation of tangent to $y^2 = 4ax$ with slope m

$y = mx + \frac{a}{m}$ ………. (I)

Equation of normal to $x^2 = 4by$ of slope m.

$y = mx + 2b + \frac{b}{m^2}$ ……. (II)

Comparing (I) and (II)

$\frac{a}{m} = 2b + \frac{b}{m^2}$

$\Rightarrow 2bm^2 - am + b = 0$

For 2 tangents $D > 0$

$\Rightarrow a^2 - 8b^2 > 0 \Rightarrow b^2 < \frac{a^2}{8}$

$\Rightarrow |b| < \frac{|a|}{2\sqrt{2}}$

37. ()

Let mid point of chord be $\left(at^2, 2at\right)$

Equation of chord

$x.at^2 + y.2at = a^2t^4 + 4a^2t^2$

$\because$ It passes through $(a, a)$

$\Rightarrow a^2t^2 + 2a^2t = a^2t^4 + 4a^2t^2$

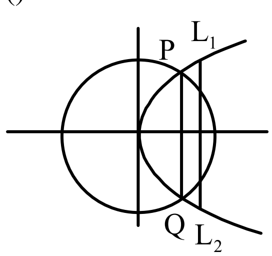
$\Rightarrow t\left(t^3 + 3t^2 - 2\right) = 0$

$t = 0$ and one real root from $t^3 + 3t^2 - 2 = 0$

$\therefore$ 2 values of t.

$\therefore$ 2 chords are possible

38. ()




$P \equiv \left(at^2, 2at\right)$

Equation of circle

$x^2 + y^2 = \frac{9a^2}{4}$

$\because$ P lies on circle also

$\Rightarrow \left(at^2\right)^2 + (2at)^2 = \frac{9}{4}a^2$

$\Rightarrow 4t^4 + 16t^2 - 9 = 0 \Rightarrow \left(2t^2 + a\right)\left(2t^2 - 1\right)$

$\Rightarrow t = \pm\frac{1}{\sqrt{2}}$

$PQ = 4at = 2\sqrt{2}a$

$L_1L_2 = 4a$

Distance between PQ and $L_1L_2 = \frac{a}{2}$

$\therefore \text{area} = \frac{1}{2} \times \frac{a}{2}\left(4a + 2\sqrt{2}a\right) = a^2\left(\frac{2+\sqrt{2}}{2}\right)$

39. ()

Equation of normal

$y + tx = 2at + at^3$

$A \equiv (\alpha, \beta)$ lies on it

$\Rightarrow \beta + t\alpha = 2at + at^3$

$\Rightarrow at^3 + (2a - \alpha)t - \beta = 0$ .......... (I)

$at^3 + (2a - \alpha)t = \beta$

Squaring

$a^2t^6 + (2a-\alpha)^2 t^2 + 2a(2a-\alpha)t^4 = \beta^2$

Focal distance $= sp = r = a(1 + t^2)$

$$\Rightarrow t^2 = \left(\frac{r-a}{a}\right)$$

$\therefore$ Equation becomes

$$a^2\left(\frac{r-a}{a}\right)^3 + (2a-\alpha)^2\left(\frac{r-a}{a}\right) + 2a(2a-\alpha)\left(\frac{r-a}{a}\right)^2 - \beta^2 = 0$$

SP. SQ. SR = (Product of roots)

$$= -\left(\frac{-a^2 - (2a-\alpha)^2 + 2a(2a-\alpha) - \beta^2}{\frac{1}{a}}\right)$$

$$= -a\left(-a^2 - 4\cancel{a^2} + 4a\alpha - \alpha^2 + 4\cancel{a^2} - 2a\alpha - \beta^2\right)$$

$$= -a\left(-a^2 + 2a\alpha - \alpha^2 - \beta^2\right)$$

$$= a\left(a^2 - 2a\alpha + \alpha^2 + \beta^2\right)$$

$$= a\left((\alpha - a)^2 + \beta^2\right)$$

$$= a(SA)^2$$

$$\therefore I = a$$

40. $P \equiv (h, k)$

Equation of chord of contact

$$yk = 2a(x + h)$$

$$\Rightarrow y = \left(\frac{2a}{k}\right)x + \left(\frac{2ah}{k}\right) \qquad \text{……….. (I)}$$

Equation of tangent to $x^2 = 4by$

$$y = mx - bm^2 \quad \text{……….. (II)}$$

From (I) and (II)

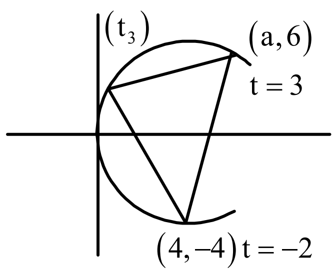
$$\frac{2ah}{k} = -\left(\frac{2a}{k}\right)^2$$

$\Rightarrow 2akh = -4a^2b$

$\Rightarrow$ locus is $xy = -2ab$

i.e. a hyperbola

41. ()




Substituting P and Q in the parabola we get $a = 1, b = 0$

Area of $\Delta = a^2\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$

$= a^2\left|(t_3 + 2)(t_3 - 3)(5)\right|$

$= 5a^2\left|t^2 - t - 6\right|$

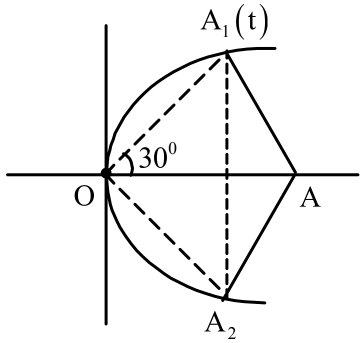
$= 5a^2\left|(t - 1/2)^2 - \frac{25}{4}\right|$

Area $= 5a^2\left(\frac{25}{4} - (t - 1/2)^2\right)$

Area will maximum when $t = 1/2$

$\therefore$ Pt is $\left(\frac{1}{4}, \frac{1}{2}\right)$

42. ()




For $OAA_1$ to be equilateral

$$\frac{2}{t} = \tan 30^0 = \frac{1}{\sqrt{3}} \Rightarrow t = 2\sqrt{3}$$

Also, equation of normal $y + tx = 2at + at^{-3}$

$\because$ A lies on it

$\Rightarrow ht = 4t + 2t^3$

$\Rightarrow h = 4 + 2t^2$

$= 4 + 2 \times 12 = 28$

(21) Hint: Common tangent Concept.

(22) Hint: Apply condition of tangency and check options.

(23) $y^2 = 2px$ —— (1)

focus $\equiv \left(\frac{p}{2}, 0\right)$, directrix $x = -\frac{p}{2}$

$\Rightarrow 2x + p = 0$

so circle is

$\left(x - \frac{p}{2}\right)^2 + y^2 = r^2$ —— (2)

As circle touches the directrix $2x + p = 0$, so

Applying condn of tangency

$$\left|\frac{2 \cdot \frac{p}{2} + p}{2}\right| = r$$

$$|p| = r$$

so eqn (2) becomes

$\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$ —— (3)

solving eqn (1) and (3) simultaneously

options

(24) Hint: direct property.

(25) Hint: Use concept of perpendicular chords.

(26) Hint: make the diagram using given information.

(27) Hint: Use concept of Normals and property.

(28) Let $(x_1, y_1) \equiv (at^2, 2at)$

Tangent at this point is $ty = x + at^2$

Any point on this tangent is $\left(h, \frac{h+at^2}{t}\right)$

Chord of contact of this point w.r.to. the circle $x^2+y^2=a^2$ is

$$hx + \left(\frac{h+at^2}{t}\right)y = a^2$$

or $$(aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$$

which is a family of st. lines passing through the point of intersection of

$$\left.\begin{array}{l} aty - a^2 = 0 \\ \text{and} \\ x + \frac{y}{t} = 0 \end{array}\right\} \text{fixed point is } \left(-\frac{a}{t^2}, \frac{a}{t}\right)$$

$$\therefore\ x_2 = -\frac{a}{t^2},\ y_2 = \frac{a}{t}$$

Clearly $x_1 x_2 = -a^2$, $y_1 y_2 = 2a^2$

Also $$\frac{x_1}{x_2} = -t^4$$

$$\frac{y_1}{y_2} = 2t^2$$

$$\Rightarrow\ 4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$$

(29) $t_2 = -t_1 - \frac{2}{t_1}$

Sd

Also $\frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$

$\Rightarrow t_1 \cdot t_2 = -4$

$\therefore -\frac{4}{t_1} = -t_1 - \frac{2}{t_1}$

$\Rightarrow t_1^2 + 2 = 4 \Rightarrow t_1 = \pm\sqrt{2}$

So point can be $(2a, \pm 2a\sqrt{2})$

(30) As a Circle can intersect a parabola in 4 points

AB → so, quadrilateral may be cyclic.

The diagonals of the quadrilateral may be equal as the quadrilateral may be isosceles trapezium

A rectangle can not be inscribed in a parabola, so (C) is wrong.

(31)
(32)
(33)
(34)
} Concept Question

(31.) Any point on the line $x+y=1$ is $(h, 1-h)$.

eq$^n$ of chord whose mid point is known w.r.t. to $y^2 = 4ax$

is $T = S_1$

$$(1-h)y - 2a(x+h) = (1-h)^2 - 4ah \quad -(1)$$

eq$^n$ (1) is passing through $(a, 2a)$, so.

$$(1-h)2a - 2a(a+h) = (1-h)^2 - 4ah$$

$$\Rightarrow 2a - \cancel{2ah} - 2a^2 - \cancel{2ah} = h^2 - 2h + 1 - \cancel{4ah}$$

$$\Rightarrow h^2 - 2h + 1 + 2a^2 - 2a = 0$$

$$D > 0 \Rightarrow B^2 - 4AC > 0$$

$$\cancel{4} - \cancel{4}(1)(1 + 2a^2 - 2a) > 0$$

$$\cancel{1} - \cancel{1} - 2a^2 + 2a > 0$$

$$-2a(a-1) > 0$$

$$\Rightarrow a(a-1) < 0$$

$$\Rightarrow 0 < a < 1$$

$$\Rightarrow \underline{0 < 4a < 4}$$

so L. R. can be (A, B, D)

(32) direct (Solve Both eqns and use condn of common tangency.)

(33) $y^2 = 4x$

Normal to the given parabola with slope m is

$y = mx - 2m - m^3$ —(1)

eqn (1) is passing through (9,6), so

$6 = 9m - 2m - m^3$

$\Rightarrow$ ~~$m^3 - 7m$~~ $m^3 - 7m + 6 = 0$

$\Rightarrow (m-1)(m^2+m-6) = 0$

$\Rightarrow (m-1)(m+3)(m-2) = 0$

$\Rightarrow m = 1, -3, 2$

options A, B, D.

(34) $A \equiv (at_1^2, 2at_1)$

$B \equiv (at_2^2, 2at_2)$

$OA \perp^r OB \Rightarrow t_1 t_2 = -4$

find OA and OB using distance formula.

Substitute $OA^2 = r_1^2$

$OB^2 = r_2^2$

then simplify the expression.

Passages

Any parabola whose axes is parallel to x-axis will be of the form

$$(y-a)^2 = 4b(x-c) \quad \text{--- (1)}$$

Now $lx+my=1$ can be rewritten as

$$y-a = -\frac{l}{m}(x-c) + \frac{1-lc-am}{m} \quad \text{--- (2)}$$

eqn (2) will touch (1) if

$$\frac{1-am-lc}{m} = -\frac{b}{l/m}$$

$$-\frac{l}{m} = \frac{bm}{1-am-lc}$$

$$\Rightarrow cl^2 - bm^2 + alm - l = 0 \quad \text{--- (3)}$$

But given that $5l^2 + 6m^2 - 4lm + 3l = 0$ --- (4)

combining (3) and (4), we get,

$$\frac{c}{5} = \frac{-b}{6} = \frac{a}{-4} = \frac{-1}{3}$$

$$\Rightarrow c = -5/3, \; b = 2 \text{ and } a = \frac{4}{3}$$

so parabola is

$\left(y-\frac{4}{3}\right)^2 = 8\left(x+\frac{5}{3}\right)$ whose focus is $\left(\frac{1}{3}, \frac{4}{3}\right)$ and directrix is $3x+11=0$

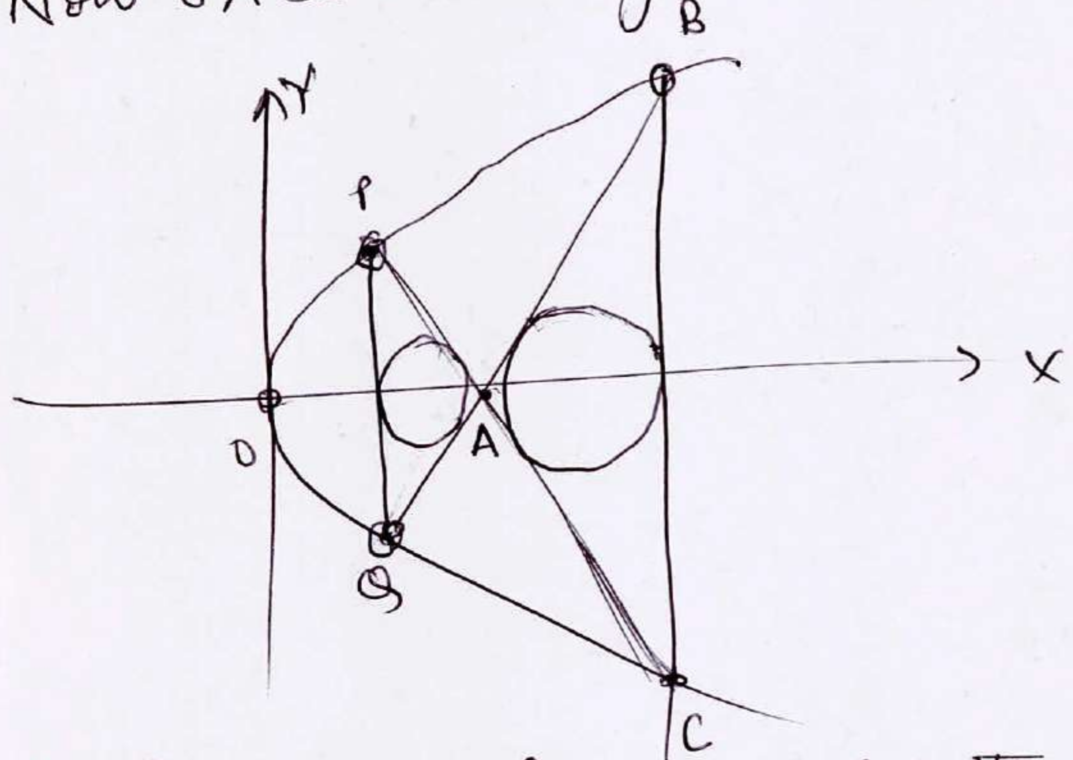
Passage-6 (b, c, d) (2.)

Solving given parabolas

$-8(x-a) = 4x \Rightarrow x = \frac{2a}{3}$

$\Rightarrow$ points of intersection are $\left(\frac{2a}{3}, \pm\sqrt{\frac{8a}{3}}\right)$

Now OABC is concyclic.




$\Delta APQ$ is isosceles right angled. Therefore, slope of PA is $-1$ and its equation is

$y-2 = -(x-1)$ or $x+y=3$

Similarly eq$^n$ of line QB is $x-y-3=0$

solving $x+y=3$ with the parabola $y^2=4x$, we have $(3-x)^2 = 4x$ or $x^2 - 10x + 9 = 0 \Rightarrow x = 1, 9$

Therefore co-ord. of B and C are $(9,-6)$ and $(9,6)$ resp.

③

∴ Area of trapezium PBCQ $= \frac{1}{2}(12+4)\times 8$

$= 64$ sq. units

let the circumcentre of trapezium PBCQ is T(h, 0). PT = TB

$\sqrt{(h-1)^2+4} = \sqrt{(h-9)^2+36}$

$\Rightarrow h = 7$

Hence radius is $\sqrt{40} = 2\sqrt{10}$

let the inradius of $\Delta PAQ$ be $r_1$, then $r_1 = \frac{\Delta_1}{s_1}$

$r_1 = \frac{\frac{1}{2}\times 4\times 2}{4+2\sqrt{4+4}} = \sqrt{2}-1$

let inradius of $\Delta ABC$ be $r_2$, then

$r_2 = \frac{\Delta_2}{s_2} = \frac{\frac{1}{2}\times 12\times 6}{12+2\sqrt{36+36}}$

$= \frac{3}{1+\sqrt{2}} = 3(\sqrt{2}-1)$

so $\frac{r_2}{r_1} = 3$

## Passage-7

$$|z-5-3i| = \frac{z+\bar{z}-6}{2}$$

Use $z = x+iy$

On simplification it gives

$$(y-3)^2 = 4(x-4)$$

with L.R. $= 4$, Vertex $(4,3)$

focus $=(5,3)$

Based on the above information options are

(53) C, (54) A (55) B.

## Passage-8

A line through $P(6,5)$ having slope m is

$y-5 = m(x-6)$

$\Rightarrow y = mx + 5 - 6m$ — (1)

eqn (1) is a tangent to $y^2=4x$ so,

$$5-6m = \frac{1}{m} \Rightarrow 6m^2-5m+1=0$$

$m = 1/2, \ 1/3$

so tangents PQ and PR are,

PQ: $2y = x + 4$

PR: $3y = x + 9$

so solving ~~PR with~~ tangent PR with $y^2 = 4x$
point R is $(9, 6)$.

Now eq^n of a family of circles touching a given ~~the PR and~~ line PR at a given point R is

$(x-9)^2 + (y-6)^2 + \lambda(3y - x - 9) = 0$ —— (2)

eq^n (2) is passing through focus $(1,0)$ so,

$64 + 36 + \lambda(-10) = 0$

$\lambda = 10$

eq^n (2) becomes

$x^2 + y^2 - 28x + 18y + 27 = 0$

whose radius is $5\sqrt{10}$.

Based on above information

Q.56 —(A)

Q.57 —(B)

Q.58 —(C)

# Solutions
# PARABOLA
# Ex. 3

Q.1

Let P be (h, k). Also let tangents from P be $t_1y = x + at_1^2$ & $t_2y = x + at_2^2$, where points of contact of these tangents being $Q(at_1^2, 2a_1)$ & $R(at_2^2, 2a_2)$.

Now point of intersection of tangents will be $h = at_1t_2, k = a(t_1 + t_2)$. Area of triangle PQR will now be given by

$$\frac{1}{2}\left\|\begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_1t_2 & a(t_1+t_2) \end{vmatrix}\right\| = 4a^2 \text{ which implies } (t_1 - t_2)^2 = 4.$$

But $h = at_1t_2, k = a(t_1 + t_2) \Rightarrow a^2(t_1 - t_2)^2 = k^2 - 4ah$, hence $k^2 - 4ah = 16a^2$.

Required locus is $y^2 = 4a(x - 4a)$ which is a parabola.

Q.2

Let P & Q be $(at_1^2, 2at_1)$ & $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

Now $OQ^2 = a(t_2^4 + 4t_2^2)$ or $OQ^2 = a^2\left(\left(t_1 + \frac{2}{t_1}\right)^4 + 4\left(t_1 + \frac{2}{t_1}\right)^2\right)$

$$\Rightarrow OQ^2 = a^2\left(\left(t_1 + \frac{2}{t_1}\right)^2 + 2\right)^2 - 4a^2. \text{ But by A.M.} \geq \text{G.M.,} \quad \left|t_1 + \frac{2}{t_1}\right| \geq 2\sqrt{2}.$$

$$\Rightarrow \left(\left(t_1 + \frac{2}{t_1}\right)^2 + 2\right)^2 \geq 100. \text{ Hence } |OQ| \geq 4a\sqrt{6}.$$

Q.3

If normal at $P(t_1)$ & $(t_2)$ meet on the parabola, then $t_1t_2 = 2$.

Also P,Q,R & N(point of intersection of normals) will form a cyclic quadrilateral and circle passing through P, Q & R will have RN as diameter as $\angle RPN = \frac{\pi}{2}$.

Now coordinates of R will be $(at_1t_2, a(t_1 + t_2))$ or $(2a, a(t_1 + t_2))$. Similarly coordinates of N will be $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$ or $(a(t_1^2 + t_2^2 + 4), -2a(t_1 + t_2))$

Now let the circum center be (h, k), then

$$h = \frac{a(t_1^2 + t_2^2 + 6)}{2} \text{ \& } k = -\frac{a(t_1 + t_2)}{2} \Rightarrow \frac{2h}{a} - 6 = t_1^2 + t_2^2 \text{ \& } \frac{4k^2}{a^2} = t_1^2 + t_2^2 + 2t_1t_2$$

Or eliminating t gives & replacing (h, k) with (x, y) gives required locus as $2y^2 = a(x - a)$.

Q.4

Substituting $y = ax^2 - b$ in $x^2 + y^2 = 1$ gives $x^2 + \left(ax^2 - b\right)^2 = 1$ or $a^2x^4 + \left(1 - 2ab\right)x^2 + b^2 - 1 = 0$.

Now for four distinct points of intersection the above equation must have four distinct real roots.
As the given equation is a biquadratic so considering $x^2 = t$ gives a quadratic in $t$ both of whose roots must be real & positive.

Hence $a^2, 2ab - 1, b^2 - 1$ must be of same sign and $\left(1 - 2ab\right)^2 > 4a^2\left(b^2 - 1\right)$.

$\Rightarrow$ $2ab > 1, b > 1, 4a^2 - 4ab + 1 > 0$.

Clearly if a > b > 1, then all the above conditions get satisfied.
(remember here that a > b > 1 is a sufficient condition and may not be necessary)

Q.5

Let P, Q, P'& Q' be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right) \& \left(at_4^2, 2at_4\right)$.

Now PQ is a focal chord & PP', QQ' are normal chords hence $t_2 = -\frac{1}{t_1}, t_3 = -t_1 - \frac{2}{t_1} \& t_4 = \frac{1}{t_1} + 2t_1$.

Slope of PQ = $\frac{2t_1 - 2t_2}{t_1^2 - t_2^2}$ or $\frac{2t_1}{t_1^2 - 1}$. Similarly

Slope of P'Q'= $\frac{2t_4 - 2t_3}{t_4^2 - t_3^2}$ or $\frac{2t_1}{t_1^2 - 1}$, hence PQ is parallel to P'Q'.

Also $PQ = a\left(t_1 + \frac{1}{t_1}\right)^2$ & P'Q' = $a\sqrt{\left(t_3^2 - t_4^2\right)^2 + 4\left(t_3 - t_4\right)^2}$ or $a\left|t_3 - t_4\right|\sqrt{\left(t_3 + t_4\right)^2 + 4}$

$\Rightarrow P'Q' = 3a\left(t_1 + \frac{1}{t_1}\right)^2$, hence P'Q' = 3 PQ.

Q.6

Let the fixed point on axis be P(h, 0), then any line passing through this point will be y = m(x – h).
Substituting $\left(at^2, 2at\right)$ this gives $amt^2 - 2at - hm = 0$.

$\Rightarrow$ $t_1 + t_2 = \frac{2}{m} \& t_1t_2 = -\frac{h}{a}$, where $t_1$ & $t_2$ are parameters of those points where this line meets the parabola. Also $t_1^2 + t_2^2 = \frac{4}{m^2} + \frac{2h}{a}$.

Now circle having this chord as diameter will be

$x^2 + y^2 - a\left(t_1^2 + t_2^2\right)x - 2a\left(t_1 + t_2\right)y + a^2t_1^2t_2^2 + 4a^2t_1t_2 = 0$.

Or $x^2 + y^2 - a\left(\frac{4}{m^2} + \frac{2h}{a}\right)x - \frac{4a}{m}y + h^2 - 4ah = 0$.

Now if we consider two such circles with $m = m_1$ & $m = m_2$, then radical axis of these circles will be

$\left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right)x + \left(\frac{1}{m_1} - \frac{1}{m_2}\right)y = 0$ or $\left(m_1 + m_2\right)x + m_1m_2y = 0$.

Clearly it passes through the origin.

Q.7

Comparing P(16, 16) with $\left(4t^2, 8t\right)$ gives t = 2.
Now tangent at P will be 2y = x + 16 & normal at P will be 2x + y = 48.
Points where these lines meet the x-axis will be A(-16, 0) & B(24, 0).
As angle APB is a right angle hence the circle passing through P, A & B will have AB as diameter.
Hence $C_1 : (x+16)(x-24)+y^2 = 0$.
Equation of common chord of $C_1$ & $C_2$ will be $6x + y + 197 = 0$.

Q.8

Let $l = at^2$ & $m = 2at$. Now vertices of the triangle are $A(0,2), B\left(0, \frac{1}{2at}\right)$ & $C\left(\frac{1-4at}{at^2}, 2\right)$.

As the triangle is right angled hence by the concept of Euler's line its circum center (x, y) will be $\left(0+0+\frac{1-4at}{at^2}, 2+\frac{1}{2at}+2\right)$.

Now $t = \frac{1}{2a(y-4)} \Rightarrow x = 4a(y-6)(y-4)$, which is equation of a parabola.

Q.9

Any tangent to $y^2 = 4ax$ will be $y = mx + \frac{a}{m}$ and any normal to $x^2 = 4by$ will be $y = mx + 2b + \frac{b}{m^2}$

Comparing the two equations gives $\frac{a}{m} = 2b + \frac{b}{m^2}$ or $2bm^2 - am + b = 0$.

For this equation to have real & distinct roots $a^2 > 8b^2$.

Q.10

Let B & C be $\left(at_1^2, 2at_1\right)$ & $\left(at_2^2, 2at_2\right)$ such that A is $\left(at_1t_2, a(t_1+t_2)\right)$. Also let another tangent be drawn at $D\left(at_3^2, 2at_3\right)$ such that P & Q are $\left(at_1t_3, a(t_1+t_3)\right)$ & $\left(at_2t_3, a(t_2+t_3)\right)$.

Now $AP = a|t_2 - t_3|\sqrt{t_1^2+1}, AQ = a|t_3 - t_1|\sqrt{t_2^2+1}$.

Also $AB = a|t_2 - t_1|\sqrt{t_1^2+1}$ & $AC = a|t_2 - t_1|\sqrt{t_2^2+1}$.

$\Rightarrow \frac{AP}{AB} + \frac{AQ}{AC} = \frac{|t_2 - t_3| + |t_3 - t_1|}{|t_2 - t_1|}$.

Now considering $t_1$, $t_3$ & $t_2$ in cyclic order we get $\frac{AP}{AB} + \frac{AQ}{AC} = 1$.

Q.11

Let the point K be (h,0) and slope of chord through K be tan θ, then any point on this line at a distance r from K will be (h + r cos θ, r sin θ).
For r = PK & r = QK, this point will satisfy the equation of parabola, hence by substituting these coordinates in the equation of the parabola we get $\left(\sin^2\theta\right)r^2 - \left(4a\cos\theta\right)r - 4ah = 0$.

Roots of this equation are PK & -QK, hence PK – QK = $\frac{4a\cos\theta}{\sin^2\theta}$, & PK.QK = $\frac{4ah}{\sin^2\theta}$.

Now $\frac{1}{PK^2}+\frac{1}{QK^2}=\frac{(PK-QK)^2+2PK.QK}{(PK.QK)^2} \Rightarrow \frac{1}{PK^2}+\frac{1}{QK^2}=\frac{16a^2\cos^2\theta+8ah\sin^2\theta}{64a^2h^2}$

Clearly if h = 2a, then $\frac{1}{PK^2}+\frac{1}{QK^2}=\frac{1}{4h^2}$.

Q.12

Any tangent to $y^2=4a(x+a)$ : $y=mx+am+\frac{a}{m}$

& an orthogonal tangent to $y^2=4b(x+b)$ : $y=-\frac{1}{m}x-\frac{b}{m}-bm$.

Arranging both the equations as quadratic equations in m gives

$bm^2+my+x+b=0$ & $(x+a)m^2-ym+a=0$.

Comparing the two equations gives $\frac{b}{x+a}=\frac{y}{-y}=\frac{x+b}{a} \Rightarrow x+a+b=0$.

Now combining $y^2=4a(x+a)$ & $y^2=4b(x+b)$ in order get a linear equation we get common chord as x + a + b = 0.

Q.13

Let the fixed parabola be $y^2=4ax$ & the variable parabola be $(y-k)^2=-4a(x-h)$ having vertex at P(h, k).

Now as the two parabolas touch hence tangent to the two parabolas at $(at^2,2at)$ must be the same.

Tangent to $y^2=4ax$ will be $y=mx+\frac{a}{m}$ touching it at $T\left(\frac{a}{m^2},\frac{2a}{m}\right)$ &

that to $(y-k)^2=-4a(x-h)$ will be $y-k=m(x-h)-\frac{a}{m}$ or $y=mx+k-mh-\frac{a}{m}$.

Comparing the two equations gives $k-mh=\frac{2a}{m}$ & substituting coordinates of T in equation of variable parabola gives $\left(\frac{2a}{m}-k\right)^2=-4a\left(\frac{a}{m^2}-h\right)$.

Or $hm^2-km+2a=0$ & $(k^2-4ah)m^2-4akm+8a^2=0$.

Comparing the two equations in order to eliminate m gives

$\frac{k^2-4ah}{h}=4a$ or $k^2=8ah$, hence required locus is $y^2=8ax$.

Q.14

Adding the two equations gives $x^2+6x-4y+13=0$ or $(x+3)^2=4(y-1)$, which means each of the points A, B, C & D lie on a parabola with vertex at (-3, 1) and focus at P(-3, 2).

Hence PA, PB, PC, PD will be focal distances of these points.

Now let any point on this parabola be $(2t-3, t^2+1)$. Substitute these coordinates in the equation $x^2-y^2+6x+16y-46=0$ to get $t^4-18t^2+24t+40=0$.

Now let the roots of this be $t_1,t_2,t_3,t_4$, then

$t_1+t_2+t_3+t_4=0, t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=-18$

Also Focal distance of a point with parameter t will be $1+t^2$, hence

PA + PB + PC + PD = 4 + $t_1^2+t_2^2+t_3^2+t_4^2$.

Now from above relations

$t_1^2+t_2^2+t_3^2+t_4^2=\left(t_1+t_2+t_3+t_4\right)^2-2\left(t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4\right)$

Therefor PA + PB + PC + PD = 40.

Q.15

Normal to $y^2=4ax$ at any point P(t) will be $tx+y=2at+at^3$.

This will meet the x-axis at $Q\left(2a+at^2,0\right)$.

The line perpendicular to normal and passing through Q will be $x-ty=2a+at^2$.

Now this equation may be rearranged as $y=m\left(x-2a\right)-\frac{a}{m}$, where $m=\frac{1}{t}$.

Clearly its in form of tangent line of slope m to the parabola $y^2=-4a\left(x-2a\right)$.

Q.16

Let mid point of any such chord be $M\left(at^2,2at\right)$.

Now using T = $S_1$, equation of chord of $x^2+y^2=16a^2$ having mid point at M may be represented as $at^2x+2aty=a^2t^4+4a^2t^2$.

As this chord is drawn through (h, 0) hence substituting these coordinates in equation of chord we get $at^2x+2aty=a^2t^4+4a^2t^2$.

Now the above equation gives three values of t, namely $0 \,\&\, \pm\sqrt{\frac{h-4a}{a^2}}$ out of which the later two values will be real & other than 0 only if h > 4a.

Also for M to be mid point of chord it must lie inside the circle hence

$a^2t^4+4a^2t^2-16a^2<0$ or $\left(\frac{4a-h}{a^2}\right)^2+4\left(\frac{4a-h}{a^2}\right)-16<0$, hence $h<\left(\sqrt{5}+1\right)2a$.

Q.17

Let A, B & P be $\left(at_1^2,2at_2\right),\left(at_2^2,2at_2\right)\&\left(at_1t_2,a\left(t_1+t_2\right)\right)$ on $y^2=4ax$.

Tangent PB will be $x-t_2y+at_2^2=0$.

Now any circle touching PB at P may be represented as family of point circle having center at P and the line PB i.e. $\left(x-at_1t_2\right)^2+\left(y-a\left(t_1+t_2\right)\right)^2+l\left(x-t_2y+at_2^2\right)=0$.

As this circle passes through F(a, 0), hence $l=-a\left(1+t_1^2\right)$.

Now the circle touching PB at P & passing through F is

$\left(x-at_1t_2\right)^2+\left(y-a\left(t_1+t_2\right)\right)^2-a\left(1+t_1^2\right)\left(x-t_2y+at_2^2\right)=0$

Substituting coordinates of A in L.H.S. of equation of circle gives

$\left(at_1^2-at_1t_2\right)^2+\left(2at_1-a\left(t_1+t_2\right)\right)^2-a\left(1+t_1^2\right)\left(at_1^2-2at_1t_2+at_2^2\right)$ or $\left(t_1-t_2\right)^2\left(t_1^2+1-\left(1+t_1^2\right)\right)$ which is zero, hence this circle passes through A.

Q.18

(i) Let P, Q & R be the vertices of a triangle formed by three tangents of $y^2 = 4ax$, then the coordinates of these points can be taken as $\left(at_1t_2, a\left(t_1 + t_2\right)\right), \left(at_2t_3, a\left(t_2 + t_3\right)\right) \& \left(at_3t_1, a\left(t_3 + t_1\right)\right)$. Also the focus is S(a, 0).

Now $m_{PQ} = \frac{1}{t_2}, m_{PR} = \frac{1}{t_1}, m_{FQ} = \frac{t_2 + t_3}{t_2t_3 - 1} \& m_{FR} = \frac{t_3 + t_1}{t_3t_1 - 1}$.

Let angle between PQ & PR be $\alpha$, then $\tan\alpha = \dfrac{\frac{1}{t_2} - \frac{1}{t_1}}{1 + \frac{1}{t_2}\frac{1}{t_1}}$ i.e. $\frac{t_1 - t_2}{t_1t_2 + 1}$.

Similarly let angle between FQ & FR be $\beta$, then $\tan\beta = \dfrac{\frac{t_3 + t_1}{t_3t_1 - 1} - \frac{t_2 + t_3}{t_2t_3 - 1}}{1 + \frac{t_3 + t_1}{t_3t_1 - 1}\frac{t_2 + t_3}{t_2t_3 - 1}}$ i.e. $\frac{t_2 - t_1}{t_1t_2 + 1}$.

(Here take care to put slopes in same cyclic order to get correct angles)

Clearly $\alpha \& \beta$ are supplementary angles, hence PQFR is a cyclic quadrilateral.

(ii) Altitude through P must be perpendicular to tangent QR, hence its slope will be $-t_3$.

Equation of this altitude will be $y - a\left(t_1 + t_2\right) = -t_3\left(x - at_1t_2\right)$.

Similarly altitude through Q will be $y - a\left(t_2 + t_3\right) = -t_1\left(x - at_2t_3\right)$.

Eliminating y between these two equations gives $x = -a$, hence orthocenter lies on directrics.

Q.19

Any circle touching the parabola at P(t) will also touch the tangent to parabola at P.

Now any circle touching the line $y = tx + at^2$ at P may be represented as family of point circle having center at P and the line PB i.e. $\left(x - at^2\right)^2 + \left(y - 2at\right)^2 + \lambda\left(x - ty + at^2\right) = 0$.

As this circle passes through F(a, 0), hence $\lambda = -a\left(1 + t^2\right)$.

Hence the circle touching the parabola at P & passing through F is

$\left(x - at^2\right)^2 + \left(y - 2at\right)^2 - a\left(1 + t^2\right)\left(x - ty + at^2\right) = 0$

Similarly the circle touching the parabola at Q & passing through F is

$\left(x - \frac{a}{t^2}\right)^2 + \left(y + \frac{2a}{t}\right)^2 - a\left(1 + \frac{1}{t^2}\right)\left(x + \frac{1}{t}y + \frac{a}{t^2}\right) = 0$, note that P & Q are end points of a focal chord.

The two equation of circles simplify to

$x^2 + y^2 - a\left(3t^2 + 1\right)x + at\left(t^2 - 3\right)y + 3a^2t^2 = 0$ & $x^2 + y^2 - a\left(\frac{3}{t^2} + 1\right)x + \frac{a}{t}\left(3 - \frac{1}{t^2}\right)y + \frac{3a^2}{t^2} = 0$.

Now $g_1 = \frac{a\left(3t^2 + 1\right)}{2}, f_1 = -\frac{at\left(t^2 - 3\right)}{2}, c_1 = 3a^2t^2$ & $g_2 = \frac{a\left(3 + t^2\right)}{2t^2}, f_2 = -\frac{a\left(3t^2 - 1\right)}{2t^3}, c_2 = \frac{3a^2}{t^2}$ gives

$2g_1g_2 + 2f_1f_2 = 2\frac{a\left(3t^2 + 1\right)}{2}\frac{a\left(3 + t^2\right)}{2t^2} + 2\frac{at\left(t^2 - 3\right)}{2}\frac{a\left(3t^2 - 1\right)}{2t^3}$ or $2g_1g_2 + 2f_1f_2 = a^2\left[\frac{3t^4 + 3}{t^2}\right] = c_1 + c_2$.

Hence the circles are orthogonal.

Q.20

Equation of line joining (1, 0) & (0, 2) is 2x + y = 2.

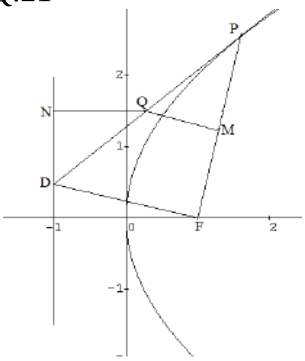
Now any curve having xy = 0 as pair of tangents and 2x + y – 2 = 0 as chord of contact may be represented as $(2x+y-2)^2+\lambda xy=0$ or $4x^2+(\lambda+4)xy+y^2-8x-4y+4=0$.

For this equation to represent a parabola $h^2=ab \Rightarrow \left(\frac{\lambda+4}{2}\right)^2=4 \Rightarrow \lambda=-8$ or $0$.

But for $\lambda=0$ the equation becomes $(2x+y-2)^2=0$.

Hence required parabola is $4x^2-4xy+y^2-8x-4y+4=0$.

Q.21




Consider the parabola $y^2=4ax$.

Let P be $(at^2,2at)$ & Q be (h, k). Also equation of PQ will be

$ty=x+at^2$, hence $k=\frac{h+at^2}{t}$.

Now slope of FP is $\frac{2t}{t^2-1}$, hence equation of QM will be

$(y-k)=\frac{1-t^2}{2t}(x-h)$.

Also QN will be parallel to x-axis thus its equation will be y = k.

Hence QN = a + h.

Now perpendicular distance of QM from F i.e. FM = $\left|\frac{k+\frac{1-t^2}{2t}(a-h)}{\sqrt{\left(\frac{1-t^2}{2t}\right)^2+1}}\right|$.

Therefor $FM=\left|\frac{\frac{h+at^2}{t}+\frac{1-t^2}{2t}(a-h)}{\sqrt{\left(\frac{1-t^2}{2t}\right)^2+1}}\right| \Rightarrow FM=\left|\frac{(h+a)(1+t^2)}{\sqrt{(1-t^2)^2+4t^2}}\right|$

Or FM = h + a = QN.

Q.22

We can get the solution by first consider a fixed parabola touching the coordinate axes and then rotating it by an angle $\theta$.

One such parabola is $x^2-2xy+y^2-2ax-2ay+a^2=0$ which touches the coordinate axes at (the end points of latus rectum i.e. (a, 0) & (0, a). Equation of its latus rectum is x + y = a.

Rotating the parabola by an angle $\theta$ transforms the equation of latus rectum into

$x(\cos\theta+\sin\theta)+y(\cos\theta-\sin\theta)=a$, which may be rearranged as

$y = -\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}x + \dfrac{a}{\cos\theta - \sin\theta}$. Now let $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = -m$, then the equation reduces to

$y = mx - \dfrac{a}{\sqrt{2}}\sqrt{1+m^2}$, which is equation of tangent to the circle $x^2 + y^2 = \dfrac{a^2}{2}$.

Q.23

Let vertices of the triangle be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right)$.

Now sides joining these will be

$y = \dfrac{2}{t_1 + t_2}x + \dfrac{2at_1t_2}{t_1 + t_2}, y = \dfrac{2}{t_2 + t_3}x + \dfrac{2at_2t_3}{t_2 + t_3}, y = \dfrac{2}{t_3 + t_1}x + \dfrac{2at_2t_3}{t_3 + t_1}$.

Let the first line touch $x^2 = 4by$, then $\dfrac{2at_1t_2}{t_1 + t_2} = -b\left(\dfrac{2}{t_1 + t_2}\right)^2$ or $at_1t_2\left(t_1 + t_2\right) = -2b$.

Similarly if the second line is a tangent then $at_2t_3\left(t_2 + t_3\right) = -2b$.

Now from these two conditions we get $t_1 + t_2 + t_3 = 0 \,\&\, t_2t_3t_1 = \dfrac{2b}{a}$.

Further $at_3t_1\left(t_3 + t_1\right) = a \times \dfrac{2b}{at_2} \times \left(-t_2\right) = -2b$, hence the third line also touch $x^2 = 4by$.

Q.24

If the triangle is equilateral, then its centroid will be same as circum center.

Let the vertices be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right)$.

Centroid will be $h = \dfrac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \,\&\, k = \dfrac{2a\left(t_1 + t_2 + t_3\right)}{3}$.

Now consider the circle $x^2 + y^2 - 2hx - 2ky + c = 0$ and put $\left(at^2, 2at\right)$ in this equation to get

$a^2t^4 + 2a\left(2a - h\right)t^2 - 4akt + c = 0$.

Now if this is the circum circle of triangle PQR, then $t_1, t_2, t_3$ will be three of its roots.

Using relations in roots and coefficients we get

$t_1 + t_2 + t_3 + t_4 = 0, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \dfrac{2\left(2a - h\right)}{a} \,\&$

$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = \dfrac{4k}{a}$.

Also $h = \dfrac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \,\&\, k = \dfrac{2a\left(t_1 + t_2 + t_3\right)}{3}$ gives $t_4 = -\dfrac{3k}{2a}$ & $t_1t_2 + t_1t_3 + t_1t_4 = \dfrac{12ah - 9k^2}{8a^2}$

Now from $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \dfrac{2\left(2a - h\right)}{a}$ we get

$t_1t_2 + t_2t_3 + t_3t_1 + \left(t_1 + t_2 + t_3\right)t_4 = \dfrac{2\left(2a - h\right)}{a}$.

Substituting the values gives $4ah - 9k^2 - 32a^2 = 0$.

Hence locus of centroid of triangle PQR is $9y^2 - 4ax + 32a^2 = 0$.

Q.25

Let extremities of the focal chord be $P\left(at^2, 2at\right), Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Point of intersection of tangents at P & Q will be $R\left(-a, 2a\left(t-\frac{1}{t}\right)\right)$.

Now area of triangle PQR will be

$$\frac{1}{2}\left\|\begin{matrix} 1 & at^2 & 2at \\ 1 & \frac{a}{t^2} & -\frac{2a}{t} \\ 1 & -a & a\left(t-\frac{1}{t}\right) \end{matrix}\right\| \text{ i.e. } \frac{a^2}{2}\left(t^2+\frac{1}{t^2}+2\right)\left|t+\frac{1}{t}\right|.$$

Similarly area of triangle OPQ will be

$$\frac{1}{2}\left\|\begin{matrix} 1 & at^2 & 2at \\ 1 & \frac{a}{t^2} & -\frac{2a}{t} \\ 1 & 0 & 0 \end{matrix}\right\| \text{ i.e. } a^2\left|t+\frac{1}{t}\right|.$$

Now ratio of these two area will be $\frac{1}{2}\left(t^2+\frac{1}{t^2}\right)+1$.

Q.26

Let slope of the variable line be $\tan\theta$.

Now any point on this line at a distance r from P(a, b) will be $\left(a + r\cos\theta, b + r\sin\theta\right)$.

These coordinates will satisfy $y^2 = 4cx$ for r = PA & r = PB.

Hence $\left(b + r\sin\theta\right)^2 = 4c\left(a + r\cos\theta\right)$ i.e. $\left(\sin^2\theta\right)r^2 + \left(2b\sin\theta - 4c\cos\theta\right)r + b^2 - 4ac = 0$ will have PA & PB as roots. Now

$$PA + PB = \frac{4c\cos\theta - 2b\sin\theta}{\sin^2\theta} \;\&\; PA \times PB = \frac{4ac - b^2}{\sin^2\theta}.$$

As given PA, PQ, PB are in H.P., hence $PQ = \frac{2 \times PA \times PB}{PA + PB} \Rightarrow PQ = \frac{2\left(4ac - b^2\right)}{4c\cos\theta - 2b\sin\theta}$.

Now let coordinate of Q be (x, y), then

$$x = a + PQ\cos\theta \;\&\; y = b + PQ\sin\theta \Rightarrow \cos\theta = \frac{x-a}{PQ} \;\&\; \sin\theta = \frac{y-b}{PQ}.$$

Substituting these in the expression of PQ we get $PQ = \frac{2\left(4ac - b^2\right)}{4c\left(x-a\right) - 2b\left(y-b\right)}PQ$

or $2cx - by = 6ac - 2b^2$.

Hence locus of Q is a fixed straight line.

Q.27

Foot of perpendicular from the focus F on tangent at P will lie on y-axis, hence let P, F & M be $\left(at^2, 2at\right), \left(a, 0\right) \& \left(0, at\right)$.

Now area of triangle PFM will be

$$\frac{1}{2}\left\|\begin{vmatrix} 1 & at^2 & 2at \\ 1 & a & 0 \\ 1 & 0 & at \end{vmatrix}\right\| = \frac{a^2}{2}\left(t^3 + t\right).$$

Now range of $t$ is $0$ to 1.

Maximum area will be for t = 1 i.e. maximum area = $a^2$.

Q.28

Let mid point of any such chord be $M\left(at^2, 2at\right)$.

Now using T = $S_1$, equation of chord of $x^2 + y^2 = 16a^2$ having mid point at M may be represented as $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

As this chord is drawn through (h, 0) hence substituting these coordinates in equation of chord we get $at^2x + 2aty = a^2t^4 + 4a^2t^2$.

Now the above equation gives three values of t, namely $0 \,\&\, \pm\sqrt{\frac{h-4a}{a^2}}$ out of which the later two values will be real & other than 0 only if h > 4a.

Also for M to be mid point of chord it must lie inside the circle hence

$a^2t^4 + 4a^2t^2 - 16a^2 < 0$ or $\left(\frac{4a-h}{a^2}\right)^2 + 4\left(\frac{4a-h}{a^2}\right) - 16 < 0$, hence $h < \left(\sqrt{5}+1\right)2a$.

Q.29

Reflection at a point P on any curved surface take place such that incident ray and reflected are reflections of each other in the normal to the curve at P.

Now y = b meets $y^2 = 4ax$ at point $P\left(\frac{b^2}{4a}, b\right)$. Comparing this with $\left(at^2, 2at\right)$ gives $t = \frac{b}{2a}$.

Normal to the parabola at this point will be

$\frac{b}{2a}x + y = 2a\frac{b}{2a} + a\left(\frac{b}{2a}\right)^3$ or $4abx + 8a^2y = 8a^2b + b^3$.

Now slope of normal is $-\frac{b}{2a}$ and y = b is parallel to x-axis so if $\theta$ is the angle between the incident ray and normal, then $\tan\theta = -\frac{b}{2a}$.

Reflected ray will make an angle $2\theta$ with y = b, hence slope of reflected ray will be

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4ab}{4a^2 - b^2}$.

Equation of the reflected ray : $y - b = \frac{4ab}{4a^2 - b^2}\left(x - \frac{b^2}{4a}\right)$ or $4abx + \left(4a^2 - b^2\right)y = 4a^2b$.

Clearly (a, 0) satisfies this equation.

Q.30

If the triangle is equilateral, then its centroid will be same as circum center.

Let the vertices be $\left(at_1^2, 2at_1\right), \left(at_2^2, 2at_2\right), \left(at_3^2, 2at_3\right)$.

Centroid will be $h = \frac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \,\&\, k = \frac{2a\left(t_1 + t_2 + t_3\right)}{3}$.

Now consider the circle $x^2 + y^2 - 2hx - 2ky + c = 0$ and put $\left(at^2, 2at\right)$ in this equation to get $a^2t^4 + 2a\left(2a - h\right)t^2 - 4akt + c = 0$.

Now if this is the circum circle of triangle PQR, then $t_1, t_2, t_3$ will be three of its roots.
Using relations in roots and coefficients we get

$$t_1 + t_2 + t_3 + t_4 = 0, t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \frac{2\left(2a - h\right)}{a} \ \&$$

$$t_1t_2t_3 + t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 = \frac{4k}{a}.$$

Also $h = \dfrac{a\left(t_1^2 + t_2^2 + t_3^2\right)}{3} \ \& \ k = \dfrac{2a\left(t_1 + t_2 + t_3\right)}{3}$ gives $t_4 = -\dfrac{3k}{2a}$ & $t_1t_2 + t_1t_3 + t_1t_4 = \dfrac{12ah - 9k^2}{8a^2}$

Now from $t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = \dfrac{2\left(2a - h\right)}{a}$ we get

$$t_1t_2 + t_2t_3 + t_3t_1 + \left(t_1 + t_2 + t_3\right)t_4 = \frac{2\left(2a - h\right)}{a}.$$

Substituting the values gives $4ah - 9k^2 - 32a^2 = 0$.

Hence locus of centroid of triangle PQR is $9y^2 - 4ax + 32a^2 = 0$.

Q.31

Given data implies that point of intersection of two normal lies on the parabola.

Let a normal be drawn at P(λ), then its equation will be $\lambda x + y = 2a\lambda + a\lambda^3$.

If it passes through $\left(at^2, 2at\right)$, then $\lambda at^2 + 2at = 2a\lambda + a\lambda^3$.

$\Rightarrow a\lambda\left(t^2 - \lambda^2\right) + 2a\left(t - \lambda\right) = 0$ or $\lambda^2 + \lambda t + a = 0$.

Q.32

Let the points on parabola be $A\left(at_1^2, 2at_1\right), B\left(at_2^2, 2at_2\right) \& C\left(at_3^2, 2at_3\right)$.

Points of intersection of tangents at these points will be

$P\left(at_1t_2, a\left(t_1 + t_2\right)\right), Q\left(at_2t_3, a\left(t_2 + t_3\right)\right) \& R\left(at_3t_1, a\left(t_3 + t_1\right)\right)$.

Now Area of $\Delta ABC = \dfrac{1}{2}\left\|\begin{matrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{matrix}\right\|$ & Area of $\Delta PQR = \dfrac{1}{2}\left\|\begin{matrix} 1 & at_2t_3 & a\left(t_2 + t_3\right) \\ 1 & at_3t_1 & a\left(t_3 + t_1\right) \\ 1 & at_1t_2 & a\left(t_1 + t_2\right) \end{matrix}\right\|$.

Now take the second determinant,

(i) Subtract $a\left(t_1 + t_2 + t_3\right)$ from third column to get

$$\text{Area of } \Delta PQR = \frac{1}{2}\left\|\begin{matrix} 1 & at_2t_3 & -at_1 \\ 1 & at_3t_1 & -at_2 \\ 1 & at_1t_2 & -at_3 \end{matrix}\right\|.$$

(ii) Multiply first row by $t_1$, second by $t_2$ & third by $t_3$ and take $t_1t_2t_3$ common from second column. Also take a common from second column and multiply 2a to first column to get

$$\text{Area of } \Delta PQR = \frac{1}{4}\left\|\begin{matrix} 2at_1 & 1 & at_1^2 \\ 2at_2 & 1 & at_2^2 \\ 2at_3 & 1 & at_3^3 \end{matrix}\right\|.$$

(iii) Now interchange first column with second and then second with third to get

$$\text{Area of } \Delta PQR = \frac{1}{4}\left\|\begin{matrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^3 & 2at_3 \end{matrix}\right\| = \frac{1}{2}\text{Area of } \Delta ABC.$$

Q.33

Let the points be $P\left(at_1^2, 2at_1\right) \& Q\left(at_2^2, 2at_2\right)$, where as given $t_2 = 2t_1$.

Now point of intersection of normal at P & Q will be

$x = a\left(t_1^2 + t_2^2 + t_1t_2 + 2\right) \& y = -at_1t_2\left(t_1 + t_2\right)$

Now $t_2 = 2t_1 \Rightarrow x - 2a = 7at_1^2 \& y = -6at_1^3$.

Eliminating $t_1$ gives $36\left(x - 2a\right)^3 = 243ay^2$.

Q.34

Let the points be $P\left(at_1^2, 2at_1\right) \& Q\left(at_2^2, 2at_2\right)$, where as given $t_1t_2 = -1$.

Now point of intersection of normal at P & Q will be

$x = a\left(t_1^2 + t_2^2 + t_1t_2 + 2\right) \& y = -at_1t_2\left(t_1 + t_2\right)$.

Now $t_1t_2 = -1$ gives $x = a\left(t_1^2 + t_2^2 + 1\right) \& y = a\left(t_1 + t_2\right)$.

Eliminating $t_1$ & $t_2$ gives $a\left(x - 3a\right) = y^2$.

Q.35

Let the points be $P\left(at_1^2, 2at_1\right) \& Q\left(at_2^2, 2at_2\right)$, where as given $t_1t_2 = 2$.

Now mid point of P & Q will be $x = \dfrac{a\left(t_1^2 + t_2^2\right)}{2} \& y = a\left(t_1 + t_2\right)$.

Now $t_1t_2 = 2$ gives $2x + 4a = a\left(t_1 + t_2\right)^2 \& y = a\left(t_1 + t_2\right)$.

Eliminating $t_1$ & $t_2$ gives $2a\left(x + 2a\right) = y^2$.