IIT – JEE: 2024 **TW TEST (ADV)** **DATE: 11/06/23**

TOPIC: FLUID MECHANICS

SOLUTIONS

1. (A)

Let ρ_c be the density of the cylinder and ρ_w that of water. Upthrust on cylinder

$$=\left(\frac{m}{\rho_c}\right)\rho_w g=\frac{mg}{\sigma}\left(\because \sigma=\frac{\rho_c}{\rho_w}\right)$$

From Newton's third law, the force exerted by the cylinder on water in the downward direction is

$$F=\frac{mg}{\sigma}$$

$\therefore$ Increase in pressure at the bottom of the vessel is

$\frac{F}{A}=\frac{mg}{\sigma A}$, which is choice (A)

2. (C)

Let m be the mass of the ball and V its volume. Its mass $m=\rho V$ the weight of the ball is

$W=mg=\rho Vg$

The volume of the liquid displaced $=V$. If σ is the density of the liquid, the weight of the liquid displaced is the upthrust U it experiences.

$$U=V\sigma g$$

$\therefore$ The net downward force acting on the body is

$$F=W-U=(\rho-\sigma)Vg$$

The initial acceleration is

$$a=\frac{F}{m}=\frac{(\rho-\sigma)Vg}{\rho V}$$

$$=\left(\frac{\rho-\sigma}{\rho}\right)g$$

Hence the correct choice is (C)

3. (C)

Let the densities of metal and water be ρ and ρ_0 respectively and let x be the length of the rod immersed in water at an instant of time t. Then, acceleration at that instant = apparent weight divided by mass of the rod, i.e

$$\frac{d\upsilon}{dt}=\frac{\pi r^2 l\rho g-\pi r^2 x\rho_0 g}{\pi r^2 l\rho}$$

$= g - \frac{gx\rho_0}{l\rho}$

$= g\left(1 - \frac{x}{\sigma l}\right)$

Or $\frac{d\upsilon}{dx}.\frac{dx}{dt} = g\left(1 - \frac{x}{\sigma l}\right)$

Or $\upsilon\frac{d\upsilon}{dt} = g\left(1 - \frac{x}{\sigma l}\right)$

Integrating, we have

$\frac{\upsilon^2}{2} = g\left|x - \frac{x^2}{2\sigma l}\right|_0^l = g\left(l - \frac{l}{2\sigma}\right)$

Or $\upsilon = \sqrt{2gl\left(1 - \frac{1}{2\sigma}\right)}$ which is choice (C)

4. (B)

Work done against gravity is

$W_1 = mgh = (\rho V) \times gh = \rho ghV$

Work done against pressure difference is

$W_2 = \Delta P \times V = h\rho gV$

$\therefore$ Total work done $W = W_1 + W_2 = 2h\rho gV$

Power $P = \frac{W}{t}$. Therefore

$t = \frac{W}{P} = \frac{2h\rho gV}{P}$

$= \frac{2 \times (30) \times (10^3) \times (10) \times (10^3 \times 10^{-3})}{10^3}$

$= 600s = 10\,\text{minutes}$

5. (C)

Let h be the depth of the hole below the free surface of water (see figure)

According to Torricelli’s theorem, the velocity of efflux v of water through the hole is given by

$\upsilon = \sqrt{2gh}$ (i)

The height through which water falls is

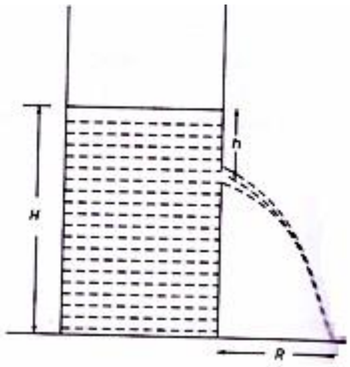
$S = H - h$

If t is the time taken by water to strike the floor, then

$S = \frac{1}{2}gt^2$

or $H - h = \frac{1}{2}gt^2$

giving $t = \sqrt{\frac{2(H-h)}{g}}$ (ii)



The distance R where the emerging stream strikes the floor is given by

$$R = \upsilon t$$

Substituting for υ and t from Eqs. (i) and (ii), we get

$$R = \sqrt{2gh} \times \sqrt{\frac{2(H-h)}{g}} = 2\sqrt{h(H-h)}$$

Hence the correct choice is (C)

6. (B, C)
Due to the upward force of buoyancy on the block exerted by the liquid, the apparent weight of the block exerted by the liquid, the apparent weight of the block will be less than 2 kg. Hence choice (C) is correct. The hanging block exerts a downward force on the liquid (and the beaker) equal magnitude to the upward buoyant force. Therefore, balance B will read more than 5 kg. Hence, choice (B) is also correct.

7. (B)
Let the mass of ice be m_1 and the mass of stone be m_2. The mass of the displaced water is equal to $(m_1 + m_2)$. If ρ is the density of water, the volume of water displaced is

$$V = (m_1 + m_2)/\rho$$

When the ice melts, additional volume of water obtained is m_1/ρ. The stone sinks in water, and displaces a volume of water equal to its own volume, which is m_2/ρ_s where ρ_s is the density of stone. Thus the total volume of extra water is

$$V' = \frac{m_1}{\rho} + \frac{m_2}{\rho_s}$$

Since $\rho_s > \rho, \frac{1}{\rho_s} < \frac{1}{\rho}$

$\therefore V' < V$. Therefore, level of water in beaker decreases.

8. (B, C)
Volume of steel block $= 10 \times 10 \times 10 = 10^3 cm^3 = 10^{-3} m^3$

(A) Let h_1 be the height of the block above the mercury surface.

Volume of mercury displaced

$= (0.1 - h_1) \times 0.1 \times 0.1 m^3$

$\therefore$ Weight of mercury displaced

$= (0.1 - h_1) \times 0.1 \times 0.1 \times 13.6 \times 10^3 g$

newton

This must be equal to the weight of block which is

$7.8\times10^{3}\times10^{-3}\times g$ newton $= 7.8\times g$ newton

$\therefore (0.1-h_1)\times0.1\times0.1\times13.6\times10^{3}\times g = 7.8\times g$

Which gives $h_1 = 0.0426\text{m} = 4.26\,\text{cm}$

(B) Let h_2 be the height of the water column required to just submerge the steel block. Thus weight of the block = weight of water displaced + weight of mercury displaced

i.e. $7.8\times g = h_2\times0.1\times0.1\times1000\times g + (0.1-h_2)\times0.1\times0.1\times13.6\times10^{3}\times g$ which gives

$h_2 = 0.046\,\text{m} = 4.60\,\text{cm}$. Hence the correct choice are (B) and (C)

9. (A, B, D)

Pressure at A and B is atmospheric. Hence choice (A) is correct. For a liquid at rest the pressure at the same horizontal level is the same. So choice (B) is also correct. Now, if P_0 is the atmospheric pressure.

$P_C = P_0 + \rho_1 h_1 g$ and $P_D = P_0 + \rho_2 h_2 g$

Since $P_C = P_D, \rho_1 h_1 = \rho_2 h_2$ or $\frac{h_1}{h_2} = \frac{\rho_2}{\rho_1}$

Since , $h_1 > h_2; \rho_1 < \rho_2$. Hence the correct choices are (A),(B) and (D)

10. (A, D)

Relative density of metal

$$= \frac{\text{weight in air}}{\text{loss of weight in water}} = \frac{2.4}{2.4-2.0} = 6$$

Relative density of liquid

$$= \frac{\text{loss of weight in liquid}}{\text{loss of weight in water}}$$

$$= \frac{2.4-1.9}{2.4-2.0} = \frac{0.5}{0.4} = 1.25$$

Hence the correct choices are (A) and (D)

11. (B, D)

Consider small element of the liquid of length dr at a distance r from O. The mass of the element $m = a\rho dr$. Therefore, the outward force(centrifugal force) acting on the element is (see figure)

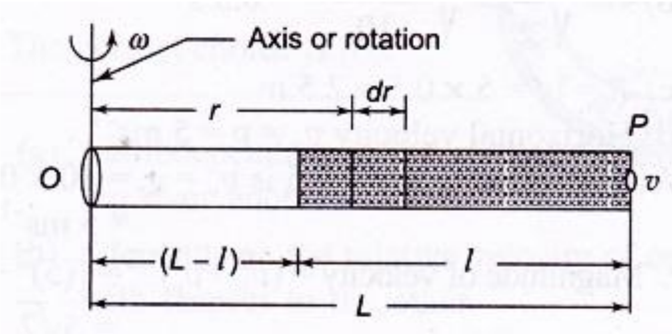
$dF = mr\omega^2 = a\rho\omega^2 r dr$

The total outward force F acting on the liquid column of length l at that instant is obtained by integrating this expression from $r = (L-l)$ to $r = L$. Thus

$$F = a\rho\omega^2 \int_{L-l}^{L} r dr = a\rho\omega^2 \left|\frac{r^2}{2}\right|_{(L-l)}^{L}$$

$$= \frac{1}{2}a\rho\omega^2\left[L^2-(L-l)^2\right] = a\rho\omega^2 l\left(L-\frac{l}{2}\right)$$

Given $l = L/2$. Therefore, $F = \frac{2a\rho\omega^2 L^2}{8}$



Outward pressure at P is $p=\frac{F}{a}=\frac{3}{8}\rho\omega^2L^2$. If υ is the velocity of efflux due to this pressure, then

$\frac{1}{2}\rho\upsilon^2=p=\frac{3}{8}\rho\omega^2L^2$

Which gives $\upsilon=\frac{\sqrt{3}\omega L}{2}$

Hence the correct choices are (B) and (D)

12. (B, D)

Equation of continuity gives $\upsilon_A<\upsilon_B$. From Bernoulli's theorem $P_A+\frac{1}{2}\rho\upsilon^2{}_A=P_B+\frac{1}{2}\rho\upsilon_B^2$. Since $\upsilon_B>\upsilon_A, p_B<p_A$. Thus the correct choices are (B) and (D).

13. (A, D)

As the aircraft moves, the streamlines of air flow curve around the wing and meet at the rear end at the same time. Hence velocity of air moving along the upper surface is higher than that moving along the lower surface. According to Bernoulli's principle, the air pressure on the upper surface is less than that on the lower surface of the wing. Hence the correct choices are (A) and (D).

14. (A, B, C, D)

(A) Area of piston $A=\frac{\pi D^2}{4}; D=\text{diameter of piston}$

Area of nozzle $a=\frac{\pi d^2}{4}; d=\text{diameter of nozzle}$ From equation of continuity $AV=a\upsilon$

$\upsilon=\frac{AV}{a}=\frac{D^2}{d^2}\times V=(5)^2\times 0.2=5\,ms^{-1}$

(B) $t=\sqrt{\frac{2h}{g}}=\sqrt{\frac{2\times 1.25}{10}}=0.5s$

(C) $R=\upsilon t=5\times 0.5=2.5m$

(D) Horizontal velocity $\upsilon_x=\upsilon=5ms^{-1}$

Vertical velocity at $t=0.5$ is $\upsilon_y=g_t=10\times 0.5=5ms^{-1}$

$\therefore$ Magnitude of velocity $=\left(\upsilon_x^2+\upsilon_y^2\right)^{1/2}=\left[(5)^2+(5)^2\right]=5\sqrt{2}ms^{-1}$

Hence all choices are correct.

15. (B, C)

According to Bernoulli's theorem

$$p+\frac{1}{2}\rho\upsilon^2=\text{constant}$$

When the cylindrical vessel is rotated at angular velocity ω about its axis, the velocity of the liquid at the sides is the maximum, given by

$$\upsilon_s=r\omega$$

Where r is the radius of the vessel. Applying Bernoulli's theorem at the sides and at the centre of the vessel, we have (Figure)

$$p_s+\frac{1}{2}\rho\upsilon_s^2=p_c+\frac{1}{2}\rho\upsilon_c^2$$

Where $p_s=$ pressure at the sides, $p_c=$ pressure at the centre and $\upsilon_c=$ velocity of the liquid at the centre. Now, since $\upsilon_c=0$, we have

$$p_c-p_s=\frac{1}{2}\rho\upsilon_s^2=\frac{1}{2}\rho r^2\omega^2 \qquad (1)$$

Since p_c is greater than p_s, the liquid rises at the sides of the vessel. Let h be the difference in the levels of the liquid at the sides and at the centre (Figure), then

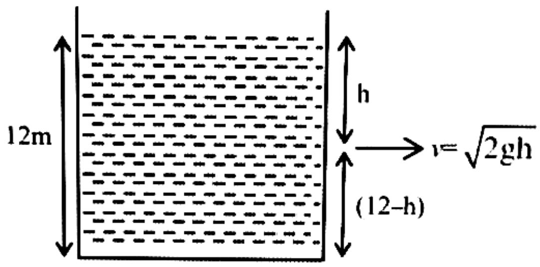
$$p_c-p_s=\rho gh \qquad (2)$$

From (1) and (2), we have

$$\rho gh=\frac{1}{2}\rho r^2\omega^2 \text{ or } h=\frac{r^2\omega^2}{2g}$$

Hence the correct choices are (A) and (C)

16. (6)



$$R=\sqrt{2gh}\times\sqrt{\frac{(12-h)\times 2}{g}}$$

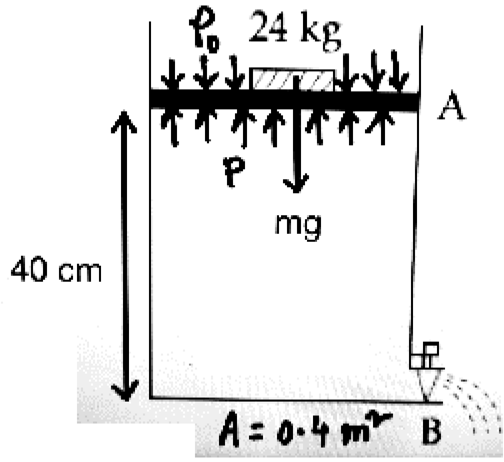
$$\sqrt{4h(12-h)}=R$$

For maximum R

$$\frac{dR}{dh}=0$$

$$\Rightarrow h=6\text{cm}$$

17. (3)



$mg + p_0A = pA$

$\Rightarrow p = p_0 + \frac{mg}{A}$

Given, area of $A = 0.4\,m^2 = 0.4\times10^4\,cm^2$ and area of $B = 1\,cm^2$ applying continuity equation

$AV_1 = av\left[V_1 = \text{velocity at point A}\right]$

$\Rightarrow V_1 = \frac{a}{A}\upsilon$

As $>>> a$ so $\frac{a}{A}$ very small

$\therefore V_1 <<< v$

So we can neglect V_1 by assuming $V_1 = 0$

On applying Bernoulli's theorem at points A and B.

$$p_0 + \frac{mg}{A} + \frac{\rho V_1^2}{2} + \rho gh = p_0 + \frac{\rho\upsilon^2}{2} + 0$$

$$\Rightarrow p_0 + \frac{mg}{A} + 0 + \rho gh = p_0 + \frac{\rho\upsilon^2}{2}$$

$$\Rightarrow \frac{mg}{A} + \rho gh = \frac{\rho\upsilon^2}{2}$$

$$\Rightarrow \frac{24\times10}{0.4} + 1000\times10\times0.4 = \frac{1000\times\upsilon^2}{2}$$

$\Rightarrow v \simeq 3\,m/s$

18. (25600)

According to Pascal's law,

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}$$

Initially, $\frac{100g}{A_1} = \frac{mg}{A_2}$ …………(i)

Finally, $\frac{Mg}{16A_1} = \frac{mg}{\left(\frac{A_2}{16}\right)}$ ……. (ii)

On dividing Eqs. (i) by (ii), we get

$$\frac{100\times16}{M} = \frac{1}{16}$$

$$\therefore M = 25600\,kg$$

19. (363)

From Bernoulli's equation

$$P_1 = \frac{1}{2}\rho\upsilon_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho\upsilon_2^2 + \rho gh_2$$

$$P_1 - P_2 + \rho g(h_1 - h_2) = \frac{1}{2}\rho\left(\upsilon_2^2 - \upsilon_1^2\right) \quad \text{………..(i)}$$

Also, from equation of continuity

$$A_1\upsilon_1 = A_2\upsilon_2$$

$$A\upsilon_1 = \frac{A}{2}\upsilon_2$$

$$\upsilon_2 - 2\upsilon_1 \text{ ………..(ii)}$$

Put equation (ii) in (i),

$$4100\times800\times10\times1 = \frac{1}{2}\times800\times\left(4\upsilon_1^2 - \upsilon_1^2\right)$$

$$4100 + 8000 = 400\times3\upsilon_1^2$$

$$\upsilon_1^2 = \frac{12100}{3\times400} = \frac{121}{12}$$

$$\upsilon_1 = \sqrt{\frac{121}{12}}$$

Now, $\frac{\sqrt{x}}{6} = \sqrt{\frac{121}{12}}$

$$\frac{x}{36} = \frac{121}{12}$$

$$x = 121\times3 = 363$$

$$\therefore x = 363$$

20. (4)

Weight = Upthrust

$$mf = F_u \Rightarrow 480\times10 = \rho Vg$$

$$480\times10 = \rho_0 e^{-\frac{h}{h_0}} \Rightarrow 480\times10 = \rho_0 e^{-\frac{100}{6000}}Vg \quad \text{……..(i)}$$

$$(480 - N\times10) = \rho' Vg$$

$$(480 - N)10 = \rho_0 e^{-\frac{150}{6000}}Vg \quad \text{………(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{480}{480-N} = e^{\left(\frac{150-100}{6000}\right)}$$

$$\frac{480}{480-N} = e^{\frac{50}{6000}} \Rightarrow \frac{480}{480-N} = e^{\frac{1}{120}}$$

Answer Key

21. (D)
22. (B)
23. (B)
24. (B)
25. (A)
26. (A, C, D)
27. (A, C, D)
28. (B, C)
29. (B, C, D)
30. (B, C)
31. (A, B, C)
32. (A, B, C, D)
33. (A, B)
34. (B)
35. (A, B, D)
36. (5)
37. (9)
38. (5)
39. (28)
40. (5)

SOLUTIONS

41. (A)

$\because$ We know that the distance between focus and directrix = 4 [$\because$ focus = (0, 0) and directrix $\rightarrow x = 4$]

$\therefore \frac{a}{e} - ae = 4$

$a\left(\frac{1}{e} - 1\right) = 4$ $\left\{\because e = \frac{1}{2}\right\}$

$\therefore a\left(1 - \frac{1}{2}\right) = 4$

$\Rightarrow a\left(\frac{3}{2}\right) = 4 \Rightarrow a = \frac{8}{3}$

$\therefore$ Length of semi major axis is $\frac{8}{3}$ units.

42. (C)

$\because$ At $t = \frac{3}{2}$ b > a

$\therefore$ Length of laustrectum $= \frac{2a^2}{b}$

$= \frac{2[t2 - 3t + 4]^2}{[3 + 5t]} = \frac{2\left[\frac{9}{4} - \frac{9}{2} + 4\right]^2}{\left[3 + \frac{15}{2}\right]}$

$\Rightarrow \frac{2[1.75]^2}{10.5} = \frac{2}{10} = \frac{1}{5}$

43. (D)

Let the director circle is $(x - h)^2 + (y - k)^2 = a^2 + b^2$ origin lies on director circle

$\Rightarrow h^2 + k^2 = a^2 + b^2 = 5$

Locus is $x^2 + y^2 = 5$

44. (A)

$\because$ A, B, C are on ellipse and vertical of equilateral triangle

Let A, b< C are have coordinates $(a\cos\theta, b\sin\theta), \left[a\cos\left(\theta+\frac{2\pi}{3}\right), b\sin\left(\theta+\frac{2\pi}{3}\right)\right]$ and $\left[a\cos\left(\theta+\frac{4\pi}{3}\right), b\sin\left(\theta+\frac{4\pi}{3}\right)\right]$

$\therefore$ we know that normal at three point from same side are concurrent

$\therefore$ A.B. C are concurrent

45. (B)

Given equation is $\frac{x^2}{25}+\frac{y^2}{16}=1$

Let roots be (α, β) $a=5$ *and* $b=4$

$\therefore e=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

$\therefore$ Focus $(\pm ae, 0)=(\pm 3, 0) \Rightarrow \alpha=3$

Length of chord whose middle part is $\left(\frac{1}{2}, \frac{2}{5}\right)$

$\therefore$ Equation of chord T = S_1

$$\Rightarrow \frac{x}{25}\left(\frac{1}{2}\right)+\frac{y}{16}\left(\frac{2}{5}\right)=\frac{\left(\frac{1}{2}\right)^2}{25}+\frac{\left(\frac{2}{5}\right)^2}{16}$$

$$\Rightarrow \frac{x}{50}+\frac{y}{40}=\frac{1}{50}$$

$$\Rightarrow \frac{x}{5}+\frac{y}{4}=\frac{1}{5}$$

$$\Rightarrow y=\frac{-4}{5}(x-1)$$

By Eq. $(i)\ \frac{x^2}{25}+\frac{\left(-\frac{4}{5}\right)^2(x-1)^2}{16}=1$

$$\Rightarrow x^2-x-12=0$$

$x_1=4$ *and* $x_2=-3$

$\therefore$ Length of chord = $\sqrt{(x_1-x_2)+(y_1-y_2)^2}$

$=|(x_1-x_2)\sqrt{1+m^2}|$

$$\Rightarrow l^2=(7)^2\left(1+\frac{16}{25}\right)=\frac{49}{25}\times 41$$

$$\Rightarrow \frac{25^{l2}}{41}=49$$

$\therefore \alpha=3$ *and* $\beta=49$

$\therefore$ Required equation is

$x^2-(3+49)x+147=0$

$x^2-52x+147=0$

46. (D)

Let the coordinates of the pole of any chord the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be (h,k)

Then, equation of the chord i.e. polar of (h, k) is

$\frac{xh}{a^2}+\frac{yk}{b^2}=1$

$b^2xh+a^2yk=a^2b^2$ …….(i)

It will touch the parabola

$ay^2=-2b^2x$

By eq. (i) $\frac{-ay^2}{2}h+a^2yk=a^2b^2$

$\Rightarrow hy^2-2ayk+2ab^2=0$ ……(ii)

If the line (i) tangent to the parabola, then the roots of (ii) are equal

$\therefore D=0$

$\Rightarrow 4a^2k^2-8ab^2h=0$

$\Rightarrow ak^2=2b^2h$

$\therefore$ Locus is $ay^2=2b^2x$

47. (A, C)

Given $\frac{x^2}{f(k^2+2k-5)}+\frac{y^2}{f(k+11)}=1$ and f(x) is decreasing function

$\therefore f(x_1)<f(x_2)\Rightarrow x_1>x_2$

Now, major axis as X-axis

$\because f(k^2+2k+5)>f(k+11)$

$\Rightarrow k^2+2k+5<k+11$

$\Rightarrow k^2+k-6<0$

$\Rightarrow (k+3)(k-2)<0$

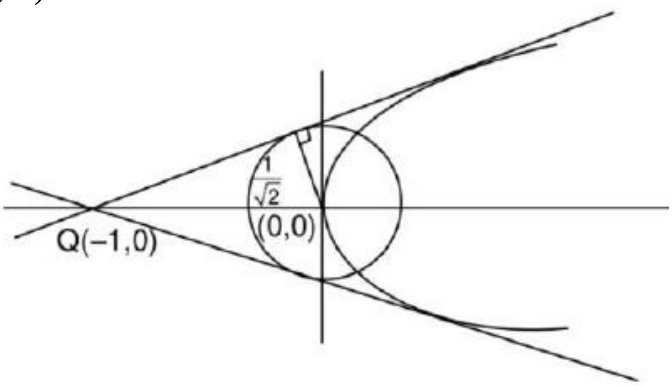
$\because k\in(-\infty,-3)\cup(2,\infty)$

48. (A, C)

The tangent and normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P.

So answer are (A) and (C)

49. (A, C)



Let equation of common tangent is $y=mx+\frac{1}{m}$

$$\therefore \left| \frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}} \right| = \frac{1}{\sqrt{2}}$$

$\Rightarrow m^4 + m^2 - 2 = 0$

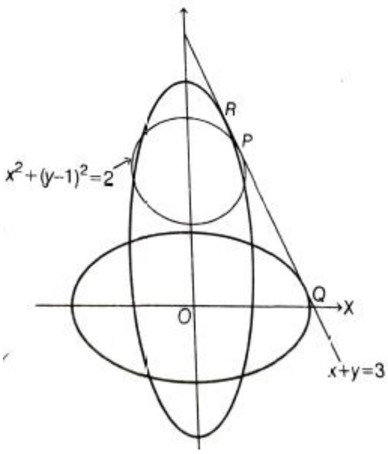
$\Rightarrow m = \pm 1$

Equation of common tangents are $y = x+1$ *and* $y = -x-1$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

$e = \sqrt{1-\frac{1}{2}} = \frac{1}{2}$ *and* $LR = \frac{2b^2}{a} = 1$

50. (A, B)



$E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$

$E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1, d > c$

and S: $x^2 + (y-1)^2 = 2$

as tangent to E_1, E_2 and S is x + y = 3

Let the point of contact of tangent be $(x_1.y_1)$ to S

$\because xx_1 + yy_1 - (y + y_1) + 1 = 2$

Or $xx_1 + yy_1 - y = 1 + y_1$ same as x + y =3

$\Rightarrow \frac{x_1}{1} = \frac{y_1 - 1}{1} = \frac{1+y_1}{3}$

$\because x_1 = 1$ *and* $y_1 = 2$

$\because P(1,2)$

$\because PR = PQ = \frac{2\sqrt{2}}{3}$

Thus by parametric form

$$\frac{x-1}{\frac{-1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = \pm \frac{2\sqrt{2}}{3}$$

$\therefore x = \frac{5}{3}, y = \frac{4}{3}$ *and* $x = \frac{1}{3}, y = \frac{8}{3}$

$\therefore Q = \left(\frac{5}{3}, \frac{4}{3}\right)$ *and* $R = \left(\frac{1}{3}, \frac{8}{3}\right)$

Now, equation of tangent at Q on ellipse E_1 is

$$\frac{x(5)}{a^2(3)}+\frac{y(4)}{b^2(3)}=1$$

On comparing with x + y = 3

$a^2=5$ and $b^2=4$

$$\because e_1^2=1-\frac{b^2}{a^2}=1-\frac{4}{5}$$

$$\Rightarrow e_1^2=\frac{1}{5}$$

Also, equation of tangent at R on ellipse is

$$\frac{x(1)}{a^2(3)}+\frac{y(8)}{b^2(3)}=1$$

On comparing with x + y =3

$a^2=1$ and $b^2=8$

$$\therefore e_2^2=1-\frac{a^2}{b^2}=1-\frac{1}{8}=\frac{7}{8}$$

Now, $e_1^2e_2^2=\frac{7}{40}$

$$\Rightarrow e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}} \text{ and } e_1^2+e_2^2=\frac{43}{40}$$

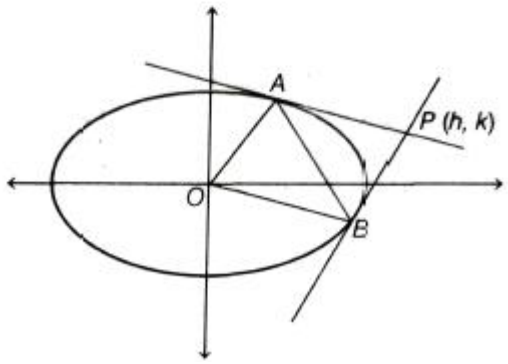
And $(e_1^2-e_2^2)=\frac{27}{40}$

51. (A, C)

Ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

Let the point of intersection of tangents at A and B be P(h,k) then equation of AB is

$$\frac{xh}{4}+\frac{yk}{1}=1 \qquad \text{….(i)}$$



Homogenizing the equation of ellipse using (i)

$$\frac{x^2}{4}+\frac{y^2}{1}=\left(\frac{xh}{4}+\frac{yk}{1}\right)^2$$

$$\Rightarrow x^2\left(\frac{h^2-4}{16}\right)+y^2(k^2-1)+\frac{2hk}{4}xy=0 \qquad \text{……(ii)}$$

Given equation of OA and OB is

$$x^2+4y^2+\alpha xy=0 \qquad \text{….(iii)}$$

Eqs.(ii) and (iii) represents same lien

$\therefore \frac{h^2-4}{16}=\frac{k^2-1}{4}=\frac{hk}{2\alpha}$

$\Rightarrow h^2-4=4(k^2-1) \Rightarrow h^2-4k^2=0$

$\Rightarrow (h-2k)(h+2k)=0$

$\because$ Locus is $(x+2y)(x+2y)=0$

$x-2y=0$ *or* $x+2y=0$

52. (B, C, D)

$S_1: 3x^2+4y^2=1$

$\Rightarrow \frac{x^2}{\frac{1}{3}}+\frac{y^2}{\frac{1}{4}}=1$

$S_2: |x+y|-1=0$

$|x+y|=1 \Rightarrow x+y=\pm 1$

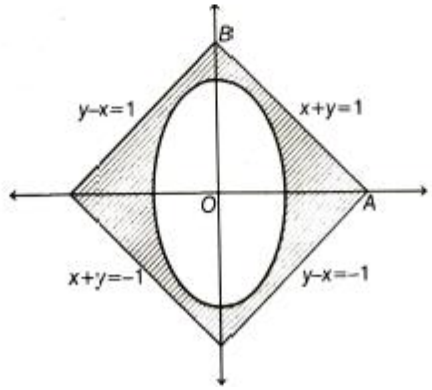
And $S_3: |y-x|-1=0$

$|y-x|=1 \Rightarrow y-x=\pm 1$

$\because$ Auxiliary circle of ellipse: $S_1: x^2+y^2=a^2$

$\Rightarrow x^2+y^2=\frac{1}{3} \Rightarrow 3x^2+3y^2=1$

Now



$\because S_1 \geq 0 \Rightarrow 3x^2+3y^2 \geq 1$

$S_2 \leq 0 \Rightarrow |x+y| \leq 1$

$S_3 \leq 0 \Rightarrow |x-y| \leq 1$

$\because$ Required area = $4\left[Area\ of\ \Delta OAB-\frac{1}{4}\pi r^2\right]$

$=4\left[\frac{1}{2}-\frac{1}{4}\pi\left(\frac{1}{\sqrt{3}}\right)^2\right]$

$=\left(2-\frac{\pi}{3}\right)$ sq. units

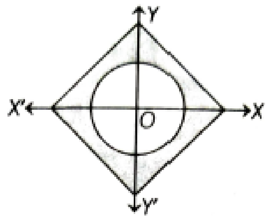
$\therefore$ Equation of director circle $x^2+y^2=a^2+b^2$

$x^2+y^2=\frac{7}{12}$

(b) Area = $4\left(\frac{1}{2}-\frac{1}{4}\pi\left(\frac{7}{12}\right)\right)=2-\frac{7\pi}{12}$

(c) Area $=4\times\frac{1}{2}\times1=2$

(d) Area of region = $4\left(\frac{1}{2}-\frac{\pi}{4}\times\frac{1}{\sqrt{3}}\times\frac{1}{2}\right)=2-\frac{\pi}{2\sqrt{3}}$



53. (A, C)

$\frac{x^2}{25}+\frac{y^2}{9}=1$ reflected ray thus will be line joining $\left(3,\frac{12}{5}\right)$ *and* $(4,0)$

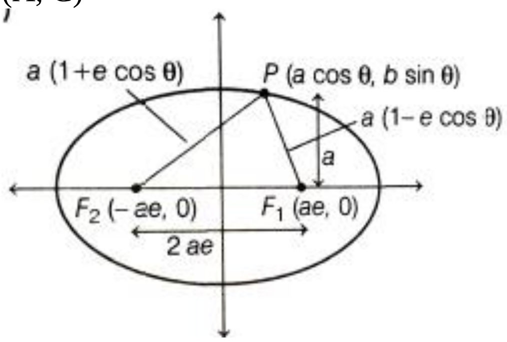
$\Rightarrow y-0=\frac{-12}{5}(x-4)$

$5y=-12x+48$

Or line joining the points $\left(3,\frac{-12}{5}\right)$ *and* $(4,0)$

$y-0=\frac{12}{5}(x-4)\Rightarrow 12x-5y-48=0$

54. (A, C)



$\because$ We know P be any point on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ i.e. $(a\cos\theta, b\sin\theta)$

$\therefore F_1=(ae,0), F_2(-ae,0), P(a\cos\theta,b\sin\theta)$,

$PF_2=a(1+e\cos\theta), PF_1=a(1-e\cos\theta)$ *and* $F_1F_2=2ae$

$F_1F_2=2ae$

$\because h=\frac{ax_1+bx_2+cx_3}{a+b+c}$

$=\frac{a^2e(1+e\cos\theta)-a^2e(1-e\cos\theta)+2a^2e\cos\theta}{2a+2ae}$

$\Rightarrow h=ae\cos\theta$ ….(i)

And $k=\frac{2aeb\sin\theta}{2a+2ae}=\frac{eb\sin\theta}{1+e}$ …..(ii)

From Eqs (i) and (ii),

$\frac{h^2}{(ae)^2}+\frac{k^2}{\left(\frac{eb}{1+e}\right)^2}=1$ $\left[\because \cos^2\theta+\sin^2\theta=1\right]$

$\therefore$ Locus of incentre is $\frac{x^2}{a^2e^2}+\frac{y^2}{\left(\frac{be}{1+e}\right)^2}=1$

$A^2=a^2e^2 \text{ and } B^2=\left(\frac{bc}{1+e}\right)^2$

$\therefore (e')^2=1-\frac{B^2}{A^2}$

$=1-\frac{\frac{(be)^2}{(1+e)^2}}{(ae)^2}=1-\frac{b^2}{a^2(1+e)^2}=1-\frac{1-e^2}{(1+e)^2} \qquad \left[\because e^2=1-\frac{b^2}{a^2}\right]$

$\Rightarrow (e')^2=1-\frac{(1-e)}{1+e}$

$\Rightarrow (e')^2=\frac{2e}{1+e}$

$\Rightarrow e'=\sqrt{\frac{2e}{1+e}}$

55. (A, B, C)

(a) Given equation is $\left(x-\frac{1}{13}\right)^2+\left(y-\frac{2}{13}\right)^2=\frac{1}{a^2}\left(\frac{5x+12y-1}{13}\right)^2$

It represent an ellipse

$\frac{1}{a^2}<1\Rightarrow a^2>1\Rightarrow a>1$

(b) $4x^2+9y^2+8x-36y=-4$

$\Rightarrow 4(x^2+2x+1)+9(y^2-4y+4)=36$

$\Rightarrow 4(x+1)^2+9(y-2)^2=36$

$\Rightarrow \frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$

$\because a=3, b=4 \text{ and } e=\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$

$\because$ (-1, 2) is centre and (1, 2) lies on major axis

$\therefore$ Minimum distance =1

(c) Equation of normal at P(θ) is

$5x\sec\theta-4y\cos ec\theta=25-16$ and it passes through $P(0,\alpha)$

$\alpha=\frac{-9}{4\operatorname{cosec}\theta}$

$\Rightarrow \alpha=\frac{-9}{4}\sin\theta$

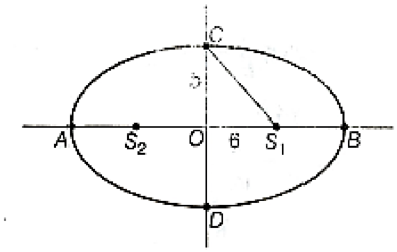
$\Rightarrow |\alpha|<\frac{9}{4}$

(d) $\therefore$ Length of latusrectum

$=\frac{2b^2}{a}=\frac{2a}{3}\Rightarrow 3b^2=a^2$

$\Rightarrow \frac{b2}{a2}=\frac{1}{3}\Rightarrow e=\sqrt{1-\frac{1}{3}}=\sqrt{\frac{2}{3}}$

56. (0.2)



$\because OS_1 = ae = 6$ *and* $OC = b$ Let also $CS_1 = a$

$\because$ Area of $\Delta OCS_1 = \frac{1}{2}(OS_1)(OC) = 3b$

$\because$ Semi-perimeter of $\Delta OCS_1 = \frac{1}{2}(OS_1 + OC + CS_1)$

$\frac{1}{2}(6 + a + b)$

$\because$ In radius of $\Delta OCS_1 = 1$

$$\frac{3b}{\frac{1}{2}(6+a+b)} = 1 \Rightarrow 5b = 6 + a$$

$\because b^2 = a^2 - a^2e^2 = a^2 - 36$

From eq(ii),

$25b^2 = 36 + 12a + a^2$

$\Rightarrow 2a^2 - a - 78 = 0$

$\Rightarrow a = \frac{13}{2}, -6$

When $a = \frac{13}{2}, then\ b = \frac{5}{2}$

$\therefore \frac{a-b}{20} = \frac{\frac{13}{2} - \frac{5}{2}}{20} = \frac{4}{20} = 0.2$

57. (0.5)

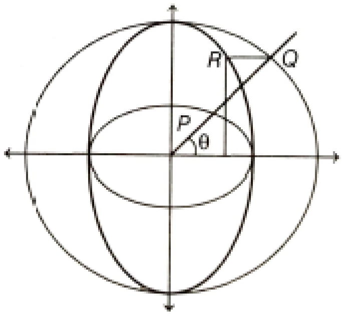
Let line OPQ makes angle θ with X-axis. So $P = (a\cos\theta, a\sin\theta), Q = (b\cos\theta, b\sin\theta)$ and let R(x, y)

So, $X = a\cos\theta, Y = b\sin\theta$

Eliminating θ we get

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Locus of R is an ellipse



Also a < b so varterics are (0, b) and (0, -b) and extremities of minor axis are $(\pm a, 0)$

So ellipse touches both inner circle and outer circle

If focii are $(0, \pm a)$ So, $a = be$ then $e = \frac{a}{b}$

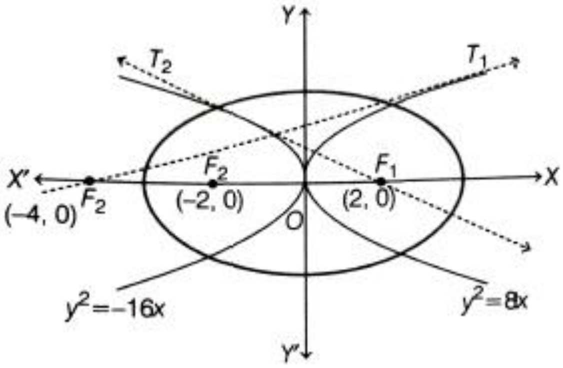
$\therefore e = \sqrt{1-e^2} \Rightarrow e^2 = 1 - e^2 \Rightarrow e = \frac{1}{\sqrt{2}}$

And ratio of radii is $\frac{a}{b} = e = \frac{1}{\sqrt{2}}$

Given that e = $\sqrt{2}\lambda$

$\lambda = \frac{1}{2} \Rightarrow \lambda = 0.5$

58. (4)



Tangent to P1 passes through $(2F_2.0) \Rightarrow (-4, 0)$

$T_1 : y = m_1 x + \frac{2}{m_1}$

$0 = -4m_1 + \frac{2}{m_1}$

$\Rightarrow m_1^2 = \frac{1}{2}$

Also tangent to P_1 passes through ($2F_2$.0) i.e., (2, 0)

T_2: y $= m_2 x + \left(\frac{-4}{m_2} \right)$

$\Rightarrow 0 = 2m_2 - \frac{4}{m_2}$

$\Rightarrow m_2^2 = 2$

$\Rightarrow \frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$

59. (10)

The tangent to the ellipse is on the form

$y = mx + \sqrt{25m^2 + 4}$

Therefore, the common tangent is

$y = mx + \sqrt{25m^2 + 4}$

$\Rightarrow m = \sqrt{\frac{r^2 - 4}{25 - r^2}}$

$\therefore A = \left(-\frac{\sqrt{25m^2 + 4}}{m}, 0 \right)$

And $B = \left(0, \sqrt{25m^2+4}\right)$

Let $M(x_1.y_1)$ be the mid-point of Ab so that

$2x_1 = \frac{-\sqrt{25m^2+4}}{m}$

and $2y_1 = \sqrt{25m^2+4}$

$\Rightarrow 4m^2x_1^2 = 4y_1^2$

$\Rightarrow m_2 = \frac{y_1^2}{x_1^2}$

Therefore $2y_1 = \sqrt{25m^2+4}$

And $m^2 = \frac{y_1^2}{x_1^2}$

$\Rightarrow 4y_1^2 = \frac{25y_1^2+4x_1^2}{x_1^2}$

$\Rightarrow 4x_1^2 + 25y_1^2 - 4x_1^2y_1^2$

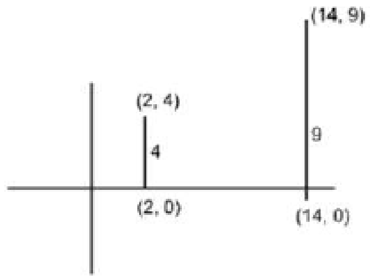
$\therefore$ Locus is $4x^2+25y^2 = 4x^2y^2 \Rightarrow \lambda = 4 \, and \, \mu = 25$

$\therefore \sqrt{\lambda\mu} = \sqrt{25\times 4} = 10$

60. (13.00)

$2ae = \sqrt{(12)^2+5^2} = 13$

$b^2 = 36 \Rightarrow ae = \frac{13}{2}$



b = 6

$a^2 = \frac{169}{4} + 36 \Rightarrow a = \frac{\sqrt{313}}{2} \, e = \frac{13}{\sqrt{313}}$