Exponents And Rationalization

Level 1

1. $[(21+15)(21-15)]^{4/3} = [36\times 6]^{4/3}$

$= 6^{3\times 4/3} = 6^4$

$= 1296$

2. $64^{\frac{2}{3}+\frac{1}{3}-\frac{5}{3}} = 64^{-2/3}$

$= 4^{-2}$

3. $[(61+11)(61-11)]^{3/2}$

$[72\cdot 50]^{3/2} = (36\times 100)^{3/2}$

$= (6\times 10)^3 = 60^3$

$= 216000$

4. $\sqrt{x^{-1}y\cdot y^{-1}z\cdot z^{-1}x} = \sqrt{x^0y^0z^0}$

$= 1$

5. $\left(\dfrac{13^{-6}}{14^{-16}}\right)^{\frac{1}{48}} = \dfrac{13^{-\frac{1}{8}}}{14^{-1/3}}$

$= \dfrac{14^{1/3}}{13^{1/8}}$

6.

$$x = 2^{2} \cdot 2^{-2} = 2^{0}$$

$$x = 1$$

$$\therefore \boxed{x^{-5} = 1}$$

7. $$\frac{5^{-2}}{7^{2}} = \frac{25}{49}$$

8. $$(6561)^{\frac{1}{8}} + (3125)^{\frac{1}{5}}$$

$$= (3^{8})^{\frac{1}{8}} + (5^{5})^{1/5}$$

$$= 3 + 5$$

$$= 8$$

9. $$4^{2^{2}} = 4^{4}$$

$$= 2^{8}$$

10. $$x^{200} < 3^{300}$$

$$x^{2} < 3^{3} = 27$$

$$\therefore x = 5$$

11. $5^{n-3} = 5^{4}$

$n = 7$

$\therefore \quad 5^{7+3} = 5^{10}$

12. $\dfrac{3^{5x} \cdot 81 \cdot 6561}{3^{2x}} = 3^{3}$

$3^{5x} \cdot 3^{4} \cdot 3^{8} \cdot 3^{-2x} = 3^{3}$

$3x + 12 = 3$

$x = -3$

13. $\sqrt[3]{\sqrt{8} - \sqrt{7}} \cdot \sqrt[3]{\sqrt{8} + \sqrt{7}}$

$\sqrt[3]{(\sqrt{8})^{2} - (\sqrt{7})^{2}}$

$= 1$

~~DA.~~ LEVEL 2

1. $\dfrac{c + \sqrt{p} + \sqrt{q} + \sqrt{r}}{23} = \dfrac{(\sqrt{7} + 2\sqrt{3})(2\sqrt{7} + \sqrt{5})}{(2\sqrt{7} - \sqrt{5})(2\sqrt{7} + \sqrt{5})}$

$\dfrac{2 \times 7 + \sqrt{35} + 4\sqrt{21} + 2\sqrt{15}}{28 - 5} = \dfrac{14 + \sqrt{35} + \sqrt{336} + \sqrt{60}}{23}$

$\therefore \quad c = 14, \quad p = 35, \quad q = 60, \quad r = 336$

(2) $a-\sqrt{b}=\sum_{K=4}^{143}\frac{1}{\sqrt{K}+\sqrt{K+1}}$

$=\sum_{4}^{143}(\sqrt{K+1}-\sqrt{K})$

$=(\sqrt{5}-\sqrt{4})+(\sqrt{6}-\sqrt{5})+\cdots\cdots+(\sqrt{144}-\sqrt{143})$

$=\sqrt{144}-\sqrt{4}$

$a-\sqrt{b}=12-2=10-\sqrt{0}$

$\therefore$ $a=10$, $b=0$

(3) 10^4

(4) $(2^{1/4}-1)(2^{3/4}+2^{1/2}+2^{1/4}+1)$

using $x^n-1=(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\cdots+x+1)$

$\therefore x=2^{1/4}$

$\therefore 2-1=1$

(5) $abc=1$

$ab=\frac{1}{c}$, $c=\frac{1}{ab}$

$\frac{1}{1+a+b^{-1}}+\frac{1}{1+b+c^{-1}}+\frac{1}{1+c+a^{-1}}=1$

(6) $\cancel{x = y^b}$ $x = y^a$

$y = z^b$

$z = x^c$

$$z = x^c = (y^a)^c = y^{ac}$$

$$z = (z^b)^{ac} = z^{abc}$$

$$\boxed{abc = 1}$$

(7) $\cancel{7^{20}} \left[7^{2^{0.16}} \right] = 7$

(8)

$$2^{-x} = 2^5$$

$$x = -5$$

$$\therefore 2x = -10$$

(9) $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{2}{9}\right)^3$

$$\frac{3^{-2}}{4^{-2}} \times \left(\frac{9}{2}\right)^3 = \frac{3^{-2}}{2^{-4}} \times \frac{3^6}{2^3}$$

$$= \frac{3^4}{2^{-1}} = 2 \times 81$$

$$= [9 \times \sqrt{2}]^2$$

$\therefore$ Reciprocal is $(9\sqrt{2})^{-2}$.

(10) $\left(\frac{-2}{3}\right)^{4} \times \left(\frac{3}{2}\right)^{3}$

$$\frac{2^4}{3^4} \times \frac{3^3}{2^3} = \frac{2}{3}$$

(11) $$5^{3/2 + a/2} = 5^{a+2}$$

$$a = 4$$

(12) $$\frac{4}{3} \times \frac{3}{4} = 1$$

13. $$\left(\frac{3}{5}\right)^{-12} = \left(\frac{3}{5}\right)^{6x}$$

$$x = -2$$

14. Given $10^{x} = 64$

$$10^{x/2+1} = 10 \cdot 10^{x/2}$$

$$= 80$$

15. $$4^{4x+6} = 4^{3x+9}$$

$$x = 3$$

EXERCISE - 2

LEVEL-1

(1) $$\frac{5\cdot 3^{4n+4} - 3^{4n+5}}{3\cdot 3^{4n} + 3^{4n}} = \frac{3^{4n+4}(5-3)}{3^{4n}(1+3)}$$

$$= 3^4 \cdot \frac{2}{4} = \frac{81}{2}$$

2: $n = 2$

$$(n+3)^{1/3} = 5^{1/3}$$

3. $$2^a \times 3^b \times 5^c = 16200$$
$$= 2^3 \times 3^4 \times 5^2$$

$a = 3$
$b = 4$
$c = 2$

(4) $$\frac{p^{ac}}{p^{bc}} \times \frac{p^{ba}}{p^{ae}} \times \frac{p^{cb}}{p^{ab}}$$

$$p^{ac-bc+ba-ac+cb-ab} = p^0 = 1$$

(5) $$\left(\frac{2}{3}\right)^{\frac{5\sqrt{x-5}}{9}} = a^0 = \left(\frac{2}{3}\right)^0$$

$x - 5 = 0$

$x = 5$

6. $$\frac{(a+b+c)^{a+b+b+c+a+c}}{(a+b+c)^{2(a+b+c)}} = 1$$

7) I: $$a^{p^2-q^2+q^2-r^2+r^2-p^2} = a^0$$
$$= 1$$

II. $$\left(\frac{a}{b}\right)^{p-q+q-r+r-p} = \left(\frac{a}{b}\right)^0 = 1$$

8. $$\frac{x^{-1}}{x^{-1}+y^{-1}} + \frac{x^{-1}}{x^{-1}-y^{-1}}$$

$$\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{y}} + \frac{\frac{1}{x}}{\frac{1}{x}-\frac{1}{y}}$$

$$\frac{\frac{1}{x}}{\frac{x+y}{xy}} + \frac{\frac{1}{x}}{\frac{y-x}{xy}}$$

$$\frac{y}{x+y} + \frac{y}{y-x} = \frac{2y^2}{y^2-x^2}$$

9 $\sqrt[5]{ab^{-1} \cdot bc^{-1} \cdot ca^{-1}}$

$\sqrt[5]{a^{1-1}\ b^{1-1}\ c^{1-1}}$

$\sqrt[5]{a^{0}\ b^{0}\ c^{0}}$

$= 1$

10 i) $(81)^{1/4} = (3^4)^{\frac{1}{4}}$

$= 3$

ii) $(625)^{-1/4} = (5^4)^{-\frac{1}{4}} = 5^{-1}$

iii) $\left[\frac{64}{125}\right]^{-2/3} = \left[\frac{4^3}{5^3}\right]^{-2/3}$

$\frac{4^{-2}}{5^{-2}} = \frac{25}{16}$

11 i) $\left[\frac{3}{12}\right]^4 \times \left[\frac{12}{4}\right]^3 \times \left[\frac{2}{3}\right]^2$

$\frac{1}{4^4} \times 3^3 \times \frac{2^2}{3^2}$

$= 3/64$

ii) $\left[\frac{6}{5}\right]^3 \times \left[\frac{-5}{3}\right]^4 \times \left[\frac{3}{2}\right]^6$

$\frac{6^3 \times 5^4 \times 3^6}{5^3 \times 3^4 \times 2^6} = \frac{3^5 \times 5}{2^3}$

12 i) $\dfrac{(243)^{2/5} \cdot (625)^{3/2}}{5^3 \times 16^{3/4}}$

$\dfrac{3^2 \times 5^6}{5^3 \cdot 2^3} = \dfrac{3^2 \times 5^3}{2^3}$

ii) $\dfrac{16 \times 2^{n+1} - 4 \times 2^n}{16 \times 2^{n+2} - 2 \times 2^{n+2}} = \dfrac{2^{n+2}(8 \quad -1)}{2^{n+2}(16-2)} = \dfrac{1}{2}$

13 i) $x^{-2}y$ (ii) $x^{-3} \cdot y^{-1}$ (iii) $x^{3/4-1} \cdot y^{\frac{1}{2}-3}$

$= x^{-1/4} \cdot y^{-5/2}$

14 $a - b^{a-b}$

$= 2 - (-2)^{2+2}$

$= 2 - 2^4$

$= -14$

15 $x = y^{2/3} + 4y^{-1/2}$, when $y = 64$

$x = ?$

$x = (64)^{2/3} + 4\ (64)^{-1/2}$

$= 4^2 + 4 \cdot 8^{-1}$

$= 16 + \frac{1}{2}$

$= 33/2 = \underline{\underline{16.5}}$

16. $$\left(\frac{2}{3}\right)^{3}\left(\frac{3}{2}\right)^{2x} = \frac{81}{16} = \left(\frac{3}{2}\right)^{4}$$

$$\left(\frac{3}{2}\right)^{2x-3} = (3/2)^{4}$$

$$2x = 7$$

$$x = 3.5$$

LEVEL (2):

Q1. i)
$$\frac{1}{3+\sqrt{2}} = \frac{3-\sqrt{2}}{(3+\sqrt{2})(3-\sqrt{2})}$$

$$= \frac{3-\sqrt{2}}{7}$$

ii)
$$\frac{1}{\sqrt{7}-\sqrt{5}} = \frac{\sqrt{7}+\sqrt{5}}{7-5} = \frac{\sqrt{7}+\sqrt{5}}{2}$$

iii)
$$\frac{17}{\sqrt{42}-5} = \frac{17(\sqrt{42}+5)}{42-25} = \sqrt{42}+5$$

iv)
$$\frac{(3\sqrt{2}+1)}{(2\sqrt{5}-3)} = \frac{(3\sqrt{2}+1)(2\sqrt{5}+3)}{20-9}$$

$$= \frac{6\sqrt{10}+9\sqrt{2}+2\sqrt{5}+3}{11}$$

Q.2 i)

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$$

$$\frac{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{b})}{a-b} + \frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{b})}{a-b}$$

$$\frac{a+b+2\sqrt{ab} + a+b-2\sqrt{ab}}{a-b}$$

$$= \frac{2(a+b)}{a-b}$$

ii) $$\frac{1}{4+\sqrt{15}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{4-\sqrt{15}}$$

$$\frac{4-\sqrt{15}}{4^2-15} + \frac{2(\sqrt{5}+\sqrt{3})}{5-3} + \frac{(4+\sqrt{15})}{4^2-15}$$

$$\frac{4-\sqrt{15}}{1} + \sqrt{5}+\sqrt{3} + \frac{4+\sqrt{15}}{1}$$

$$= 8+\sqrt{5}+\sqrt{3}$$

Q3 i) $a-b\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Rationalising the denominator,

$$a-b\sqrt{3} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{3-1}$$

$$= \frac{3+1-2\sqrt{3}}{2}$$

$$a-b\sqrt{3} = 2-\sqrt{3}$$

$\therefore a = 2$

$b = +1$

II $$a - b\sqrt{77} = \frac{\sqrt{11} - \sqrt{7}}{\sqrt{11} + \sqrt{7}}$$

$$= \frac{(\sqrt{11} - \sqrt{7})(\sqrt{11} - \sqrt{7})}{11 - 7}$$

$$= \frac{11 + 7 - 2\sqrt{11}\sqrt{7}}{4}$$

$$= \frac{18 - 2\sqrt{77}}{4}$$

$$= \frac{18}{4} - \frac{2\sqrt{77}}{4}$$

$$a - b\sqrt{77} = \frac{9}{2} - \frac{1}{2}\sqrt{77}$$

$$a = \frac{9}{2}$$

$$b = \frac{1}{2}$$

III $$a + b\sqrt{3} = \frac{5 + 3\sqrt{3}}{4\sqrt{3} + 7}$$

$$= \frac{(5 + 3\sqrt{3})(4\sqrt{3} - 7)}{(4\sqrt{3} + 7)(4\sqrt{3} - 7)}$$

$$= \frac{20\sqrt{3} - 35 + 36 - 21\sqrt{3}}{48 - 49}$$

$$a + b\sqrt{3} = \frac{1 - \sqrt{3}}{-1} = -1 + \sqrt{3}$$

$$\therefore \begin{pmatrix} a = -1 \\ b = 1 \end{pmatrix}$$

(4) i) $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

By rationalising the denominator,

$$\frac{\sqrt{a}(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})} + \frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}$$

$$= \frac{\sqrt{a}(\sqrt{a}-\sqrt{b})}{a-b} + \frac{\sqrt{b}(\sqrt{a}+\sqrt{b})}{a-b}$$

$$= \frac{a-\sqrt{ab}+b+\sqrt{ab}}{a-b}$$

$$= \frac{a+b}{a-b}$$

ii) $\frac{\sqrt{7}-2}{\sqrt{7}+2} + \frac{\sqrt{7}+2}{\sqrt{7}-2}$

$$\frac{(\sqrt{7}-2)(\sqrt{7}-2)}{(\sqrt{7}+2)(\sqrt{7}-2)} + \frac{(\sqrt{7}+2)(\sqrt{7}+2)}{(\sqrt{7}-2)(\sqrt{7}+2)}$$

$$\frac{7+4-4\sqrt{7}}{7-4} + \frac{7+4+4\sqrt{7}}{7-4}$$

$$\frac{11-4\sqrt{7}}{3} + \frac{11+4\sqrt{7}}{3}$$

$$\frac{22}{3}$$

5 $\frac{1}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})}$

$= \frac{1-\sqrt{2}}{-1} = \sqrt{2}-1$

$\frac{1}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{2}-\sqrt{3}}{2-3} = \sqrt{3}-\sqrt{2}$

$\frac{1}{\sqrt{3}+\sqrt{4}} = \frac{\sqrt{3}-\sqrt{4}}{3-4} = \sqrt{4}-\sqrt{3}$

$\vdots$

$\frac{1}{\sqrt{99}+\sqrt{100}} = \frac{\sqrt{99}-\sqrt{100}}{99-100} = \sqrt{100}-\sqrt{99}$

$\therefore$ After adding, we get

$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$

$= \sqrt{2}-1 + \sqrt{3}-\sqrt{2} + \sqrt{4}-\sqrt{3} + \cdots + \sqrt{100}-\sqrt{99}$

$= \sqrt{100}-1$

$= 10-1$

$= 9$

(6) $\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-\sqrt{4}}$

By rationalising the denominator,

$$\frac{3+\sqrt{8}}{9-8} - \frac{(\sqrt{8}+\sqrt{7})}{8-7} + \frac{\sqrt{7}+\sqrt{6}}{7-6} - \frac{(\sqrt{6}+\sqrt{5})}{6-5} + \frac{(\sqrt{5}+\sqrt{4})}{5-4}$$

$$3+\sqrt{8} - \sqrt{8} - \sqrt{7} + \sqrt{7} + \sqrt{6} - \sqrt{6} - \sqrt{5} + \sqrt{5} + \sqrt{4}$$

$$3+\sqrt{4} = 3+2 = 5$$

(7) Given $a = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$, $b = \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$

$$\therefore\ ab = 1$$

$$a+b = \frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}} + \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$$

$$= \frac{(\sqrt{7}+\sqrt{5})(\sqrt{7}+\sqrt{5})}{7-5} + \frac{(\sqrt{7}-\sqrt{5})(\sqrt{7}-\sqrt{5})}{7-5}$$

$$= \frac{7+5+2\sqrt{35} + 7+5-2\sqrt{35}}{2}$$

$$= \frac{24}{2} = 12$$

$$\therefore\ a+b+ab = 13$$

(8) Given $2^m \times \frac{1}{2^m} = \frac{1}{4} = 2^{-2}$

$$2^{-2m} = 2^{-2}$$

$$\therefore m=1$$

And

$$\frac{1}{7}\left\{\left((\sqrt{16})^m\right)^{1/2} + \left(\frac{1}{5^m}\right)^{-1}\right\}$$

$$\frac{1}{7}\left\{4^{m/2} + 5^m\right\}$$

$$\frac{1}{7}\left\{4^{1/2} + 5^1\right\} \text{, but } m=1$$

$$\frac{1}{7}\left\{2+5\right\} = \frac{7}{7} = 1$$

(9) Given $a=2$, $b=4$

$$\Rightarrow \left(\frac{a}{b}\right)^{a-b} + \left(\frac{b}{a}\right)^{b-a}$$

$$\Rightarrow \left(\frac{2}{4}\right)^{2-4} + \left(\frac{4}{2}\right)^{4-2}$$

$$\left(\frac{1}{2}\right)^{-2} + 2^2$$

$$= 2^2 + 2^2 = 4+4$$

$$= 8$$

(10)

$$\frac{5^{n+2}-6\times5^{n+1}}{13\times5^{n}-2\times5^{n+1}}$$

$$\frac{5^{n+1}(5-6)}{5^{n}(13-2\times5)}=\frac{5(-1)}{13-10}$$

$$=-5/3$$

(11) Given $5^{x-3}\times3^{2x-8}=225$

$$5^{x-3}\times3^{2x-8}=5^{2}\times3^{2}$$

on Comparing, we get

$$x-3=2$$

$$x=5$$

(12) Given

$$\frac{(81)^{n/2}\times3^{2}\times3^{n}-27^{n}}{3^{3m}\times2^{3}}=\frac{1}{27}$$

$$\frac{3^{2n+2+n}-3^{3n}}{3^{3m}\times2^{3}}=3^{-3}$$

$$\frac{3^{3n}(3^{2}-1)}{3^{3m}\times8}=3^{-3}$$

$$\Rightarrow 3^{3n-3m} \frac{\times 8}{8} = 3^{-3}$$

$$\Rightarrow 3^{3n-3m} = 3^{-3}$$

$$\Rightarrow \boxed{n-m=-1}$$

$$\Rightarrow \boxed{m=1+n}$$ Hence Proved

13 Given $x = 2+\sqrt{3}$.

$$\frac{1}{x} = 2-\sqrt{3}$$

$$x+\frac{1}{x} = 4$$

on cubic, we get

$$\left(x+\frac{1}{x}\right)^3 = 4^3 = 64$$

$$x^3+\frac{1}{x^3}+3\left(x+1/x\right) = 64$$

$$x^3+\frac{1}{x^3} = 64-12$$

$$= 52$$

(14)

$$x = 3+\sqrt{8}$$

$$\therefore \frac{1}{x} = 3-\sqrt{8}$$

$$x+\frac{1}{x} = 6$$

on squaring, we get

$$\left(x+\frac{1}{x}\right)^2 = 6^2 = 36$$

$$x^2+\frac{1}{x^2}+2 = 36$$

$$x^2+\frac{1}{x^2} = 34$$

(15) i) $\sqrt{7+\sqrt{40}}$

$$= \sqrt{7+2\sqrt{10}} = \sqrt{2+5+2\cdot\sqrt{2}\cdot\sqrt{5}}$$

$$= \sqrt{5}+\sqrt{2}$$

ii) $\sqrt{17+12\sqrt{2}} = \sqrt{17+2\times 6\sqrt{2}}$

$$= \sqrt{9+8+2\sqrt{36\times 2}} = \sqrt{9+8+2\sqrt{72}}$$

$$= \sqrt{9+8+2\sqrt{9}\sqrt{8}}$$

$$= 3+\sqrt{8}$$

iii) $\sqrt{11+6\sqrt{2}} = 3+\sqrt{2}$