Applications Of Derivatives
EXERCISE 1

1. $y = 2x^3 + 13x^2 + 5x + 9$

$\therefore \quad \frac{dy}{dx} = 6x^2 + 26x + 5$

Let the point be (h, k)

$\therefore \quad k = 2h^3 + 13h^2 + 5h + 9$

$\therefore \quad \frac{y - (2h^3 + 13h^2 + 5h + 9)}{x - h} = 6h^2 + 26h + 5$

substituting $(0, 0)$

$\therefore \quad 2h^3 + 13h^2 + 5h + 9 = 6h^3 + 26h^2 + 5h$

$\therefore \quad 4h^3 + 13h^2 - 9 = 0$

$\Rightarrow \quad h = -1, k = 15.$

2. $x = a(t + \sin t \cos t)$

$y = a(1 + \sin t)^2$

$\therefore \quad \frac{dx}{dt} = a(1 + \cos 2t)$

$\frac{dy}{dt} = 2a(1 + \sin t)\cos t$

$\therefore \quad \frac{dy}{dx} = \frac{2\cos t + \sin 2t}{1 + \cos 2t}$

$$= \frac{2\cos t(1 + \sin t)}{2\cos^2 t} = \frac{\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right)^2}{\cos^2\frac{t}{2} - \sin^2\frac{t}{2}}$$

$$= \frac{1 + \tan\frac{t}{2}}{1 - \tan\frac{t}{2}} \qquad = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

3. $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\therefore \quad \frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}$

At (a, b)

$$\frac{dy}{dx} = -\frac{b}{a}$$

$\therefore \quad b(x-a)+a(y-b)=0$

$\therefore \quad bx+ay=2ab$

$\therefore \quad \frac{x}{a}+\frac{y}{b}=2.$

4. $y = be^{-x/a}$

$x = 0 \quad \Rightarrow \quad y = b$

$y' = -\frac{b}{a}e^{-x/a}$

$\therefore \quad \left.\frac{dy}{dx}\right|_{(0,b)} = -\frac{b}{a}$

$\therefore \quad a(y-b) = -bx$

$\therefore \quad bx+y=ab$

$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}=1.$

5. $y = 3x^2 + bx + 2$

$x = 0 \quad \Rightarrow \quad y = 2$

$\frac{dy}{dx} = 6x + b$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=0} = b = 4$ .... Given

$\therefore \quad b = 4$

6. $y = \frac{8-x^2}{2}$

$\therefore \quad \frac{dy}{dx} = -x = -2$ [Given]

$\therefore \quad x = 2$

$\Rightarrow \quad y = 2$

$\therefore \quad y-2+2(x-2)=0$

$\therefore \quad 2x+y-6=0.$

7. Points are (p, ap^2+bp+c) & (q, aq^2+bq+c)

$$\therefore \quad \frac{y-(ap^2+bp+c)}{x-p} = \frac{a(q^2-p^2)+b(q-p)}{q-p}$$

$$\therefore \quad \frac{y-\left(ap^2+bp+c\right)}{x-p}=aq+ap+b$$

$$\therefore \quad y=\left(aq+ap+b\right)x-apq+c$$

$$\therefore \quad m=aq+ap+b$$

$$\& \quad m=2ax+b=\frac{dy}{dx}$$

$$\therefore \quad x=\frac{p+q}{2}$$

8. $\sqrt{x}+\sqrt{y}=\sqrt{a}$

$$\therefore \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \qquad x-x=\sqrt{xy}$$

$$\therefore \quad \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

$$\therefore \quad \frac{Y-y}{X-x}=-\sqrt{\frac{y}{x}}$$

$$X=0 \;\Rightarrow\; Y=y+\sqrt{xy}$$

$$Y=0 \;\Rightarrow\; X=x+\sqrt{xy}$$

$$x+y+2\sqrt{xy}=OA+OB=\left(\sqrt{x}+\sqrt{y}\right)^2=a$$

9. $x^{2/3}+y^{2/3}=a^{2/3}$

$$\therefore \quad \frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore \quad \frac{Y-y}{X-x}=-\left(\frac{y}{x}\right)^{1/3}$$

$$\therefore \quad Y=y+x^{2/3}y^{1/3} \qquad \text{When } X=0$$

$$X=x+x^{1/3}+y^{2/3} \qquad \text{When } Y=0$$

$$Y^2+X^2=y^2+x^{4/3}y^{2/3}+2x^{2/3}y^{4/3}+x^2+2x^{4/3}y^{2/3}+x^{2/3}y^{4/3}$$

$$=x^2+y^2+3x^{4/3}y^{2/3}+3x^{2/3}y^{4/3}$$

$$=x^2+x^{4/3}y^{2/3}+y^2+x^{2/3}y^{4/3}+2\left(x^{4/3}y^{2/3}+x^{2/3}y^{4/3}\right)$$

$$=\left(x^{4/3}+y^{4/3}\right)a^{2/3}+\left(2x^{2/3}y^{2/3}\right)a^{2/3}$$

$$=a^{2/3}\left(x^{2/3}+y^{2/3}\right)$$

$$=a^2$$

10. $xy^n = a^{n+1}$

$\therefore \quad y^n = nxy^{n-1}\dfrac{dy}{dx} = 0 \quad , \; n \neq -1$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{y}{nx}$

$\therefore \quad \dfrac{Y-y}{X-x} = -\dfrac{y}{nx}$

$X = 0 \;\Rightarrow\; Y = y + \dfrac{y}{n}$

$Y = 0 \;\Rightarrow\; X = x + nx$

$\therefore \quad \Delta = \dfrac{1}{2}XY = \dfrac{1}{2}xy(1+n)\left(1+\dfrac{1}{n}\right)$, Here n is a constant.

Δ is constant only when xy is constant but xy^n is constant

$\therefore \quad n = 1.$

11. $f(x) = 2x^3 - 9x^2 + 12x - 3$

$\therefore \quad f'(x) = 6x^2 - 18x + 12 > 0$

$\therefore \quad x^2 - 3x + 2 > 0$

$\therefore \quad x \in (-\infty, 1) \cup (2, \infty)$

12. $f(x) = x^3 - ax^2 + 48x + 19$

$f'(x) = 3x^2 - 2ax + 48 \geq 0 \quad \forall\, x$

$\therefore \quad (2a)^2 - 4(3)(48) \leq 0$

$\therefore \quad a^2 - 144 \leq 0$

$\therefore \quad a \in [-12, 12]$

13. $f(x) = 2x^3 - 9x^2 - 60x + 81$

$\therefore \quad f'(x) = 6x^2 - 18x - 60 < 0$

$\therefore \quad x^2 - 3x - 10 < 0$

$\therefore \quad x \in (-2, 5)$

14. $f(x) = \dfrac{x^2}{x+2}$

$\therefore \quad f'(x) = \dfrac{2x(x+2) - x^2}{x+2}$

$= \dfrac{x^2 + 4x}{x+2} < 0$

$\therefore \quad \frac{x(x+4)}{(x+2)} < 0 \qquad \therefore \quad x \in (-\infty, -4) \cup (-2, 0)$

15. $f(x) = x^2$

$\therefore \quad f'(x) = x^x(1 + \ell n\, x) = 0$

$\therefore \quad 1 + \ell n\, x = 0$

$\therefore \quad x = \frac{1}{e}$

For $x < \frac{1}{e}, f'(x) < 0$

$\therefore$ Function decreases in $\left(0, \frac{1}{e}\right)$.

16. $f(x) = \frac{\log x}{x}$

$\therefore \quad f'(x) = \frac{1 - \log x}{x^2} < 0$

$\therefore \quad \log x > 1 \quad \Rightarrow \quad x > e$

$\therefore \quad x \in (e, \infty)$

17. $f(x) = 2|x-2| + |x-3|$

For $x < 2$

$f(x) = 2(2-x) + 3 - x$

$= 7 - 3x$ is a decreasing function.

For $2 < x < 3$

$f(x) = 2(x-2) + 3 - x$

$= x - 1$ is increasing function.

For $x > 3$

$f(x) = 3x - 7$ is an increasing function.

$\therefore \quad x \in (2, \infty)$

18. $f(x) = \cos x - \sin x$

$\therefore \quad f'(x) = -\sin x - \cos x < 0 \qquad \forall\, x \in \left(0, \frac{\pi}{2}\right)$

$g(x) = \cos x + \sin x$

$g'(x) = \cos x - \sin x > 0 \qquad \forall\, x \in \left(0, \frac{\pi}{4}\right)$

$< 0 \qquad \forall\, x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$h(x) = \frac{\sin x}{x}$$

$$h'(x) = \frac{x\cos x - \sin x}{x^2} = 0 \quad \text{at} \quad x = 0,$$

$$h''(x) = \frac{x^2(-x\sin x) - 2x(x\cos x - \sin x)}{x^2} < 0 \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

$\therefore \quad h'(x) < 0 \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \frac{x}{\sin x}$ being reciprocal of $\frac{\sin x}{x}$ is an increasing function.

19. $$f(x) = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$$

$$f'(x) = \frac{(a\cos x - b\sin x)(c\sin x + d\cos x) - (a\sin x + b\cos x)(c\cos x - d\sin x)}{(c\sin x + d\cos x)^2}$$

$$= \frac{ad - bc}{(c\sin x + d\cos x)^2} > 0 \quad \forall\, x \quad \text{iff} \quad ad - bc > 0.$$

20. $\sin x - bx + c = f(x)$

$f'(x) = \cos x - b \le 0 \quad \forall \; b \ge 1$

21. $y = 2x^3 - 3x^2 - 36x + 10 = f(x)$

$f'(x) = 6x^2 - 6x - 36 = 0$

$\therefore \quad x^2 - x - 6 = 0$

$\therefore \quad x = 3 \;\text{ or }\; x = -2$

$f(3) = 2(27) - 3(9) - 36(3) + 10$

$= -71$

$f(-2) = 2(-8) - 3(4) - 36(-2) + 10$

$= -28 + 82 = 54$

22. $f(x) = x^2 - 3x + 3$

$\therefore \quad f'(x) = 2x - 3 = 0$

$\therefore \quad x = \frac{3}{2} \qquad f''(x) = 2 > 0$

$\therefore \quad f(x)$ has minima at $x = \frac{3}{2}$

$f\left(\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} + 3 = \frac{3}{4}$

23. $f(x) = 2x^3 - 3x^2 - 12x + 8$

$f'(x) = 6x^2 - 6x - 12 = 0$

$\therefore \quad x = 2 \quad \text{or} \quad x = -1$

$f''(x) = 12x - 6 \quad < 0 \quad \text{for} \quad x = -1$

and $\quad > 0 \quad \text{for} \quad x = 2$

$\therefore \quad x = 2$ is minima

24. $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x = f(x)$

$f'(x) = 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x = 0$

$f(x)$ can have minima only as maxima is ∞.

$\therefore \quad a^2 \sec^2 x \tan x = b^2 \operatorname{cosec}^2 x \cot x$

$\therefore \quad a^2 \frac{\sin x}{\cos^3 x} = b^2 \frac{\cos x}{\sin^3 x}$

$\therefore \quad \tan^4 x = \frac{b^2}{a^2}$

$\therefore \quad \tan^2 x = \left|\frac{b}{a}\right| , \ \cot^2 x = \left|\frac{a}{b}\right|$

For $a, b > 0$

$\sec^2 x = \frac{a+b}{a}, \operatorname{cosec}^2 x = \frac{a+b}{b}$

$\therefore \quad a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$

$= a^2 + ab + ab + b^2 = (a+b)^2$

25. $f(x) = \sin x + \sin x \cos x$

$f'(x) = \cos x + \cos^2 x - \sin^2 x = 0 \qquad = \cos x + \cos 2x$

$\therefore \quad \cos x + 2\cos^2 x - 1 = 0$

$\therefore \quad \cos x = -1 \quad \Rightarrow \quad x = \pi \quad \text{or} \quad \cos x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$

$f''(x) = -\sin x - 2\sin 2x \quad < 0 \quad \text{for} \quad x = \frac{\pi}{3}$

$> 0 \quad \text{for} \quad x = -\frac{\pi}{3}$

$\therefore \quad x = \frac{\pi}{3}$

26. $f(x) = x + \sin x$

$f'(x) = 1 + \cos x \qquad f''(x) = -\sin x$

When $f'(x) = 0$, $f''(x) = 0$

$\therefore$ $f(x)$ has neither minimum nor maximum.

27. $$\Delta = \frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ a\cos\theta & b\sin\theta & 1 \\ a\cos\theta & -b\sin\theta & 1 \end{vmatrix}$$

$= \frac{1}{2}(2ab\sin\theta - 2ab\cos\theta\sin\theta)$

$= ab(\sin\theta - \sin\theta\cos\theta)$

$\therefore \quad \frac{d\Delta}{d\theta} = ab(\cos\theta - \cos 2\theta) = 0$

$\therefore \quad 2\cos^2\theta - \cos\theta - 1 = 0$

$\cos\theta = 1 \quad \text{or} \quad \cos\theta = -\frac{1}{2}$

Now for a triangle,as it will be a st.line, $\theta \neq 0$, $\therefore \quad \cos\theta \neq 1$

$\therefore \quad \cos\theta = -\frac{1}{2}$

$\therefore \quad \theta = \frac{2\pi}{3} \qquad \Rightarrow \qquad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \Delta = ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \frac{3\sqrt{3}ab}{4}$

28. $f'(x) = 3x^2 - 6x + 6 \quad > 0 \qquad \text{for all } x$

$\therefore$ Neither minimum nor maximum.

29. $f(x) = (x+6)^4(8-x)^3$

Let $x + 6 = t$

$\therefore \quad 8 - x = 14 - t$

$\therefore \quad f(t) = t^4(14-t)^3 \qquad \& \ f'(x) = f'(t)$

$\therefore \quad f'(t) = 4t^3(14-t)^3 - 3t^4(14-t)^2 = 0$

$\therefore \quad t^3(14-t)^2\left[4(14-t) - 3t\right] = 0$

$\therefore \quad t^3(4-t)^2\left[56 - 7t\right] = 0$

$\therefore \quad t = 0$ or $t = 14$ or $t = 8$

Now $t \neq 0$ & $t \neq 14$ as product will become zero

$\therefore \quad t = 8$ & $f(t) = 8^4.6^3$

30. $x^2-(a-2)x-(a+1)=0$

$\alpha+\beta=a-2,\ \alpha\beta=-(a+1)$

$\therefore \quad \alpha^2+\beta^2=a^2-4a+4+2a+2$

$\therefore \quad f(a)=a^2-2a+6$

$\therefore \quad f'(a)=2a-2=0 \qquad f''(a)>0$

$\therefore \quad a=1$

$\therefore \quad f(a)=5=\min(\alpha^2+\beta^2)$

31. Function must be continuous and differentiable to apply Rolle's theorem.

32. Function must be continuous and differentiable to apply Rolle's theorem.

33. $f(x)=\log(\sin x) \qquad \text{in } \left[\frac{\pi}{6},\frac{5\pi}{6}\right]$

$f\left(\frac{\pi}{6}\right)=f\left(\frac{5\pi}{6}\right)$

$f'(x)=\cot x=0 \text{ at } x=\frac{\pi}{2}$

$\therefore \quad c=\frac{\pi}{2}$

34. $f(x)=\log\left(\frac{x^2+ab}{x(a+b)}\right) \text{ in } [a,b]$

$f'(x)=\frac{2x}{x^2+ab}-\frac{(a+b)}{x(a+b)}=0$

$\therefore \quad 2x^2=x^2+ab$

$\therefore$ x = GM of a & b.

35. $f(x)=x^3+bx^2+ax$ satisfies Rolle's theorem on $[1,3]$

$c=2+\frac{1}{\sqrt{3}}$

$\therefore \quad f(1)=f(3)$

$\therefore \quad 1+a+b=27+9b+3a$ and $3\left(1+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$

Solving , we get $(a,b)=(11,-6)$

36. $f(x)=\log x$ in $[1,e]$

$f'(x)=\frac{1}{x} \qquad f'(c)=\frac{1}{c}$

$\therefore \quad \frac{1}{c}=\frac{\log e-\log 1}{e-1}=\frac{1}{e-1}$

$\therefore \quad c=e-1.$

37. Here, $f(0) = f(2) = 0$ in $[0,2]$

$\therefore$ $f'(c) = 0$

$\therefore$ $(c-2)^2 + 2c(c-2) = 0$

$\therefore$ $(c-2)[3c-2] = 0$

$\therefore$ $c = \frac{2}{3}$ as $c \in (0,2)$

38. $f(x) = \ell x^2 + mx + n$ in

$\therefore$ $f'(c) = 2\ell x + m = \frac{\ell(b^2 - a^2) + m(b-a)}{b-a}$

$\therefore$ $2\ell x + m = \ell(b+a) + m$

$\therefore$ $x = \frac{a+b}{2}$

39. Function should be diffrentiable in domain.

40. $\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x} + \sqrt{x+1}}$

Now, $x > N^2$,

$\therefore$ $x + 1 > N^2$

$\therefore$ $\sqrt{x} > N, \sqrt{x+1} > N$

$\therefore$ $\sqrt{x} + \sqrt{x+1} > 2N$

$\therefore$ $\frac{1}{\sqrt{x} + \sqrt{x+1}} < \frac{1}{2N}$

APPLICATIONS OF DERIVATIVES
EXERCISE 2

1. Let P be $\left(ct, \frac{c}{t}\right)$, then

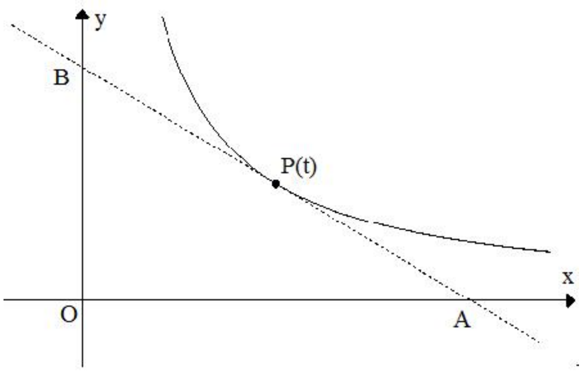
$xy = c^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ or $\frac{dy}{dx} = -\frac{1}{t^2}$

tangent at P will be

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ or $x + t^2 y = 2ct$

Now $OA = |2ct|, OB = \left|\frac{2c}{t}\right|$

$\Delta = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 2c^2$



2. $x^2 = 4y$

$\therefore \quad \frac{dy}{dx} = \frac{x}{2}$

$\therefore \quad -\frac{dx}{dy} = -\frac{2}{x} = -2$ at $(1,2)$

FInd equation of line passing through (1,2) with slope -2.

3. $\frac{dy}{dx} = \frac{a\theta \sin\theta}{a\theta \cos\theta} = \tan\theta$

$\therefore \quad -\frac{dx}{dy} = -\cot\theta$

$\therefore \quad \frac{Y - a(\sin\theta - \theta\cos\theta)}{X - a(\cos\theta + \theta\sin\theta)} = -\cot\theta$

$\therefore \quad Y - a\sin\theta + a\theta\cos\theta = -\cot\theta X + a\cos\theta\cot\theta + a\theta\cos\theta$

$\therefore \quad Y + X\cot\theta - a(\sin\theta + \cos\theta\cot\theta) = 0$

$\therefore \quad$ Distance from origin $= \frac{|a(\sin\theta + \cos\theta\cot\theta)|}{\sqrt{1 + \cot^2\theta}}$

$= a$

4. $\frac{dy}{dx} = \frac{2ae^\theta \cos\theta}{2ae^\theta \sin\theta} = \cot\theta$

$-\frac{dy}{dx} = -\tan\theta$

Equation of tangent is

$$\frac{y-ae^{\theta}(\sin\theta+\cos\theta)}{x-ae^{\theta}(\sin\theta-\cos\theta)}=\cot\theta$$

$\therefore \quad y-ae^{\theta}\sin\theta-ae^{\theta}\cos\theta=x\cot\theta-ae^{\theta}\cos\theta+ae^{\theta}\cos\theta\cot\theta$

$\therefore \quad \frac{x\cos\theta}{\sin\theta}-y+ae^{\theta}\sin\theta+ae^{\theta}\frac{\cos^2\theta}{\sin\theta}=0$

$\therefore \quad x\cos\theta-y\sin\theta+ae^{\theta}=0$

$\therefore \quad p=\frac{|ae^{\theta}|}{1}=ae^{\theta}$

Equation of normal is

$$\frac{y-ae^{\theta}(\sin\theta+\cos\theta)}{x-ae^{\theta}(\sin\theta-\cos\theta)}=-\tan\theta$$

$\therefore \quad y-ae^{\theta}\sin\theta-ae^{\theta}\cos\theta=-x\tan\theta+ae^{\theta}\sin\theta\tan\theta-ae^{\theta}\sin\theta$

$\therefore \quad y\cos\theta+x\sin\theta-ae^{\theta}=0$

$\therefore \quad q=\frac{|-ae^{\theta}|}{1}=ae^{\theta}$

$\therefore \quad p=q$

5. $x^{2/3}+y^{2/3}=a^{2/3}$

$\therefore \quad \frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$

$\therefore \quad -\frac{dx}{dy}=\left(\frac{x}{y}\right)^{1/3}$

Equation of tangent is

$$\frac{Y-y}{X-x}=-\left(\frac{y}{x}\right)^{1/3}$$

$\therefore \quad x^{1/3}Y-x^{1/3}y=-y^{1/3}X+xy^{1/3}$

$\therefore \quad y^{1/3}X+x^{1/3}Y-x^{1/3}y-xy^{1/3}=0$

$$p=\frac{|x^{1/3}y+xy^{1/3}|}{\sqrt{x^{2/3}+y^{2/3}}}$$

$$=|x^{1/3}y^{1/3}a^{1/3}|$$

Equation of normal is

$$\frac{Y-y}{X-x}=\left(\frac{x}{y}\right)^{1/3}$$

$\therefore \quad y^{1/3}Y-y^{4/3}=x^{1/3}X-x^{4/3}$

$\therefore \quad x^{1/3}X-y^{1/3}Y-x^{4/3}+y^{4/3}=0$

$\therefore \quad q=\frac{\left|y^{4/3}-x^{4/3}\right|}{\sqrt{x^{2/3}+y^{2/3}}}$

$=\left|\left(x^{2/3}-y^{2/3}\right)a^{1/3}\right|$

$\therefore \quad 4p^2+q^2=4x^{2/3}y^{2/3}+a^{2/3}\left(x^{4/3}-2x^{2/3}y^{2/3}+y^{4/3}\right)$

$=a^{2/3}\left(x^{2/3}+y^{2/3}\right)$

$=a^2.$

6. $y^2=2x, x^2+y^2=8$

$\therefore \quad x^2+2x-8=0$ and $x>0$ as $x=\frac{y^2}{2}$

$\therefore \quad x=2$

$\Rightarrow \quad y=2$

For $y^2=2x$

$\therefore \quad 2y\frac{dy}{dx}=2$

$\therefore \quad m_1=\frac{1}{y}=\frac{1}{2}$

For $x^2+y^2=8$

$\therefore \quad 2x+2y\frac{dy}{dx}=0$

$\therefore \quad m_2=-\frac{x}{y}=-1$

$\therefore \quad \tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=3$

7. $y=\frac{x+3}{x^2+1}$

$\therefore \quad \frac{dy}{dx}=\frac{\left(x^2+1\right)-\left(2x^2+6x\right)}{\left(x^2+1\right)^2}=\frac{-x^2-6x+1}{\left(x^2+1\right)^2}$

$\therefore \quad m_1 = \frac{-4-12+1}{25} = \frac{-3}{5}$

$y = \frac{x^2-7x+11}{x-1}$

$\therefore \quad \frac{dy}{dx} = \frac{(2x^2-9x+7)-(x^2-7x+11)}{(x-1)^2} = \frac{x^2-2x-4}{(x-1)^2}$

$\therefore \quad m_2 = \frac{4-4-4}{1} = -4$

$\therefore \quad \tan\alpha = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = 4$

8. $x^3-3xy^2+2=0$

$\therefore \quad 3x^2-3xy^2-6xy\frac{dy}{dx}=0$

$\therefore \quad x^2-y^2 = 2xy\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x^2-y^2}{2xy} = m_1$

$3x^2y-y^3+2=0$

$\therefore \quad 6xy+3x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=0$

$\therefore \quad \frac{dy}{dx}(x^2-y^2) = -2xy$

$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2-y^2} = m_2$

$m_1m_2 = -1$

9. $x = y^2$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2y} = m_1$

$xy = k$

$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} = m_2$

$m_1m_2 = -1$

$\therefore \quad \frac{-1}{2x} = -1$

$\therefore \quad x = \frac{1}{2}$

$\therefore \quad y = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad k = \pm\frac{1}{2\sqrt{2}}$

10. $ST = \frac{3}{8}, SN = 24$

$y_0^2 = ST.SN$

$= \frac{3}{8} \times 24 = 9$

$\therefore \quad y_0 = \pm 3$

11. $by^2 = (x+a)^3$

$\therefore \quad 2by\frac{dy}{dx} = 3(x+a)^2$

$\therefore \quad \frac{dy}{dx} = \frac{3(x+a)^2}{2by} = \tan\theta$

$\cot\theta = \frac{2by}{3(x+a)^2}$

$ST = |y\cot\theta| = \left|\frac{2by^2}{3(x+a)^2}\right|$

$SN = |y\tan\theta| = \left|\frac{3(x+a)^2}{2b}\right|$

$\therefore \quad \frac{3p(x+a)^2}{2b} = \frac{4qb^2y^4}{9(x+a)^4}$

$\therefore \quad \frac{p}{q} = \frac{8b^3y^4}{27(x+a)^6} = \frac{8b}{27}\frac{(by^2)^2}{(x+a)^6} = \frac{8b}{27}$

12. $xy^n = a^{n+1}$

$\therefore \quad \frac{dy}{dx} = \frac{-y}{nx} = \tan\theta$

$\therefore \quad SN = |y\tan\theta|$

$= \left|\frac{-y^2}{nx}\right| = \text{constant}$

But xy^n = constant

$\Rightarrow \quad n = -2$

13. $x^2y^2 = a^5$

$\therefore \quad 2xy^2 + 2x^2y\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{-y}{x} = \tan\theta$

$\therefore \quad \cot\theta = \frac{-x}{y}$

$\therefore \quad ST = |y\cot\theta| = |-x|$

14. $x^m y^n = a^{m+n}$

$\therefore \quad mx^{m-1}y^n + nx^m y^{n-1}\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{my}{nx}$

$\therefore \quad \cot\theta = -\frac{nx}{my}$

$\therefore \quad ST = \left|-\frac{nx}{m}\right|$

15. From informationin Q.22,

$(ST)^2 \propto (SN)$

16. $-\frac{dx}{dy} = -\frac{a\theta\cos\theta}{a\theta\sin\theta} = -\cot\theta$

As shown in Q.14 of this exercise it is at a constant distance from origin.

17. $ax + by + c = 0$ normal to $xy = 1$

For $xy = 1, \frac{dy}{dx} = -\frac{y}{x}$

As xy is positive, $\frac{dy}{dx} < 0 \quad \forall\, x, y$

$\therefore \quad -\frac{dx}{dy} < 0 \quad \forall\, x, y$

$\therefore$ Slope of normal is positive.

$\therefore \quad a > 0, b < 0$ or $a < 0, b > 0$

18. $(3 - a)x + ay + a^2 - 1 = 0$

$\therefore \quad -\left(\frac{a}{3-a}\right) > 0$

$\therefore \quad a \in (-\infty, 0) \cup (3, \infty)$

19. $f(x) = 2x^2 - \log|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x} < 0$

$\therefore \quad \frac{4x^2 - 1}{x} < 0$

$\therefore \quad \frac{(2x+1)(2x-1)}{x} < 0 \qquad \therefore \quad x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

20. $f(x) = \frac{x}{\sin x}, \ g(x) = \frac{x}{\tan x}$

$f'(x) = \frac{\sin x - x\cos x}{x^2} > 0 \qquad \forall\, x \in (0,1)$

$g'(x) = \frac{\tan x - x\sec^2 x}{x^2} < 0 \quad \forall\, x \in (0,1)$

21. $f(x) = \tan^{-1}(\sin x + \cos x)$

Let $\quad g(x) = \tan^{-1} x$

$\therefore \quad g'(x) = \frac{1}{1+x^2} > 0 \qquad \forall\, x$

$\therefore \quad f(x)$ increases when $\sin x + \cos x$ increases

Let $\quad h(x) = \sin x + \cos x$

$\therefore \quad h'(x) = \cos x - \sin x > 0$

$\therefore \quad \cos x > \sin x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \quad x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

22. $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x < 0$

23. $f(x) = |x| - |x-1|$

$x < 0$	$\Rightarrow$	$f(x) = -x + 1 - x = 1 - 2x$	MD
$0 < x < 1$	$\Rightarrow$	$f(x) = x + 1 - x = 1$	Constant
$x > 1$	$\Rightarrow$	$f(x) = 2x - 1$	MI

24. $f(x) = x(a^2 - 2a - 2) + \cos x$

$f'(x) = a^2 - 2a - 2 - \sin x > 0 \quad \forall\, x$

$\therefore \quad a^2 - 2a - 2 > 1$

$\therefore \quad a \in (-\infty, -1) \cup (3, \infty)$

25. $\phi(x) = 3f\left(\frac{x^3}{3}\right) + f(3 - x^2) \ \ \forall\ x \in (-3, 4)$

$f''(x) > 0$

$\therefore \quad \phi'(x) = 3x^2 f'\left(\frac{x^3}{3}\right) - 2x f'(3 - x^2) > 0$

$x^2 f'\left(\frac{x^3}{3}\right) > 2x\ f'(3 - x^2)$

For $x > 0$

$x f'\left(\frac{x^3}{3}\right) > 2 f'(3 - x^2)$

$\therefore \quad \frac{x}{2} f'\left(\frac{x^3}{3}\right) > f'(3 - x^2)$

26. $f'(x) \ge 0$, $g'(x) \le 0$

$\therefore \quad h'(x) = f'(\underset{>0}{g(x)}\ \underset{<0}{g'(x)}) \le 0$

$\therefore \quad h(2) = 1$ as $h(1) = 1$

27. $y = a \log|x| + bx^2 + x$

$\therefore \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1 = 0$

$\therefore \quad \frac{a + 2bx^2 + x}{x} = 0 \qquad \alpha = \frac{-4}{3}, \beta = 2$

$\therefore \quad \alpha + \beta = \frac{2}{3} = \frac{-1}{2b}$

$\therefore \quad b = \frac{-3}{4}$

$\alpha\beta = \frac{-8}{3} = \frac{a}{2b}$

$\therefore \quad a = \frac{-8}{3} \times 2 \times \frac{-3}{4} = 4$

28. Point on $y^2 = 4x$ is $\left(t^2, 2t\right)$

Distance between point & (2,1) is

$d = \sqrt{\left(t^2 - 2\right)^2 + \left(2t - 1\right)^2}$

$d^2 = \left(t^2 - 2\right)^2 + \left(2t - 1\right)^2$

$= t^4 - 4t + 5 \qquad = f(t)$

$\therefore \quad f'(t) = 4t^3 - 4 = 0$

$\therefore \quad t = 1$ we can show that $t = 1$ is minima

$\therefore \quad$ Point is $(1, 2)$.

28. Point nearest to the required line will have common normal.

$\therefore \quad \frac{dy}{dx} = 3 = 2x + 7$

$\therefore \quad x = -2, y = -8$

point is $(-2, -8)$

30. $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$x = a\cos\theta, y = 2\sin\theta$

$\sqrt{a^2\cos^2\theta + 4\left(1 - \sin\theta\right)^2} = d$

$d^2 = f(\theta) = a^2\cos^2\theta + 4 - 8\sin\theta + 4\sin^2\theta$

$= a^2 + 4 + \left(4 - a^2\right)\sin^2\theta - 8\sin\theta$

$\therefore \quad f'(\theta) = 2\left(4 - a^2\right)\sin\theta\cos\theta - 8\cos\theta = 0$

$\therefore \quad \cos\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{2}$

$\therefore \quad$ point is (0,2).

31. $r\theta + 2r = k$

$\therefore \quad r(\theta + 2) = k$

$\therefore \quad \theta = \frac{k - 2r}{r}$

$\therefore \quad A = \frac{1}{2} r^2 \times \frac{(k-2r)}{r}$

$= \frac{kr - r^2}{2}$

$\therefore \quad \frac{dA}{dr} = 0$

$\therefore \quad r = \frac{k}{4}$

$\therefore \quad \theta = \frac{k - \frac{k}{2}}{\frac{k}{4}} = 2^c$

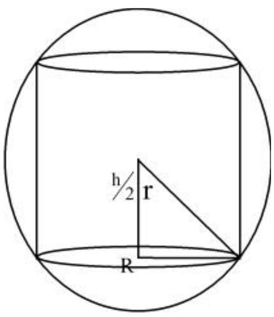
32. From above example, $\theta = 2^c$

$\therefore \quad 2r + 2r = 20$

$\therefore \quad r = 5$

$\therefore \quad A = \frac{1}{2} \times 25 \times 2 = 25$ sq.cm.

33.



$R^2 + \frac{h^2}{4} = r^2$

$v = \pi R^2 h$

$= \pi h \left(r^2 - \frac{h^2}{4} \right)$

$= \pi r^2 h - \frac{\pi h^3}{4}$

$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4} = 0$

$\therefore \quad h^2 = \frac{4r^2}{3} \qquad h = \frac{2r}{\sqrt{3}}$

34. $s = 2\pi r(r+h)$

$$= 2\pi r\left(r + \frac{v}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2v}{r}$$

$$\frac{ds}{dr} = 0 \qquad \therefore \qquad 4\pi r - \frac{2v}{r^2} = 0$$

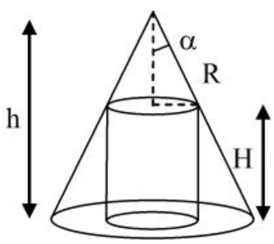
$$\therefore \quad r^3 = \frac{v}{2\pi} \qquad \therefore \qquad r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$$

$$\pi r^2 = \left(\frac{\pi v^2}{4}\right)^{\frac{1}{3}}$$

$$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right)^{\frac{1}{3}} = h$$

$h = 2r$

35. $\frac{R}{h-H} = \tan\alpha$



$R = \tan\alpha(h-H)$

curved surface area

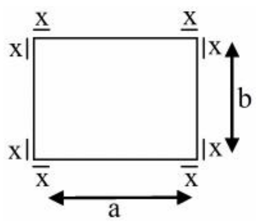
$$= S_c = 2\pi RH$$

$$= 2\pi\tan\alpha\left(hH - H^2\right)$$

$$\therefore \quad \frac{dS_c}{dH} = 2\pi\tan\alpha(h-2H) = 0$$

$$\therefore \quad H = \frac{h}{2}.$$

36.



$$V = (a-2x)(b-2x)x$$

$$V = 4x^3 - 2(a+b)x^2 + abx$$

$$\therefore \quad \frac{dV}{dx} = 12x^2 - 4(a+b)x + ab = 0$$

$\therefore \quad x = \dfrac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$

$= \dfrac{(a+b) \pm \sqrt{a^2 + b^2 - ab}}{6}$

But $\quad x < a\ , x < b$

$\therefore \quad x = \dfrac{(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}}{6} = \dfrac{1}{6}\left[(a+b) - (a^2 + b^2 - ab)^{\frac{1}{2}}\right]$

37. $v = x(a-2x)^2$

$\therefore \quad v = 4x^3 - 4ax^2 + a^2x$

$\therefore \quad \dfrac{dv}{dx} = 12x^2 - 8ax + a^2 = 0 \qquad \therefore \quad x = \dfrac{a}{2} \quad \text{or} \quad x = \dfrac{a}{6}$

But $x = \dfrac{a}{2}$ will make volume zero.

$\therefore \quad x = \dfrac{a}{6}$

38. $a^2h = 32$

$a^2 + 4ah$ has to be minimised

$\therefore \quad h = \dfrac{32}{a^2}$

$\therefore \quad f(a) = a^2 + \dfrac{128}{a}$

$\therefore \quad f'(a) = 2a - \dfrac{128}{a^2} = 0$

$\therefore \quad a = 4 \ \& \ h = 2$

$\therefore \quad \text{Area} = 16 + 32 = 48$

39. Line is $(y-4) = m(x-3)$

$x = 0 \quad \Rightarrow \quad y = 4 - 3m$

$y = 0 \quad \Rightarrow \quad x = 3 - \dfrac{4}{m}$

$\Delta = \dfrac{1}{2}(4-3m)\left(3 - \dfrac{4}{m}\right)$

$= \dfrac{1}{2}\left(24 - 9m + \dfrac{16}{m}\right)$

$\therefore \quad \dfrac{d\Delta}{dm} = \dfrac{-9}{2} + \dfrac{16}{2m^2} = 0$

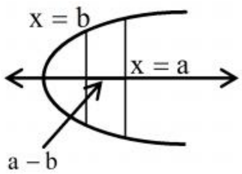
$\therefore \quad \frac{8}{m^2} = \frac{9}{2}$

$\therefore \quad m^2 = \frac{16}{9}$

$\therefore \quad m = \frac{-4}{3}$ as $m > 0 \quad \Rightarrow$ no Δ is formed

$\therefore \quad \Delta = \frac{1}{2}(8)(6) = 24$

40.



$x = b \quad \Rightarrow \quad y^2 = 4ab \qquad y = \pm\sqrt{4ab} = \pm 2\sqrt{ab}$

$\therefore \quad |2y| = \pm 4\sqrt{ab}$

$A = \frac{1}{2}(a-b)\left(4a + 4^2\sqrt{ab}\right)$

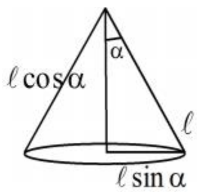
$= 2a^2 + 2a^{\frac{3}{2}}b^{\frac{1}{2}} - 2ab - 2a^{\frac{1}{2}}b^{\frac{3}{2}}$

$\frac{dA}{sb} = \frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}} - 2a - 3a^{\frac{1}{2}}b^{\frac{1}{2}} = 0$

$\therefore \quad -3a^{\frac{1}{2}}b - 2ab^{\frac{1}{2}} + a^{\frac{3}{2}} = 0$

$\Rightarrow \quad b = \frac{a}{9}$

41. $\therefore \quad V = \frac{1}{3}\pi\,\ell^3 \sin^2\alpha\cos\alpha$



$\frac{dV}{d\alpha} = \frac{1}{3}\pi\ell^2\left[2\sin\alpha\cos^2\alpha - \sin^3\alpha\right] = 0$

$\therefore \quad \sin\alpha = 0$ or $2\cos^2\alpha = \sin^2\alpha$
Rejected

$\therefore \quad \tan\alpha = \sqrt{2}$ as $\alpha \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \alpha = \tan^{-1}\left(\sqrt{2}\right)$

42. $V = \frac{1}{3}\pi r^2 h = \text{constant}$

$S_c = \pi r\sqrt{r^2 + h^2}$ has to be maximized

$\therefore \quad h = \dfrac{3V}{\pi r^2}$

$\therefore \quad S_c = \pi r^2 \sqrt{r^2 + \dfrac{9V^2}{\pi^2 r^4}}$

$$= \sqrt{\pi^2 r^4 + \frac{9V^2}{r^2}}$$

$\dfrac{dS_c}{dr} = \dfrac{dS_c^2}{dr} = 0$

$\therefore \quad 4\pi^2 r^3 - \dfrac{18V^2}{r^3} = 0$

$\therefore \quad 4\pi^2 r^6 = 18V^2$

$\therefore \quad r^6 = \dfrac{9V^2}{2\pi^2}$

$\therefore \quad r = \dfrac{3^{\frac{1}{3}}}{2^{\frac{1}{6}}} \; \dfrac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$

$$r^2 = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{3}}$$

$h = \dfrac{3V}{\pi r^2} = \dfrac{3V}{\pi} \times \left(\dfrac{2\pi^2}{9V^2}\right)^{\frac{1}{3}}$

$$= \frac{3V}{\pi} \times \frac{2^{\frac{1}{3}}\pi^{\frac{2}{3}}}{9^{\frac{1}{3}}V^{\frac{2}{3}}} = \frac{\left(3^{1/3}\right)\left(2^{1/3}\right)V^{1/3}}{\pi^{1/3}}$$

$\dfrac{h}{r} = 2^{1/3} \times 2^{1/6} = \sqrt{2}$

43. $\ell^2 = h^2 + r^2$

$r = \sqrt{\ell^2 - h^2}$

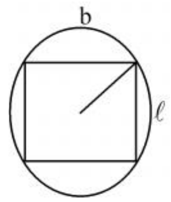
$V = \dfrac{1}{3} r^2 h = \dfrac{1}{3}\left(\ell^2 - h^2\right)h$

$$= \frac{\ell^2 h}{3} - \frac{h^3}{3}$$

$\therefore \quad \dfrac{dV}{dh} = \dfrac{\ell^2}{3} - h^2 = 0$

$\therefore \quad h = \frac{\ell}{\sqrt{3}}$

44.



$b^2 + \ell^2 = 4r^2$

$b = \sqrt{4r^2 - \ell^2}$

$S = kb\ell^3$

$= k\ell^3 \sqrt{4r^2 - \ell^2}$

$\therefore \quad \frac{dS}{d\ell} = 3k\ell^2\sqrt{4r^2 - \ell^2} - \frac{k\ell^4}{\sqrt{4r^2 - \ell^2}} = 0$

$\therefore \quad 3k\ell^2\left(4r^2 - \ell^2\right) = k\ell^4$

$\therefore \quad \ell = 0$ Rejected or

$3\left(4r^2 - \ell^2\right) = \ell^2$

$\therefore \quad 12r^2 = 4\ell^2$

$\therefore \quad \ell = \sqrt{3}r$

$\therefore \quad b = r$

45. $b^2 + d^2 = 4r^2$

$d^2 = 4r^2 - b^2$

$\therefore \quad S = k\,b\,d^2$

$= kb\left(4r^2 - b^2\right) = 4kbr^2 - kb^3$

$\therefore \quad \frac{dS}{dr} = 4kr^2 - 3kb^2 = 0$

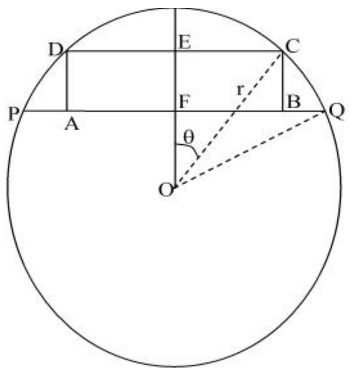
$\therefore \quad b = \frac{2r}{\sqrt{3}}$

$\therefore \quad d^2 = 4r^2 - \frac{4r^2}{3} = \frac{8r^2}{3}$

$\Rightarrow \quad d = \frac{2\sqrt{2}\,r}{3}$

$\therefore \quad d = \sqrt{2}\,b = 2\sqrt{\frac{2}{3}}r$

46. Let OABC be the sheet of paper as



The corner A of the rectangular sheet OABC is folded over along PQ so as to reach the opposite edge OC at R.
Let the crease PQ be of length x.
Let $\angle APQ = \theta$. Then $\angle PQR = \theta$ and $\angle OPR = \pi - 2\theta$.
In ΔAPQ , we have

$$\cos\theta = \frac{AP}{PQ}$$

$\Rightarrow \quad AP = x\cos\theta$
In ΔOPR, we have

$$\cos(\pi - 2\theta) = \frac{OP}{RP}$$

$\Rightarrow \quad -\cos 2\theta = \frac{OP}{AP} \qquad [\because \ AP = RP]$

$\Rightarrow \quad OP = -AP\cos 2\theta = -x\cos\theta\cos 2\theta$
Now,

$a = OA = OP + AP$

$\Rightarrow \quad a = x\cos\theta - x\cos 2\theta\cos\theta$

$\Rightarrow \quad x = \frac{a}{\cos\theta - \cos\theta\cos 2\theta} \qquad \text{...(i)}$

$\Rightarrow \quad \frac{a}{x} = \cos\theta - \cos\theta\cos 2\theta$

Let $y = \frac{a}{x}\cdot$ Then y is maximum when x is minimum.
Now,
$y = \cos\theta - \cos\theta\cos 2\theta$

$\Rightarrow \quad \frac{dy}{d\theta} = -\sin\theta + \sin\theta\cos 2\theta + 2\cos\theta\sin 2\theta$

For maximum or minimum values of y we must have $\frac{dy}{d\theta} = 0$

$\Rightarrow \quad -\sin\theta + \sin\theta\cos 2\theta + 4\sin\theta\cos^2\theta =$

$\Rightarrow \quad -\sin\theta(1 - \cos 2\theta) + 4\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow \quad -2\sin^3\theta + 4\sin\theta - 4\sin^3\theta = 0$

$\Rightarrow \quad 4\sin\theta = 6\sin^3\theta$

$\Rightarrow \quad \sin^2\theta = \frac{2}{3}$ or $\sin\theta = 0$

$\Rightarrow \quad \sin\theta = \sqrt{\frac{2}{3}}$ or $\theta = 0.$

Now,

$$\frac{d^2y}{d\theta^2} = -\cos\theta + \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta + 4\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta$$

$$\Rightarrow \quad \frac{d^2y}{d\theta^2} = -\cos\theta + 5\cos\theta\cos 2\theta - 4\sin\theta\sin 2\theta$$

For $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{2}{3}}$, we have

$$\frac{d^2y}{d\theta^2} = -\frac{1}{\sqrt{3}} + 5\times\sqrt{\frac{2}{3}}\times\left(\frac{2}{3}-1\right) - 4\times\sqrt{\frac{2}{3}}\times 2\sqrt{\frac{2}{3}}\times\sqrt{\frac{1}{3}} < 0.$$

So, y is maximum when $\sin\theta = \sqrt{\frac{2}{3}}$

Hence, x is minimum when $\sin\theta = \sqrt{\frac{2}{3}}$

Putting $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \sqrt{\frac{1}{3}}$ in (i), we get

Length of the crease $= x = \dfrac{a}{\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\left(1 - 2\times\frac{2}{3}\right)} = \dfrac{3\sqrt{3}a}{4}$

47. Let speed of boat be v & walking speed be v sec α

$$\therefore \quad t = \frac{\sqrt{a^2 + (b-x)^2}}{v} + \frac{x\cos\alpha}{v}$$

$$= \frac{\sqrt{a^2 + (b-x)^2} + x\cos\alpha}{v}$$

$$\therefore \quad v\frac{dt}{dx} = \cos\alpha + \frac{1}{2\sqrt{a^2 + (b-x)^2}} \times -2(b-x) = 0$$

$$\therefore \quad \cos\alpha = \frac{b-x}{\sqrt{a^2 + (b-x)^2}}$$

$$\therefore \quad (b-x)^2 = \cos^2\alpha\left(a^2 + b^2 - 2bx + x^2\right)$$

$\therefore \quad (b-x)^2 = a^2 \cot^2 \alpha$

$\therefore \quad x = b - a\cot\alpha = \dfrac{b\sin\alpha - a\cos\alpha}{\sin\alpha}$

48. $\therefore \quad T = \dfrac{\sqrt{d^2+x^2}}{u} + \dfrac{1-x}{v}$

$\therefore \quad \dfrac{dT}{dx} = \dfrac{x}{u\sqrt{d^2+x^2}} - \dfrac{1}{v} = 0$

$\therefore \quad xv = u\sqrt{d^2+x^2}$

$\therefore \quad x^2\left(v^2-u^2\right) = u^2d^2$

$\therefore \quad x = \dfrac{ud}{\sqrt{v^2-u^2}}$

For students. [Think for solution if $u > v$]

49. $2\ell + 2\pi r = 440$

$\therefore \quad \ell + \pi r = 220$ & $\ell = 220 - \pi r$

$A = 2\left(220r - \pi r^2\right) \quad = 2\ell r$

$\dfrac{dA}{dr} = 2\left(220 - 2\pi r\right) = 0$

$\therefore \quad r = 35 \text{ ft}$

$\Rightarrow \quad 2r = 70\text{ft}$ & $\ell = 110\text{ft}$

50. $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_nx^{2n}$

$0 < a_1 < a_2 < \ldots\ldots < a_n$

$\therefore \quad P'(x) = 2na_nx^{2n-1} + \ldots. + 4a_2x^3 + 2a_1x$

$= 0$ only at $x = 0$ &

$P''(x) > 0 \quad \forall \; x \in R$

$\therefore \quad P(x)$ has only one minimum.

51. $x = a\sec\theta, y = b\,\text{cosec}\,\theta$

Minimum radius vector = ?

$r = \sqrt{x^2+y^2} = \sqrt{a^2\sec^2\theta + b^2\,\text{cosec}^2\,\theta}$

From (Q.4),

Minimum value of $r = \sqrt{(a+b)^2} = a + b$

52. From (Q.18)

$s = 2\pi r(r+h)$

$$= 2\pi r\left(r + \frac{v}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2v}{r}$$

$$\frac{ds}{dr} = 0 \qquad \therefore \qquad 4\pi r - \frac{2v}{r^2} = 0$$

$$\therefore \qquad r^3 = \frac{v}{2\pi} \qquad \therefore \qquad r = \left(\frac{v}{2\pi}\right)^{\frac{1}{3}}$$

$$\pi r^2 = \left(\frac{\pi v^2}{4/3}\right)^{\frac{1}{3}}$$

$$\frac{v}{\pi r^2} = \left(\frac{4v}{\pi}\right) = h$$

$h = 2r$, form this statement $h : r = 2 : 1$

53. $f(x) = (x-1)^p (x-2)^q$

$$\therefore \qquad f'(x) = p(x-1)^{p-1}(x-2)^q + q(x-1)^p (x-2)^{q-1}$$

$$f''(x) = p(p-1)(x-1)^{p-2}(x-2)^q + 2pq(x-1)^{p-1}(x-2)^{q-1} + q(q-1)(x-1)^p (x-2)^{q-2}$$

If we go on taking derivatives, we find that the condition given in the question holds when (even)[th] derivative is non-zero for it, p & q should be even.

54. $f(x) = xe^x$

$$f'(x) = xe^x + e^x = 0$$

$$\therefore \qquad x = -1$$

$f''(x) = xe^x + 2e^x > 0$ for $x = -1$

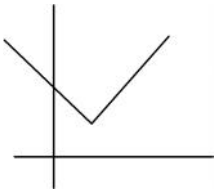
$\therefore \qquad x = -1$ is a minimum

55. Time required $= T = \left(\frac{N}{x}\right)(\alpha + \beta x^2)$

$$\therefore \qquad T = N\left(\frac{\alpha}{x} + \beta x\right)$$

$$\therefore \qquad \frac{dT}{dx} = N\left(\beta - \frac{\alpha}{x^2}\right) = 0 \qquad \therefore \qquad x = \sqrt{\frac{\alpha}{\beta}}$$

56. $f(x) = \max\{x, x+1, 2-x\}$

By graph, $f(x) =$ 

$$f(x) = 2 - x, x \le +\frac{1}{2}$$

$$x + 1, x > \frac{1}{2}$$

$\therefore$ $x = \frac{1}{2}$ is point of minima and minimum value is $\frac{3}{2}$.

57. $f(\alpha) = \left(1 + \frac{1}{\sin^n x}\right)\left(1 + \frac{1}{\cos^n \alpha}\right)$

$$= \left(1 + \sec^n \alpha\right)\left(1 + \operatorname{cosec}^n \alpha\right)$$

$$= 1 + \sec^n \alpha + \operatorname{cosec}^n \alpha + \sec^n \alpha \operatorname{cosec}^n \alpha$$

$\therefore$ $f'(\alpha) = n \sec^n \alpha \tan \alpha - n \operatorname{cosec}^n \alpha \cot \alpha$

$$+n \sec^n \alpha \operatorname{cosec}^n \alpha (\tan \alpha - \cot \alpha) = 0$$

$\therefore$ $\sec^n \alpha \tan \alpha \left(1 + \operatorname{cosec}^n \alpha\right) = \operatorname{cosec}^n \alpha \cot \alpha \left(1 + \sec^n \alpha\right)$

$\therefore$ $\frac{(\sec^n \alpha)(\sec^2 \alpha - 1)}{1 + \sec^n \alpha} = \frac{\operatorname{cosec}^n \alpha}{1 + \operatorname{cosec}^n \alpha}$

$\therefore$ $\frac{(\cos^n \alpha)(\sin^2 \alpha)}{(\cos^n \alpha + 1)(\cos^2 \alpha)} = \frac{\sin^n \alpha}{1 + \sin^n \alpha}$

$\therefore$ $\frac{\sin^{n-2} \alpha}{1 + \sin^n \alpha} = \frac{\cos^{n-2} \alpha}{1 + \cos^n \alpha}$

$\Rightarrow$ $\sin \alpha = \cos \alpha$

For minima,

$$\sin \alpha = \cos \alpha = \frac{+1}{\sqrt{2}}$$

$\therefore$ Minimum value $= \left(1 + 2^{n/2}\right)^2$

58. $f(x) + f\left(\frac{1}{x}\right) = \frac{1}{x}$

$\therefore$ $f(x) + f\left(\frac{1}{x}\right) = x$

$\Rightarrow$ $x = \frac{1}{x}$

$\Rightarrow \quad x = \pm 1$ Only

Here, $x = -1 \quad \Rightarrow \quad f(x) = -\frac{1}{2}$ and

$x = +1 \quad \Rightarrow \quad f(x) = +\frac{1}{2}$

$\therefore \quad f(x)$ has maximum value $\frac{1}{2}$.

59. $f(x) = \cos 2\pi x + \{x\}$

At non-integral points,

$f'(x) = -2\pi \sin 2\pi x + 1$

It tends to achieving maximum values at points infinitesimally close to and less than integers but it has a discontinuity.

$\therefore$ It has no maxima.

$f(x) = x - x^2$

$x_1 \& x_2 \in \quad y = x - x^2 \quad$ in $(0,1)$

maximum value of expression

$= \max\left(x - x^2\right) = \frac{1}{4}$

61. $f(x) = x^2, \qquad x \in [-2,-1] \cup [1,2]$

$2 - x^2, \qquad x \in (-1,1)$

$\therefore$ Function has maximum at $x = 0$ & local as well as global minima at $x = \pm 1$

62. $x^3 - ax^2 + bx - 6 = 0$ has roots real and positive

$\therefore \quad \alpha\beta\gamma = 6, \alpha + \beta + \gamma = a, \alpha\beta + \beta\gamma + \gamma\alpha = b$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{b}{6}$

Now, sum is minimum when each of them is equal

$$\frac{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}}{3} \geq \left(\frac{1}{\alpha\beta\gamma}\right)^{\frac{1}{3}}$$ [AM-GM inequality]

$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \geq \frac{3}{6^{1/3}} \qquad \therefore \quad b \geq \frac{3 \times 6}{6^{1/3}} = 3(36)^{1/3}$

63. $f'(x) = \frac{2}{3(6-x)^{\frac{1}{3}}}$

Which is not diff. at $x = 6$

$\therefore$ Theorems are not applicable.

64. By definition.

65. $f(0) = -6$, $f(4) = +6$

$\therefore$ $f'(x) = (x-2)(x-3) + (x-1)(x-2) + (x-1)(x-3)$

$f'(c) = \frac{6+6}{4-0} = 3$

$\therefore$ $3x^2 - 12x + 11 = 3$ and $x = c$

$\therefore$ $3c^2 - 12c + 8 = 0$

$c = 12 \pm \frac{\sqrt{144-96}}{6}$

$= 12 \pm \frac{\sqrt{48}}{6} = 6 \pm \frac{2\sqrt{3}}{3} = 2 \pm \frac{2}{\sqrt{3}}$

66. $f(x) = x^{\alpha} \log x$

$f'(x) = x^{\alpha-1}(1 + \alpha \log x) = 0$

$c = e^{-1/\alpha} \in (0,1)$

$\therefore$ $\alpha > 0$

67. $a + b + c = 0$

$f(x) = ax^3 + bx^2 + cx$

Has roots 0 & 1

$\therefore$ $3ax^2 + 2bx + c = 0$

has at least one root in $(0,1)$.

68. $f'(c) = \frac{13-5}{2} = 4$

69. Refer (Q.28) (above)

$a + b + c = 0$

$f(x) = ax^3 + bx^2 + cx$

Has roots 0 & 1

$\therefore$ $3ax^2 + 2bx + c = 0$

70. $x^3 - 3x + a = 0$ has two roots in $[0,1]$

$f'(x) = 3x^2 - 3 \neq 0$ in $(0,1)$

$\therefore$ There is no value of a satisfying the conditions.