EXERCISE – 1(B)

1. (B)

$\Rightarrow \cos^2 h\theta - \sin^2 h\theta = 1$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

$\Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$

2. (D)

We know $\cos^2 h\theta - \sin^2 h\theta = 1$

$\Rightarrow \cos h\,\theta = \frac{x+y}{a} \,\&\, \sin h\,\theta = \frac{x-y}{a}$

$\Rightarrow \cos^2 h\theta - \sin^2 h\theta = 1$

$\Rightarrow (x+y)^2 - (x-y)^2 = a^2$

$\Rightarrow xy = \frac{a^2}{4}$

3. (A)

Let mid – point is (h, k). Equation of chord is T = Q!. So $xh - yk - 4 = h^2 - k^2 - 4$. Comparing with $x + 2y + 3 = 0$

$\Rightarrow \frac{h}{1} = \frac{-k}{2} = \frac{k^2 - h^2}{3}$

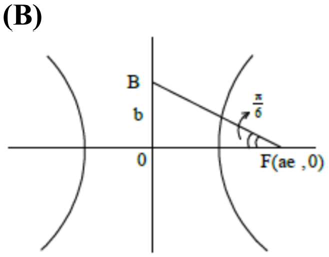
On solving h = 1, k = –2

4. (A)

Let P is (h, k) equation of chord with P as middle points is T = S'. Slope obtained is $\frac{3h+2}{2k+3}$ which is equal to 2.

So, 3h – 4k = 4

5. (B)



$\Rightarrow \frac{\pi}{6} = \frac{b}{ae} \,\&\, b^2 = a^2\left(e^2 - 1\right)$

On solving we get $e = \sqrt{\frac{3}{2}}$

6. (C)

Let midpoint of a chord be P(h, k) then by 'T = S_1' its equal will be

$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

As it passes through (α,β) hence

$$\frac{\alpha h}{a^2}-\frac{\beta k}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}$$

Required locus is

$$\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}=\frac{x^2}{a^2}-\frac{y^2}{b^2} \text{ or } \frac{\left(x-\frac{\alpha}{2}\right)^2}{a^2}-\frac{\left(y-\frac{\beta}{2}\right)^2}{b^2}=\frac{\alpha^2}{4a^2}-\frac{\beta^2}{4b^2}$$

Which is hyperbola having center at $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$.

7. **(A)**

Area of triangle formed by any tangent and the asymptotes is (ab)

Now $ab=a^2\tan\lambda \Rightarrow \tan\lambda=\frac{b}{a}$

Hence $e=\sqrt{1+\frac{b^2}{a^2}} \Rightarrow e=\sqrt{1+\tan^2\lambda}$ or $e=|\sec\lambda|$

8. **(D)**

Locus of feet of perpendicular from foci on any tangent is the auxiliary circle.

Hence required locus is $x^2+y^2=\frac{1}{16}$

9. **(D)**

Let A (1, –1) & B (2, 1) be two fixed points and P (x, y) be a moving point, then

$|Z-1+i|-|Z-2-i|=3 \Rightarrow PA-PB=3$

Hence locus will be no real curve as $AB=\sqrt{5}<3$

10. **(B)**

Eccentricity of $\frac{x^2}{5}-\frac{y^2}{5\cos^2\alpha}=1, e_1=\sqrt{1+\cos^2\alpha}$

Eccentricity of $\frac{x^2}{25\cos^2\alpha}-\frac{y^2}{25}=1, e_2=\sqrt{1-\cos^2\alpha}$

Given $e_1=\sqrt{3}e_2$ hence

$1+\cos^2\alpha=3\sin^2\alpha$ or $\sin^2\alpha=\frac{1}{2}$

A value of α is $\frac{\pi}{4}$

11. **(B)**

Equation of tangents with slope m to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=mx\pm\sqrt{a^2m^2-b^2}$

Slope of tangent perpendicular to y = x is –1

Hence equation of tangents with slope –1 to $\frac{x^2}{18}-\frac{y^2}{9}=1$ are $x+y=\pm 3$.

12. **(B)**

For the hyperbola $\frac{x^2}{3}-y^2=1,\left(\sqrt{3},0\right)$ is one the vertices, hence tangent at this point will be equally inclined to the asymptotes.

Also the asymptotes are $y=\frac{1}{\sqrt{3}}x$ & $y=-\frac{1}{\sqrt{3}}x$ hence angle between the asymptotes is 60^0.

The tangent and the asymptotes must from an equilateral triangle.

13. (D)

Any tangent to $xy = c^2$ is $x + t^2y = 2ct$

Now foot of perpendicular on this tangent from (0, 0) will be given by

$\frac{x-0}{1}=\frac{y-0}{t^2}=-\frac{0+0-2ct}{1+t^2}$ or $x=\frac{2ct}{1+t^4}$ & $y=\frac{2ct^3}{1+t^4}$

Eliminating t between x & y gives the required locus as $\left(x^2+y^2\right)^2=4c^2xy$

14. (C)

Standard result in geometrical properties.

15. (B)

Equation of asymptotes $bx - ay = 0$ & $x + ay = 0$

Any point P on hyperbola (a sec t, b tan t).

Product of perpendicular from P on asymptotes $\left|\frac{ab\left(\sec t-\tan t\right)}{\sqrt{a^2+b^2}}\right|\times\left|\frac{ab\left(\sec t+\tan t\right)}{\sqrt{a^2+b^2}}\right|$

i.e., $\frac{a^2b^2}{a^2+b^2}=6$, but given $e^2=\frac{a^2+b^2}{a^2}=3$, hence $b^2 = 18$ & $a^2 = 9$.

16. (B)

For a rectangular hyperbola, eccentricity is $\sqrt{2}$ & independent of ‘c’.

Hence $e_1+e_2=\sqrt{2}+\sqrt{2}$ i.e., $2\sqrt{2}$

17. (A)

$\sqrt{3}x-y-4\sqrt{3}t=0\Rightarrow t=\frac{\sqrt{3}x-y}{4\sqrt{3}}$

Now $\sqrt{3}\,tx+ty-4\sqrt{3}=0\Rightarrow\left(\sqrt{3}x-y\right)\left(\sqrt{3}x+y\right)=48$ or $\frac{x^2}{16}-\frac{y^2}{48}=1$

Hence $e^2=\frac{a^2+b^2}{a^2}\Rightarrow e^2=\frac{16+48}{16}$ i.e. $e = 2$.

18. (C)

Any tangent of $y^2 = 8x$: $y=mx+\frac{2}{m}$

If this is a tangent to $xy=-1$ as well then $x\left(mx+\frac{2}{m}\right)=-1$ must have real & equal roots.

Now discriminant of $m^2x^2+2x+m=0, 4-4m^3=0\Rightarrow m=1$.

19. (C)

Tangent to $xy = c^2$ at P(h, k) : $kx + hy = 2c^2$.

x – intercept, $x_1 = \frac{2c^2}{k}$, y – intercept, $y_1 = \frac{2c^2}{h}$.

Normal to $xy = c^2$ at P(h, k) : $hx - ky = h^2 - k^2$.

x – intercept, $x_2 = \frac{h^2 - k^2}{h}$, y – intercept, $y_2 = \frac{k^2 - h^2}{k}$.

Clearly $\frac{x_2}{y_1} + \frac{y_2}{x_1} = 0$ or $x_1x_2 + y_1y_2 = 0$

20. **(B)**

$xy = hx + ky \Rightarrow (x - k)(y - h) = hk$

Hence the center is (k, h).

21. **(C)**

22. **(D)**

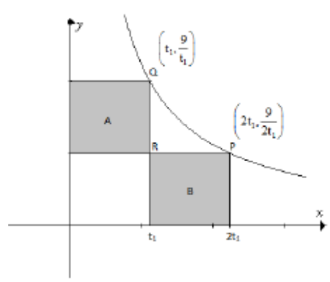
23. **(A)**

Coordinates of R : $\left(t_1, \frac{9}{2t_1}\right)$

Area $A = \left(\frac{9}{2t_1} - \frac{9}{t_1}\right) \times t_1$ i.e. $\frac{9}{2}$ &

Area $B = \frac{9}{2t_1}(2t_1 - t_1)$ i.e. $\frac{9}{2}$

Hence A = B



24. **(A)**

Polar of a pole is chord of contact from the given point. Let pole is (h, k) equation of polar is T = 0

$\Rightarrow 3hx - 5ky - 15 = 0$

Comparing with 2x +5y – 5 = 0 we get,

$\Rightarrow \frac{3h}{2} = \frac{-5k}{5} = \frac{-15}{-5}$

$\Rightarrow$ h = 2, k = –3

25. **(D)**

Let pole of 3x – y + 1 = 0 is (h, k) on comparing it with (5h) x – (6k) y – 15 = 0.

We get, $\Rightarrow \frac{5h}{3} = \frac{6k}{1} = -15$

$\Rightarrow h = -9; k = \frac{-5}{2}$

This $\left(-9, \frac{-5}{2}\right)$ satisfies 2x – ky + 3 = 0

So, $2(-9) - k\left(\frac{-5}{2}\right) + 3 = 0$

$\Rightarrow$ k = 6

26. **(A)**

Let mid points is (h, k). Equation of chord is T = Q !.

So $xh - yk - 4 = h^2 - k^2 - 4$.
Comparing with $x + 2y + 3 = 0$

$\Rightarrow \frac{h}{1} = \frac{-k}{2} = \frac{k^2 - h^2}{3}$

On solving $h = 1, k = -2$

27. **(B)**
Centre of hyperbola is

$\Rightarrow \frac{\delta s}{\delta y} = 0 \Rightarrow 6x - 5y + 17 = 0$

$\Rightarrow \frac{\delta s}{\delta y} = 0 \Rightarrow -5x - 4y + 1 = 0$

On solving : we get centre $\left(\frac{-9}{7}, \frac{13}{7}\right)$

Equation of asymptotes(s) differ from that of hyperbola by a constant. Let the Asymptotes(s) are $3x^2 - 5xy - 2y^2 + 17x + y + \lambda = 0$. It satisfies $\left(\frac{-9}{7}, \frac{13}{7}\right)$

On solving we get $\lambda = 10$

28. **(D)**
Let other is $2x - y + \lambda = 0$ (Equation of hyperbola & asymptote differ by a constant)
So, $(2x - y + \lambda)(x + 2y - 3) = 2x^2 + 3xy - 2y^2 - 7x + y + \lambda$
Compare co – efficient of $x \Rightarrow \lambda - 6 = -7 \Rightarrow \lambda = -1$
So equation is $2x - y - 1 = 0$

29. **(B)**
Let other is $x - y + \lambda = 0$
So, $(x + y + 1)(x - y + \lambda) = x^2 - y^2 + x - y + \lambda$
Comparing coefficient of $y \Rightarrow \lambda - 1 = -1 \Rightarrow \lambda = 0$
So equation is $x - y = 0$

30. **(D)**

31. **(D)**

$\Rightarrow y = mx + \frac{2}{m}; y = mx + \sqrt{m^2 - 3}$

Comparing $\frac{4}{m^2} = m^2 - 3 \Rightarrow m = \pm 2$

So tangents are $2x - y + 1 = 0$ & $2x + y + 1 = 0$

32. **(B)**
A point (a, b) can be taken on $x^2 - y^2 = a^2 - b^2$. A tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with slope 'm' is $y = mx \pm \sqrt{a^2m^2 + b^2}$. If it passes through (a, b) then we have $b = am \pm \sqrt{a^2m^2 + b^2}$
$\Rightarrow 2amb = 0; m = 0$

33. **(A)**
Any tangent to $x^2 + y^2 = a^2$ is $x\cos\theta + y\sin\theta = a$

Let P is its pole w.r.t. $x^2 - y^2 = a^2$
So comparing $xh - yk = a^2$
$\Rightarrow h = a\cos\theta, k = -a\sin\theta \Rightarrow x^2 + y^2 = a^2$

34. **(B)**
Tangent to $4x^2 - 3y^2 = a^2$ is $(2\sec\theta)x - y(\sqrt{3}\tan\theta) = a$
Let P(h, k) is its pole w.r.t. $y^2 = 4ax$
So polar is yk = 2x (x + h)
On comparing we get $h = \frac{-a\cos\theta}{2}$ and $k = a\sqrt{3}\sin\theta$
So we have $2h^2 + k^2 = 3a^2$

35. **(B)**
Given hyperbola are $\frac{x^2}{9} - \frac{y^2}{16} = 1$ … (i) and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ …(ii)
Any tangent to (i) having slope m is $y = mx \pm \sqrt{9m^2 - 16}$ … (iii)
Putting in (ii), we ger $16\left[mx \pm \sqrt{9m^2 - 16}\right]^2 - 9x^2 = 144$
$(16m^2 - 9)x^2 \pm 32m(\sqrt{9m^2 - 16})x + 144m^2 - 256 - 144 = 0$
$\Rightarrow (16m^2 - 9)x^2 \pm 32m(\sqrt{9m^2 - 16})x + (144m^2 - 400) = 0$ …(iv)
If (iii) is a tangent to (ii), then the roots of (iv) are real and equal.
$\therefore$ Discriminant = 0; $32 \times 32m^2(9m^2 - 16) = 0(16m^2 - 9)(144m^2 - 400) = 64(16m^2 - 9)(9m^2 - 25)$
$16m^2(9m^2 - 16) = (16m^2 - 9)(9m^2 - 25) \Rightarrow 144m^4 - 256m^2 = 144m^4 - 481m^2 + 225$
$\Rightarrow 225m^2 = 225 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$

36. **(A)**
Let the point of intersection of tangent be $P(x_1, y_1)$.
Then the equation of pair of tangents from $P(x_1, y_1)$ to the given hyperbola is
$(4x^2 - 9y^2 - 36)(4x_1^2 - 9y_1^2 - 36) = [4x_1x - 9y_1y - 36]^2$ …(i)
From $SS_1 = T^2$ or $x^2(y_1^2 + 4) + 2x_1y_1xy + y^2(x_1^2 - 9) + ... = 0$ …(ii)
Since angle between the tangents is $\pi/4$

$$\therefore \tan(\pi/4) = \frac{2\sqrt{\left[x_1^2y_1^2 - (y_1^2 + 4)(x_1^2 - 9)\right]}}{y_1^2 + 4 + x_1^2 - 9}$$

Hence locus of $P(x_1, y_1)$ is $(x^2 + y^2 - 5)^2 = 4(9y^2 - 4x^2 + 36)$

37. **(A)**
The equation of normal at $(a\sec\phi, b\tan\phi)$ to the given hyperbola is $ax\cos\phi + by\cot\phi = (a^2 + b^2)$
This meet the transverse axis i.e., x – axis at G.
So the co – ordinates of the vertices A and A' are A(a, 0) and $A'(-a, 0)$ respectively.

$$\therefore AG.A'G = \left(-a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)\left(a + \left(\frac{a^2+b^2}{a}\right)\sec\phi\right)$$

$$= \left(\frac{a^2+b^2}{a}\right)\sec^2\phi - a^2 = \left(ae^2\right)^2\sec^2\phi - a^2 = a^2\left(e^4\sec^2\phi - 1\right)$$

38. **(A)**

Let (x_1, y_1) be the required point.

Then the equation of the chord of contact of tangents drawn from (x_1,y_1) to the given hyperbola is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ …(i)

The given line is $lx + my + n = 0$ …(ii)

Equation (i) and (ii) represent the same line

$\therefore \frac{x_1}{a^2 l} = -\frac{y_2}{b^2 m} = \frac{1}{-h} \Rightarrow x_1 = \frac{-a^2 l}{n}, y_1 = \frac{b^2 m}{n};$

Hence the required point is $\left(-\frac{a^2 l}{n}, \frac{b^2 m}{n}\right)$

39. **(A)**

The given hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$ …(i)

Any tangent to (i) is $y = mx + \sqrt{16m^2 - 9}$ …(ii)

Let (x_1, y_1) be the midpoint of the chord of the circle $x^2 + y^2 = 16$

Then equation of the chord is $T = S_1$ i.e., $xx_1 + yy_1 - \left(x_1^2 + y_1^2\right) = 0$ …(iii)

Since (ii) and (iii) represents the same line.

$\therefore \frac{m}{x_1} = \frac{-1}{y_1} = \frac{\sqrt{16m^2-9}}{-\left(x_1^2+y_1^2\right)}$

$\Rightarrow m = -\frac{x_1}{y_1}$ and $\left(x_1^2 + y_1^2\right)^2 = y_1^2\left(16m^2 - 9\right)$

$\Rightarrow \left(x_1^2 + y_1^2\right)^2 = 16.\frac{x_1^2}{y_1^2}y_1^2 - 9y_1^2 = 16x_1^2 - 9y_1^2$

$\therefore$ Locus of $\left(x_1, y_1\right)$ is $\left(x^2 + y^2\right) = 16x^2 - 9y^2$

40. **(A)**

Let (x_1, y_1) be the given point.

Its polar w.r.t. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

i.e., $y = \frac{b^2}{y_1}\left(1 - \frac{xx_1}{a^2}\right) = -\frac{b^2 x_1}{a^2 y_1}x + \frac{b^2}{y_1}$

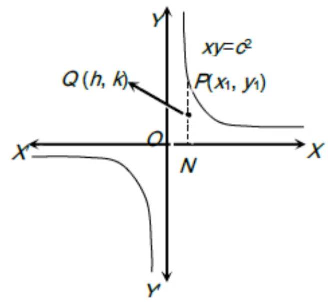
This touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\left(\frac{b^2}{y_1}\right) = a^2.\left(\frac{b^2 x_1}{a^2 y_1}\right) - b^2$

$\Rightarrow \frac{b^4}{y_1^2} = \frac{a^2 b^4 x_1^2}{a^4 y_1^2} - b^2 \Rightarrow \frac{b^2}{y_1^2} = \frac{b^2 x_1^2}{a^2 y_1^2} - 1 \Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

$\therefore$ Locus of (x_1, y_1) is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Which is the same hyperbola.

41. **(B)**
Coordinates of P and D are $(a\sec\phi, b\tan\phi)$ and $(a\tan\phi, b\sec\phi)$ respectively.
Then, $(CP)^2 - (CD)^2 = a^2\sec^2\phi + b^2\tan^2\phi - a^2\tan^2\phi - b^2\sec^2\phi$
$= a^2(\sec^2\phi - \tan^2\phi) - b^2(\sec^2\phi - \tan^2\phi)$
$= a^2(1) - b^2(1) = a^2 - b^2$

42. **(D)**
Let $xy = c^2$ be the rectangular hyperbola, and let $P(x_1, y_1)$ be a point on it. Let Q(h, k) be the mid – point of PN. Then the coordinates of Q are $\left(x_1, \frac{y_1}{2}\right)$



$\therefore x_1 = h$ and $\frac{y_1}{2} = k \Rightarrow x_1 = h$ and $y_1 = 2k$
But (x_1, y_1) lies on $xy = c^2$.
$\therefore h.(2k) = c^2 \Rightarrow hk \Rightarrow c^2/2$
Hence, the locus of (h, k) is $xy = c^2/2$, which is hyperbola

43. **(C)**
Let the hyperbola be $xy = c^2$.
Tangent at any point t is $x + yt^2 - 2ct = 0$
Putting y = 0 and then x = 0 intercept on the axes are $a_1 = 2ct$ and $b_1 = \frac{2c}{t}$
Normal is $xt^3 - yt - ct^4 + c = 0$
Intercepts as above are $a_2 = \frac{c(t^4-1)}{t^3}, b^2 = \frac{-c(t^4-1)}{t}$
$\therefore a_1a_2 + b_1b_2 = 2ct \times \frac{c(t^4-1)}{t^3} + \frac{2c}{t} \times \frac{-c(t^4-1)}{t} = \frac{2c^2}{t^2}(t^4-1) - \frac{2c^2}{t^2}(t^4-1) = 0$;
$\therefore a_1a_2 + b_1b_2 = 0$

44. **(B)**
Let t_1, t_2, t_3, t_4 be the parameter of the points P, Q, R and S respectively.
Then the coordinates of P, Q, R and S are
$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right)$ and $\left(ct_4, \frac{c}{t_4}\right)$ respectively

Now, $PQ \perp RS \Rightarrow \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1$

$\Rightarrow -\frac{1}{t_1t_2} \times -\frac{1}{t_3t_4} = -1 \Rightarrow t_1t_2t_3t_4 = -1$ …(i)

$\therefore$ Product of the slopes of CP, CQ, CR and CS

$$\frac{1}{t_1^2}\times\frac{1}{t_2^2}\times\frac{1}{t_3^2}\times\frac{1}{t_4^2}=\frac{1}{t_1^2t_2^2t_3^2t_4^2}=1 \qquad \text{[Using (i)]}$$

45. **(B)**

Let the equation of circle be $x^2+y^2=a^2$ … (i)

Parametric equation of rectangular hyperbola is $x=ct, y=\frac{c}{t}$

Put the values of x and y in equation (i) we get $c^2t^2+\frac{c^2}{t^2}=1$

EXERCISE – 2(B)

MULTIPLE CHOICE QUESTIONS

1. (AD)

Given $9(x^2 + 2x + 1) - 16(y^2 - 2y + 1) = 144$ or $\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1$

Now a = 4 , b = 3 & Center : (–1, 1)

$\frac{b^2}{a^2} = e^2 - 1 \Rightarrow \frac{9}{16} = e^2 - 1$ or $e = \frac{5}{4}$

$\therefore$ focus $(-1 \pm 5, 1)$

$\therefore$ (–4, 1) & (–6, 1)

2. (ABD)

$x^2 - y^2 = \cos^2\alpha$

Vertices $\equiv (\pm\cos\alpha, 0)$

Abscissae of foci $\equiv \pm\cos\alpha\sqrt{2} - 0$

$e = \sqrt{2}$

Equation of directrics : $x = \pm\frac{\cos\alpha}{\sqrt{2}}$

3. (BCD)

For hyperbola $2a = \frac{1}{2}$ & given Ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

As the curve are confocal hence $2.\frac{1}{4}.e = 2\sqrt{a^2 - b^2} = 2$

$\Rightarrow e = 4 \rightarrow (B)$

$\therefore b^2 = a^2(e^2 - 1) = \frac{1}{16}(16 - 1) = \frac{15}{16}$

Hyperbola: $\frac{x^2}{\frac{1}{16}} - \frac{y^2}{\frac{15}{16}} = 1$

$\Rightarrow \frac{x^2}{1} - \frac{y^2}{15} = \frac{1}{16}$

Distance between directrices $= \frac{2a}{3} = \frac{\frac{1}{2}}{4} = \frac{1}{8} \quad \rightarrow (C)$

L.R. $= \frac{2b^2}{a} = \frac{2.\frac{15}{16}}{\frac{1}{4}} = \frac{15}{2} \qquad \rightarrow (D)$

4. (AB)

$\left|\sqrt{x^2 + (y-1)^2} - \sqrt{x^2 + (y+1)^2}\right| = k$

Clearly, it is of the form

$|SP - S'P| = 2a$ where $2a < SS'$

$\Rightarrow$ k < distance between (0, 1), (0, –1)

$\Rightarrow$ k < 2

Obviously k > 0

$\Rightarrow$ exhaustive values of k are (0, 2)

5. **(AB)**

$(a\cos\theta, b\sin\theta), (a\cos\phi, b\sin\phi)$ & $(ae, 0)$ are collinear, hence

$$\begin{vmatrix} 1 & a\cos\theta & b\sin\theta \\ 1 & a\cos\phi & b\sin\phi \\ 1 & ae & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & \cos\theta & \sin\theta \\ 1 & \cos\phi & \sin\phi \\ 1 & e & 0 \end{vmatrix} = 0 \Rightarrow e\sin\phi - e\sin\theta + \sin(\theta-\phi) = 0 \Rightarrow \frac{1}{e} = \frac{\sin\theta - \sin\phi}{\sin(\theta-\phi)}$$

$$\Rightarrow \frac{2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}}{2\sin\frac{\theta-\phi}{2}\cos\frac{\theta-\phi}{2}} = \frac{1}{e} \Rightarrow \frac{\cos\frac{\theta-\phi}{2}}{\cos\frac{\theta+\phi}{2}} = e$$

$$\Rightarrow \frac{\cos\frac{\theta-\phi}{2} - \cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2} + \cos\frac{\theta+\phi}{2}} = \frac{e-1}{e+1} \Rightarrow \tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{e-1}{e+1} \qquad \ldots(B)$$

Similarly $(a\cos\theta, b\sin\theta), (a\cos\phi, b\sin\phi)$ & (–ae, 0) are collinear, hence

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{e+1}{e-1} \qquad \ldots(A)$$

6. **(AC)**

Slope of required tangent m = 3

$y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow y = 3x \pm \sqrt{1.9 - 3}$

$\Rightarrow y = 3x \pm \sqrt{6}$

7. **(CD)**

$y = mx \pm \sqrt{a^2m^2 - b^2}$

Let it pass through (h, k)

$\Rightarrow (k - mh)^2 = a^2m^2 - b^2$

$\Rightarrow m^2(h^2 - a^2) - (2hk)m + (k^2 + b^2) = 0$

Now $m_1m_2 = -1 \Rightarrow k^2 + b^2 = a^2 - b^2$

$\Rightarrow h^2 + k^2 = a^2 - b^2$

Will not have perpendicular tangent if $a^2 - b^2 < 0$ or $a^2 < b^2$

Now $e = \sqrt{1 + \frac{b^2}{a^2}} > \sqrt{2}$

8. **(AB)**

Let $y = mx + \frac{8}{m}$ be the tangent to $y^2 = 32x$ and

$y = mx \pm \sqrt{\frac{8}{9}}\sqrt{m^2 - 1}$ be that of $\frac{x^2}{\frac{8}{9}} - \frac{y^2}{\frac{8}{9}} = 1$

Comparing $\left(\frac{8}{m}\right)^2 = \frac{8}{9}\left(m^2 - 1\right)$

$\Rightarrow m^4 - m^2 - 72 = 0 \Rightarrow m = \pm 3$

Hence equation of common tangents are

$y = 3x + \frac{8}{3}$ & $y = -3x - \frac{8}{3}$ or $9x - 3y + 8 = 0$ & $9x + 3y + 8 = 0$

9. **(ABC)**

$xy = 2 \;\Rightarrow y = \frac{2}{x} \qquad \frac{dy}{dx} = \frac{-2}{x^2}$

Equation of Normal: $y - y_i = \frac{x_i^2}{2}\left(x - xi\right)$

$\Rightarrow 8x_i - 4 = 2x_i^3 - x_i^4$

$\Rightarrow x_i^4 - 3x_i^3 + 8x_i - 4 = 0$

Clearly $\sum x_i = 3$ & $\pi x_i = -4$

Replacing x_i with $\frac{2}{y_1} \Rightarrow \sum y_i = 4 \;\&\; \prod y_i = -4$

10. **(ACD)**

11. **(BC)**

Foci of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ are $\left(0, \pm 3\right)$

Also e of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is $\frac{3}{5} \Rightarrow \boxed{\frac{3}{5} \times e_{hyp} = 2}$

$\therefore$ e of hyperbola is $\frac{10}{3}$

Since hyperbola passes through foci and has axes along the coordinates axes hence let the hyperbola be $\frac{y^2}{9} - \frac{x^2}{a^2} = 1$

$\because e = \frac{10}{3} \qquad \Rightarrow a^2 = 91 \qquad$ … (B)

$L.R. = \frac{2.a^2}{3} = \frac{2.91}{3} = \frac{182}{3} \qquad$ …(C)

12. **(AB)**

Confocal ellipse and hyperbola are always orthogonal

Clearly in option (A) 31 + 41 = 91 – 19

And in option (B) 71 – 17 = 31 + 23

13. **(D)**

Let $\left(t_1, \frac{1}{t}\right)$ and $\left(t_2, \frac{1}{t_2}\right)$ be the points

Now m = 4 $\Rightarrow \frac{-1}{t_1 t_2} = 4$

Given (h, k) divides the line segment in the ratio 1 : 2

$\Rightarrow (h, k) = \left(\frac{t_2 + 2t_1}{3}, \frac{t_1 + 2t_2}{3t_1 t_2}\right)$

$3h = t_2 + 2t_1$ …(1)

$\frac{-3k}{4} = t_1 + 2t_2$ …(2)

Using $t_2 = -\frac{1}{4t_1}$ we get

$3h = -\frac{1}{4t_1} + 2t_1 \ \& -\frac{3k}{4} = t_1 - \frac{1}{2t_1}$

or $2h + k = \frac{1}{2t_1} \ \& \ 8h + k = 4t_1$

$\Rightarrow (2h + k)(8h + k) = 2$

Required locus is $16x^2 + 10xy + k^2 = 2$

14. **(ABC)**

Let e be a root of $x^2 - ax + 2 = 0$, then

$e^2 - ae + 2 = 0$ has both the roots greater than 1.

Now let $P(e) = e^2 - ae + 2$, then

(i) $P(e) > 1 \Rightarrow a < 3$

(ii) $\frac{a}{2} > 1 \Rightarrow a > 2$

(iii) $a^2 - 8 \geq 0 \Rightarrow a \leq -2\sqrt{2}$ or $a \geq \sqrt{2}2$

Hence $2\sqrt{2} < a < 3$

15. **(AD)**

16. **(AD)**

17. **(BD)**

Given hyperbola is $\frac{x^2}{9} - \frac{y^2}{3} = 1$

Angle between Asymptotes: $2\tan^{-1}\frac{b}{a} = 2\tan^{-1}\frac{1}{\sqrt{3}} = 60^0$

$\Rightarrow$ acute angle = 60^0

$e \Rightarrow 3 = 9(e^2 - 1) \Rightarrow e = \frac{2}{\sqrt{3}}$ …(B)

L.R.: $\frac{2.3}{3} = 2$

Asymptotes: $\frac{x}{a} \pm \frac{y}{b} = 0$

Product of 1st from $(a\sec\theta, b\tan\theta) = \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{9}{4} > 2$ …(D)

18. **(ABCD)**

Solving $x^2 + y^2 = a^2$ and $xy = c^2$

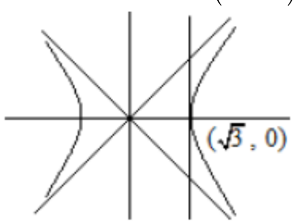
$\Rightarrow x^2 + \frac{c^4}{x^2} = a^2$

$\Rightarrow x^4 - a^2x^2 + c^4 = 0$

$\Rightarrow \sum x_i = 0 \;\&\; \prod x_i = c^4 \;\&\; \sum y_i = 0 \;\&\; \prod y_i = c^4$

19. **(BC)**

Clearly vertex is $(\sqrt{3}, 0)$



Solving with Asymptotes $x^2 - 3y^2 = 0$

$\Rightarrow (\sqrt{3}, 1)$ and $(\sqrt{3}, -1)$

$\therefore$ Triangle is formed by (0, 0), $(\sqrt{3},1), (\sqrt{3},-1)$

$\Rightarrow$ Equilateral triangle

Area $= \frac{1}{2} ab\sin\theta = \frac{1}{2}.2.2.\frac{\sqrt{3}}{2} = \sqrt{3}$

20. **(ABC)**

Given $2\sqrt{2} < e_1 + e_2 < 3\sqrt{2} \;\&\; e_1e_2 = 2$

$\Rightarrow e_1^2 - 3\sqrt{2}e_1 + 2 < 0 \;\&\; e_1^2 - 2\sqrt{2}e_1 + 2 > 0$

$\Rightarrow \frac{3\sqrt{2}-\sqrt{10}}{2} < e_1 < \frac{3\sqrt{2}+\sqrt{10}}{2}$

21. **(BC)** **22.** **23.** **(BCD)**

24. **(AD)**

25. **(ACD)**

26. **(AB)** **27.** **(CD)** **28.** **(BCD)** **29.** **(ABCD)** **30.** **(AC)**

COMPREHENSION TYPE

$\frac{a}{5(5-b)} \cdot \frac{a}{5(5-c)} = \frac{1}{2}$

$$\frac{5(5-a)(5-b)(5-c)}{5^2(5-b)(5-c)}=\frac{1}{2}$$

$2(5-9)=s \Rightarrow ab+c=3a$

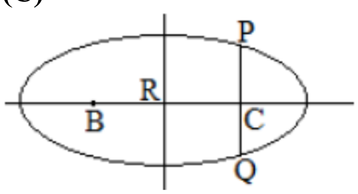
AC + AB = 3BC = 12 (BC = 4)

$\therefore$ locus of A will be ellipse with foci B(2, 4) & C(6, 4) & with length of major axis = 12

$\therefore$ 2ae = 4_1 2a = 12

$\therefore e=\frac{1}{3}$

1. **(C)**



Area of ΔPQR

$=\frac{1}{2}PQ.CR=\frac{1}{2}.\frac{2b^2}{2}.(ae)$

$=\left\{a^2\left(1-e^2\right)\right\}.\frac{1}{3}$

$=\left(6^2-2^2\right).\frac{1}{3}=\frac{32}{3}$

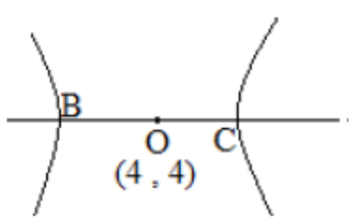
2. **(A)**

ΔPBC is right angled

$\therefore$ circum radius = $=\frac{1}{2}PB=\frac{1}{2}\sqrt{\left(2ae^2\right)+\left(\frac{b^2}{9}\right)^2}=\frac{1}{2}.\frac{20}{3}=\frac{16}{3}$

3. **(D)**

2a = BC = 4 (where 2a = length of transverse axis)



$\therefore \frac{(x-4)^2}{4}-\frac{(y-4)^2}{b^2}=1$

Passing through (O, C)

$\therefore 4-\frac{4}{b^2}=1 \quad \therefore b^2=\frac{4}{3}$

$\therefore 4\left(e^2-1\right)=\frac{4}{3}\therefore e=\frac{2}{\sqrt{3}}$

4. **(A)**

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2e^2$$

$$G\left[ae^2\sec\theta,0\right]$$

$$g\left[0,\frac{a^2e^2\tan\theta}{b}\right]$$

$$P\left[a\sec\theta,b\tan\theta\right]$$

$$PG^2=\left(ae^2\sec\theta-a\sec\theta\right)^2+b^2\tan^2\theta$$

$$PG^2=a^2\sec^2\theta\left[\left(e^2-1^2\right)\right]+b^2\tan^2\theta$$

$$L^2=a^2\left(1+\tan^2\theta\right)\left[\left(e^2-1\right)^2+b^2\tan\theta\right]$$

$$L^2\min=a^2\left[1\times\left(e^2-1\right)\right]$$

$$=a^2\left(e^2-1\right)=\frac{b^2}{a^2}$$

$$=a^2\left[\left(\frac{b^2}{a^2}\right)^2\right]$$

$$\min=\frac{b^2}{a}$$

5. **(A)**

$$pg^2=a^2\sec^2\theta+\left(\frac{a^2e^2\tan\theta}{b}-b\tan\theta\right)^2$$

$$=a^2\sec^2\theta+\tan^2\theta\frac{\left(a^2e^2-b^2\right)^2}{b^2}$$

$$=a^2\sec^2\theta+\frac{a^4}{b^4}\tan^2\theta$$

6. **(B)**

$$PG^{\theta}.Pg^{0}=b^2$$

$\therefore$ G.M. of PG. Pg = b

7. **(A)**

$2x^2 + 3xy - 2y^2 + 5 = 0$

Clearly $2x^2 + 3xy - 2y^2 = 0$ are pair of asymptote

8. **(C)**

Given hyperbola: $x^2 + 6x - 2y^2 + 4x + 2 = 0$

$\Rightarrow$ Pain of Asymptotes: $x^2 + 6xy - 2y^2 + 4x + 2 + \lambda = 0$

$$\Rightarrow \begin{vmatrix} 1 & 3 & 2 \\ 3 & -2 & 0 \\ 2 & 0 & 2+\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2(-2-\lambda) - 3(6+2\lambda) + 2(4) = 0$$

$$\lambda = \frac{-14}{11}$$

9. **(C)**

$2x^2 + 3xy - 2y^2 + 5 = 0$

$\Rightarrow$ Pair of Asymptotes: $2x^2 + 3xy - 2y^2 = 0$

Angle between Asymptotes $= 2\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$

$$\Rightarrow 2\tan^{-1}\sqrt{e^2 - 1} = \frac{\pi}{2}$$

$$\Rightarrow e = \sqrt{2}$$

10. **(A)**

Solving $x^2 + y^2 + 2gc + 2fy + k = 0$ and $\left(ct, \frac{c}{t}\right)$

$$\Rightarrow c^2t^2 + \frac{c^2}{t^2} + 2gct + \frac{2fc}{t} + k = 0$$

$$\Rightarrow c^2t^4 + c^2 + 2gct^3 + 2fct + kt^2 = 0 \qquad \ldots(1)$$

$$\sum \frac{1}{t_1} = \frac{\sum t_1t_2t_3}{t_1t_2t_3t_4} = \frac{-2fc}{c^2} = \frac{-2f}{c}$$

11. **(C)**

Form (1)

$$t_1 + t_2 + t_3 + t_4 = \frac{-29}{c} \quad (\because t_1t_2t_3t_3 = 1)$$

$$\Rightarrow t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3} = \frac{-29}{c}$$

$$\Rightarrow -g = \frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right) \& -f = \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1t_2t_3\right)$$

12. **(B)**

$$\sum t_1 = (\text{from}(1)) = \frac{-29}{c}$$

13. **(D)**

14. **(B)**

15. **(A)**

16. **(B)**

17. **(A)**

18. **(B)**

MATRIX MATCH

1. **A-q; B-s; C-r; D-p**

(A) $e=\sqrt{\dfrac{a^2+b^2}{a^2}}$

$e'=\sqrt{\dfrac{a^2+b^2}{b^2}}$

$\Rightarrow \dfrac{1}{e^2}+\dfrac{1}{(e')^2}=1$ (A – q)

(B) $e_1=e_2=\sqrt{2}$ (B – s)

(C) $\dfrac{(2y-x-3)^2}{20}-\dfrac{9(2x+y-1)}{80}=1$

Now $a^2=4$ $b^2=\dfrac{16}{9}$

$\Rightarrow b^2=a^2(e^2-1)$

$\Rightarrow \boxed{e=\dfrac{\sqrt{13}}{3}}$ c – r

(D) $\dfrac{\sqrt{3}x-y}{4\sqrt{3}}=k$ and $\dfrac{4\sqrt{3}}{\sqrt{3}x+y}=k$

2. **A-r; B-r; C-q,s; D-r**

(A) angle between Asymptotes: $2\tan^{-1}\dfrac{b}{a}=\dfrac{\pi}{3}$

$2\tan^{-1}\sqrt{e^2-1}=\dfrac{\pi}{3}$

$\sqrt{e^2-1}=\dfrac{1}{\sqrt{3}}$

$\Rightarrow e^2=\dfrac{4}{3}$

We know that $\dfrac{1}{e^2}+\dfrac{1}{e'^2}=1$

$\Rightarrow \dfrac{3}{4}+\dfrac{1}{(e')^2}=1$

$\Rightarrow e'=2$ A – 2

(B) $x=\dfrac{4\sqrt{3}m^2+4\sqrt{3}m}{2m\sqrt{3}}$

$=2\dfrac{(m^2-1)}{m}$

$$y=\frac{(m^2-1)2\sqrt{5}}{m}$$

(C) x + y = k touches $x^2 - 2y^2 = 18$

Put x = k – y

$\Rightarrow x^2 = k^2 + y^2 - 2ky$

$\Rightarrow -y^2 - 2ky + (k^2 - 18) = 0$

$\Rightarrow$ D = 0

$\Rightarrow 4k^2 + 4(k^2 - 18) = 0$

$k = \pm 3$ C – q, s

(D) $\frac{x^2}{4a^2}+\frac{y^2}{ab^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are can focal

$\Rightarrow 4a^2e_1^2 = a^2e_2^2$

$\Rightarrow 4a^2 - 4b^2 = a^2 + b^2$

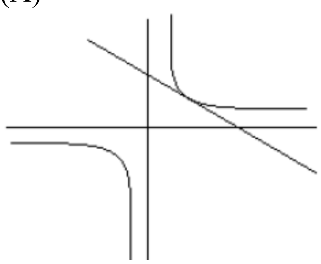
$\Rightarrow 3a^2 = 5b^2$

Now, $e_1^2 = 1-\frac{b^2}{a^2}=1-\frac{3}{5}=\frac{2}{5}$

$e_2^2 = 1+\frac{b^2}{a^2}=1+\frac{3}{5}=\frac{8}{5}$

3. **(A-r), (B-s), (C-r), (D-q)**

(A)



Let P be $\left(2\sqrt{2}t, \frac{2\sqrt{2}}{t}\right)$

Equation of tangent : $\frac{x}{x_1}+\frac{y}{y_1}=2$

Area $=\frac{1}{2}(2x_1)(2y_1)=2x_1y_1=16$

A – 2

(B) $\frac{x^2}{5}-\frac{y^2}{5\cos^2\theta}=1$ $e_1=\sqrt{1+\cos^2\theta}$

$\frac{x^2}{25\cos^2\theta}+\frac{y^2}{25}=1$ $e_2=\sin\theta$

Given $\sqrt{1+\cos^2\theta}=\sqrt{3}\sin\theta$

$\Rightarrow 1+\cos^2\theta = 3\sin^2\theta$

$\Rightarrow 2 = 4\sin^2\theta$

$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$

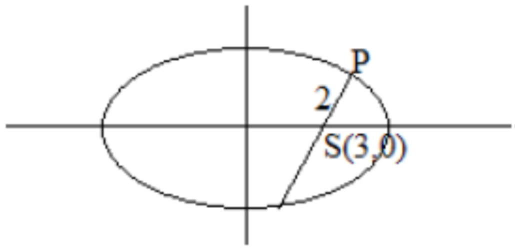
$\Rightarrow \theta = \frac{\pi}{4}$

B – s

(C) $P_1P_2 = b^2$ $\Rightarrow$ Product perpendicular = 16

C – r

(D)



$\text{L.R.} = \frac{2.16}{5}$

$\text{Semi L.R.} = \frac{16}{5}$

We know that PS, semi – L.R. and SQ are in H.P.

$\therefore \frac{16}{5} = \frac{2.PS.SQ}{PS+SQ} = \frac{2.(2)SQ}{2+SQ}$

$\Rightarrow$ SQ = 8

$\therefore$ PQ = 10

4. **A – p; B – s; C – r; D – s**

5. **A – p; B – q; C – r; D – s**

HYPERBOLA

EXERCISE – 2(C)

Q.1

$$\frac{2b^2}{a} = 2b \quad \Rightarrow \frac{b}{a} = 1;$$

So slopes of asymptotes are ± 1.

$\therefore$ asymptotes perpendicular

Q.2

Let the hyperbolas be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

A line parallel to y – axis, say x = k, meets these in P & Q.

Now $\frac{k^2}{a^2} - \frac{y^2}{b^2} = 1$ & $x = k \Rightarrow y = b\sqrt{\frac{k^2}{a^2} - 1}$, hence P is $\left(k, b\sqrt{\frac{k^2}{a^2} - 1}\right)$

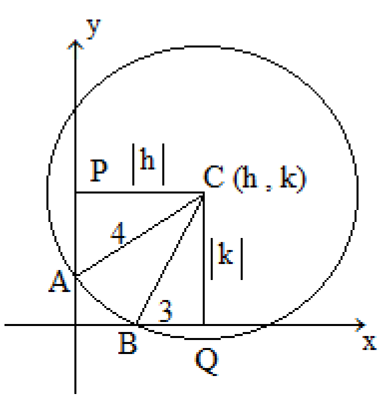
and $\frac{k^2}{a^2} - \frac{y^2}{b^2} = -1$ & $x = k \Rightarrow y = b\sqrt{\frac{k^2}{a^2} + 1}$, hence Q is $\left(k, b\sqrt{\frac{k^2}{a^2} + 1}\right)$

Normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P will be $\left(\frac{x-k}{k}\right)a^2 + \left(\frac{ay - b\sqrt{k^2 - a^2}}{b\sqrt{k^2 - a^2}}\right)b^2 = 0 \quad \ldots(1)$

Normal to $\frac{k^2}{a^2} - \frac{y^2}{b^2} = -1$ at Q will be $\left(\frac{x-k}{k}\right)a^2 + \left(\frac{ay - b\sqrt{k^2 + a^2}}{b\sqrt{k^2 + a^2}}\right)b^2 = 0 \ldots(2)$

Putting y = 0 in (1) & (2) gives $x = \frac{b^2 k}{a^2} + k$, hence the point of intersection lies on x – axis.

Q.3



CA = CB $\Rightarrow h^2+16=k^2+9$

$\Rightarrow \boxed{y^2-x^2=7}$

Now foci of $x^2-y^2=-a^2$ are $\left(0,\pm\sqrt{2}a\right)$, hence the foci are $\left(0,\pm\sqrt{14}\right)$.

Q.4

Let P be $\left(a\sec\theta, b\tan\theta\right)$.

Normal at P : $\dfrac{ax}{\sec\theta}+\dfrac{by}{\tan\theta}=a^2+b^2$

It meets transverse axis(x – axis) at $G\left(\dfrac{a^2+b^2}{a}\sec\theta, 0\right)$

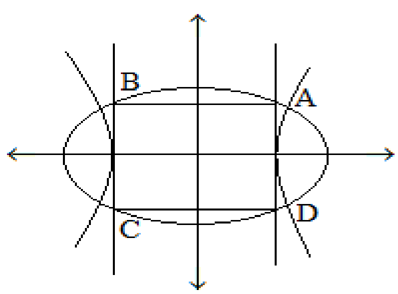
Slope of one of the asymptotes = $\dfrac{b}{a}$

Now GL will be $y=-\dfrac{a}{b}\left(x-\dfrac{a^2+b^2}{a}\sec\theta\right)$

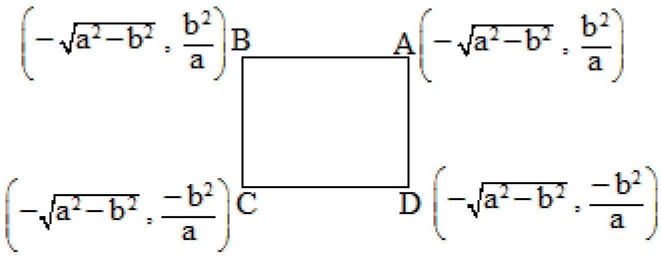
It will meet the asymptote $bx=ay$ at $L\left(a\sec\theta, b\sec\theta\right)$

Clearly slope of LP = 0.

Q.5



AD : $x=\sqrt{a^2-b^2}$ & CB : $x=-\sqrt{a^2-b^2}$

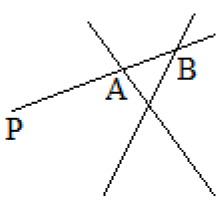


So area of rectangle ABCD is $=\left(2\sqrt{a^2-b^2}\right)\left(\frac{2b^2}{a}\right)$

$$=\frac{4b^2\sqrt{a^2-b^2}}{a}$$

Q.6

Let 'P' is $\left(\frac{\sec\theta}{b}\,,\,\frac{\tan\theta}{a}\right)$. Combined equation of asymptotes is $b^2x^2-a^2y^2=0$



By parametric from a point on line PA will be

$$\left(\frac{\sec\theta}{b}+r\cos\alpha\,,\,\frac{\tan\theta}{a}+r\sin\alpha\right)$$

It lies on asymptotes then

$$b^2\left(\frac{\sec\theta}{b}+r\cos\alpha\right)^2-a^2\left(\frac{\tan\theta}{a}+r\sin\alpha\right)^2=0$$

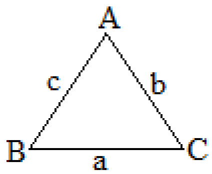
$$\Rightarrow r^2(b^2\cos^2\alpha-a^2\sin^2\alpha)+2(b\sec\theta\cos\alpha-a\tan\theta\sin\alpha)r+(\sec^2\theta-\tan^2\theta)=0$$

So $PA\cdot PB=r_1\cdot r_2=\frac{\sec^2\theta-\tan^2\theta}{b^2\cos^2\alpha-a^2\sin^2\alpha}$

$PA\cdot PB=\frac{1}{b^2\cos^2\alpha-a^2\sin^2\alpha}$ (independent of θ)

Hence $PA\cdot PB$ is independent of point P.

Q.7



BC = a is fixed

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} \quad \& \quad \tan\frac{c}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} = \frac{s-c}{s-b} = k \Rightarrow \frac{a+b-c}{a+c-b} = k$$

$$\Rightarrow a+b-c = ka+kc-kb$$

$$\Rightarrow (k+1)c - (k+1)b = (1-k)a$$

$$\Rightarrow c-b = \left(\frac{1-k}{1+k}\right) a \;= a \text{ constant}$$

$\Rightarrow$ BA – CA is constant.

So locus of A is a hyperbola with B and C as foci.

Q.8

Let any tangent to $y^2 = 4ax$ with slope m be $y = mx + \frac{a}{m}$.

Now if two tangents are drawn from P(h, k), then slopes of these tangents $(m_1 \& m_2)$ will be the roots of $k = mh + \frac{a}{m}$ or $hm^2 - km + a = 0$.

$$\therefore \; m_1 + m_2 = \frac{k}{h} \; \& \; m_1 m_2 = \frac{a}{h}$$

Now $\tan 45^o = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| \Rightarrow (m_1 - m_2)^2 = (1+m_1 m_2)^2$

$$\Rightarrow (m_1 + m_2)^2 - 4m_1 m_2 = (1+m_1 m_2)^2$$

$$\Rightarrow \frac{k^2}{h^2} - \frac{4a}{h} = \left(1 + \frac{a}{h}\right)^2$$

$$\Rightarrow k^2 - 4ah = h^2 + 2ah + a^2$$

$$\Rightarrow (h + 3a)^2 - k^2 = 8a^2$$

Hence required locus is a hyperbola.

Q.10

Factorizing $x^2 + 2xy - 3y^2 = 0$ gives x + 3y & x – y as factors.

Now let the asymptotes be x + 3y + a = 0 & x – y + b = 0

Pair of asymptotes will be $x^2 + 2xy - 3y^2 + (a + b)x + (3b - a)y + ab = 0$

Comparing it with $x^2 + 2xy - 3y^2 + x + 7y + c = 0$ gives

a + b = 1, 3b – a = 7 & ab = c.

$\Rightarrow a = -1, b = 2, c = -2$.

Hence the asymptotes are x + 3y – 1 = 0 & x – y + 2 = 0.

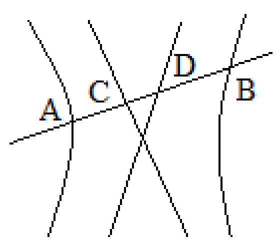
$$m_1 = -3, m_2 = 1 \Rightarrow \tan\theta = \frac{-3-1}{1-1\times 3} = 2$$

$\therefore$ angle between the asymptotes $= \tan^{-1} 2$.

Q.11

Let the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let the line is $y = mx + c$



Solving $y = mx + c$ & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We have $\frac{x^2}{a^2} - \frac{(mx+c)}{b^2} = 1$

$$\Rightarrow \left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)x^2 - \left(\frac{2mc}{b^2}\right)x - \left(1+\frac{c^2}{b^2}\right) = 0$$

Let A, B are (x_1, y_1) & (x_2, y_2) mid – point of AB is $\left(\frac{x_1+x_2}{2}, \frac{m(x_1+x_2)}{2} + c\right)$

$x_1 + x_2$ is sum of roots of above quadratic.

Equation of asymptotes are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

On solving we have the quadratic

$$\left(\frac{1}{a^2} - \frac{m^2}{c^2}\right)x^2 - \left(\frac{2mc}{b^2}\right)x - \frac{c^2}{b^2} = 0$$

So again $x_1 + x_2$ is same, hence mid – pint is same.

Q.12

Normal at any point $\left(t, \frac{1}{t}\right)$ is $(ty-1) = t^3(x-t)$. It passes through (h , k).

So, $(tk-1) = t^3(h-t) \Rightarrow t^4 - ht^3 + tk - 1 = 0$

We have four roots of above equation. Let variable line is ax + by + c = 0

We have $\frac{a(x_1+x_2+x_3+x_4) + b(y_1+y_2+y_3+y_4) + 4c}{\sqrt{a^2+b^2}} = c$

$$\sum x = \sum t \;,\; \sum y = \sum \frac{1}{t}$$

$$\sum t = h \;;\; \sum \frac{1}{t} = \frac{\sum t_1t_2t_3}{t_1t_2t_3t_4} = \frac{-k}{-1} = k$$

$$\Rightarrow ah + bk + 4c = 0$$

$\Rightarrow a\left(\frac{h}{4}\right) + b\left(\frac{k}{4}\right) + c = 0$

So line passé through $\left(\frac{h}{4}, \frac{h}{4}\right)$

Q.13

Conjugate hyperbola : $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

Let P is $(a\tan\theta, b\sec\theta)$.

Equation of tangent is $\boxed{\frac{y\sec\theta}{b} - \frac{x\tan\theta}{a} = 1}$

And chord to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with P as mid – point is T = S'

$\Rightarrow \frac{x\tan\theta}{a} - \frac{y\sec\theta}{b} = \tan^2\theta - \sec^2\theta$

$\Rightarrow \frac{a\tan\theta}{a} - \frac{y\sec\theta}{a} = -1$

$\Rightarrow \boxed{\frac{y\sec\theta}{b} - \frac{x\tan\theta}{a} = 1}$ which is same as above equation.

Q.14

A chord joining $A(t_1)$ & $B(t_2)$ on the curve $xy = c^2$ subtends right angle at $P(t_3)$

So we have slope of $AP = \frac{-1}{t_1 + 3}$ & slope of $BP = \frac{-1}{t_2 t_3}$

$\Rightarrow \frac{1}{t_1 t_2 t_3 2} = -1$ as $\angle APB = 90^\circ$

slope of AB : $\frac{-1}{t_1 t_2}$

slope of normal at P : t_3^2

we have : $t_3^{\,2} = \dfrac{-1}{t_1\, t_2}$

Q.15

Polar of $(x_1\,,\,y_1)$ with respect to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$. This line is tangent to $xy = c^2$.

On sharing we have $(x_1)x + \dfrac{(c^2 y_1)}{x} = a^2$

$\Rightarrow (x_1)x^2 - (a^2)x + (c^2 y_1) = 0$

$D = a^4 - 4x_1 y_1 c^2 = 0$

$\Rightarrow x_1 y_1 = \dfrac{a^4}{4c^2}$

So $(x_1\,,\,y_1)$ lies on $xy = \dfrac{a^4}{c^4}$ (A concentric rectangular hyperbola to $xy = c^2$)

Q.16

$xy = c^2 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{-c^2}{x^2}$

$\Rightarrow \dfrac{-dx}{dy} = \dfrac{x^2}{c^2}$

Slope of normal at $P(t_1)$ is t_1^2. Equation of normal is $\left(y - \dfrac{c}{t_1}\right) = t_1^2\,(x - c\,t_1)$

$\Rightarrow (t_1 y - c) = t_1^3 (x - c\,t_1)$

$\Rightarrow (t_1^3)x - (t_1)y + c(1 - t_1^4) = 0$

Let is meets curve at $\left(ct\,,\,\dfrac{c}{t}\right)$.So

$\Rightarrow (c\,t_1^3)t - (t_1)\dfrac{c}{t} + c(1 - t_1^4) = 0$

$\Rightarrow t^2 (t_1^3) + t(1 - t_1^4) - (t_1) = 0$

The two roots are t_1 & t_2. So

$$t_1 t_2 = \frac{-t}{t_1^3}$$

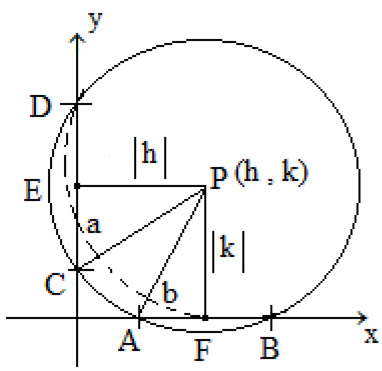
$$\Rightarrow t_1^3 t_2 = -1$$

Q.17

Let two mutually perpendicular lines are x = 0 & y = 0

Let AB = 2a and CD = 2b

Let centre is P(h , k)



$$PE = |h|$$

$$PF = |k|$$

$$PC = \sqrt{h^2 + a^2}$$

$$PA = \sqrt{k^2 + b^2}$$

$$PC^2 = PA^2$$

$$\Rightarrow h^2 + a^2 = k^2 + b^2$$

$$\Rightarrow \boxed{x^2 - y^2 = b^2 - a^2} \qquad \text{(rectangular hyperbola)}$$

Q.18

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ and the hyperbola be $xy = a^2$.

Any point on the hyperbola will be $\left(at, \frac{a}{t}\right)$.

Substitute these coordinates in the equation of the circle and rearrange the terms to get

$$a^2t^4 + 2gat^3 + ct^2 + 2fat + a^2 = 0$$

Now $t_1 + t_2 + t_3 + t_4 = -\frac{2g}{a}, \sum t_1t_2t_3 = -\frac{2f}{a}, t_1t_2t_3t_4 = 1$

Hence $\frac{at_1 + at_2 + at_3 + at_4}{4} = -\frac{g}{2}$ & $\frac{\frac{a}{t_1} + \frac{a}{t_2} + \frac{a}{t_3} + \frac{a}{t_4}}{4} - \frac{f}{2}$.

Clearly its midpoint of line joining (0, 0) & (-g, -f)

Q.19

Equation of circle on the line joining foci (ae, 0) and (–ae, 0) as diameter is

(x – ae) (x + ae) + (y – 0) (y – 0) = 0

i.e. $x^2 + y^2 = a^2e^2 = a^2 + b^2$ ... (i) [$a^2e^2 = a^2 + b^2$]

Let chord of contact of P (x_1, y_1) touch the circle (i)

Equation of chord of contact of P is [T = 0]

$xx_1/a^2 – yy_1/b^2 = 1$ i.e., $b^2x_1x – a^2 y_1y – a^2 b^2 = 0$ ... (ii)

$$\therefore \frac{a^2b^2}{\sqrt{(b^4x_1^2 + a^4y_1^2)}} = \pm\sqrt{(a^2 + b^2)}$$

Hence locus of P (x_1, y_1) is $(b^4x^2 + a^4y^2)(a^2 + b^2) = a^4b^4$.

Q.20

Let hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

A normal to it is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

A is $\left(\frac{(a^2 + b^2)}{a}\sec\theta, 0\right)$

B is $\left(0, \frac{(a^2 + b^2)\tan\theta}{b}\right)$

Let mid – point of AB is P(h , k)

So $h = \dfrac{(a^2+b^2)\sec\theta}{2a}$; $k = \dfrac{(a^2+b^2)}{2b}\tan\theta$

Eliminating 'θ' we have

$$\Rightarrow (2ah)^2 - (2bk)^2 = (a^2+b^2)^2$$

$$\Rightarrow \frac{x^2}{\frac{a^4e^4}{4a^2}} - \frac{y^2}{\frac{a^4e^4}{4b^2}} = 1$$

$$e = \sqrt{1+\frac{a^4e^4}{4b^2}\times\frac{4a^2}{a^4e^4}}$$

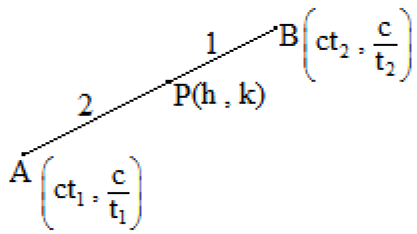
$$e = \sqrt{1+\frac{1}{e^2-1}}$$

$$e = \frac{e}{\sqrt{e^2-1}}$$

Q.21

Let chord is $A(t_1)$ to $B\ (t_2)$.

$$\text{Slope AB} = \frac{-1}{t_1+t_2} = 4$$



$$h = \frac{(2t_2+t_1)c}{3}$$

$$k = \frac{\left(\frac{2}{t_2} + \frac{1}{t_1}\right)c}{3}$$

We have, $2t_2 + t_1 = \frac{3h}{c}$ & $2t_1 + t_2 = \frac{-3k}{4c}$

Eliminating t_1 & t_2 we get,

$16x^2 + 10xy + y^2 = 2c^2$

Q.22

A tangent to $x^2 = 4ay$ is $x = my + \frac{a}{m}$. It meets $xy = c^2$

So $x = \frac{mc^2}{x} + \frac{a}{m}$

$\Rightarrow mx^2 = m^2c^2 + ax$

$\Rightarrow mx^2 - ax + m^2c^2 = 0$

Let P & Q are (x_1 , y_1) & (x_2 , y_2)

$\Rightarrow$ mid – point be R (h , k)

$h = \frac{x_1 + x_2}{2}$; $k = \frac{y_1 + y_2}{2}$

$x_1 + x_2 = \frac{a}{m}$; $y_1 + y_2 = \frac{-a}{m_2}$

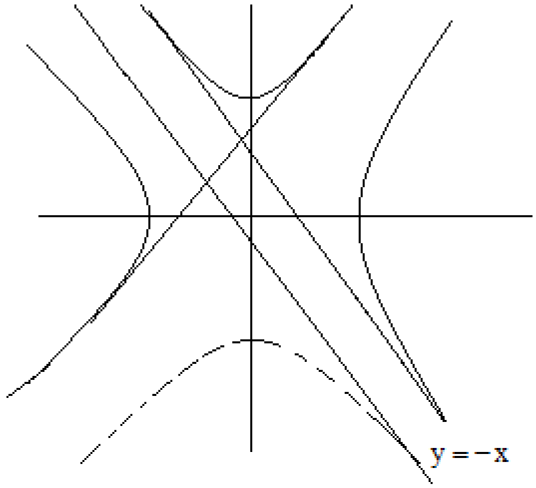
$2h = \frac{a}{m}$; $2k = \frac{-a}{m^2}$

$\frac{4h^2}{a^2} = \frac{-2k}{a}$

$\Rightarrow 2x^2 = -ay \Rightarrow y = \frac{-2x^2}{a}$ (a parabola)

Q.23

The hyperbolas are conjugate to each other so the common tangent will be the ones with slope ± 1



So equation of tangent with slope '1'

$$\boxed{y = x \pm \sqrt{a^2 - b^2}}$$

Let point of tangency is (h , k)

So $x - y - \sqrt{a^2 - b^2} = 0$

$$\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = 0$$

So $\frac{a^2}{h} = \frac{b^2}{k} = + \sqrt{a^2 - b^2}$

Point is $\left(\frac{a^2}{\sqrt{a^2 - b^2}} \; , \; \frac{b^2}{\sqrt{a^2 - b^2}} \right)$

Length is twice of its distance from asymptote y + x = 0

So length is $\frac{\sqrt{2}\left| a^2 + b^2 \right|}{\sqrt{a^2 - b^2}}$

Q.24

Let the normal be $tx - \frac{y}{t} = ct^2 - \frac{c}{t^2}$.

As it passes through $\left(ct_1, \frac{c}{t_1} \right)$ hence $t_1 t - \frac{1}{t_1 t} = t^2 - \frac{1}{t^2}$

$\Rightarrow (t_1 - t)\, t = \dfrac{t - t_1}{t_1 t^2}$ or $t_1 t^3 = -1$

Q.25

We have $\dfrac{2b^2}{a} = \dfrac{32\sqrt{2}}{5}$

And point of intersection of lines is

$$\left.\begin{array}{l} 7x + 13y - 87 = 0 \\ 5x - 8y + 7 = 0 \end{array}\right] \Rightarrow x = 5 \ , \ y = 4$$

So we have $\dfrac{25}{a^2} - \dfrac{16}{b^2} = 1$

On solving we have $a = \dfrac{5}{\sqrt{2}}$ & b = 4

Q.26

Hyperbola is $16x^2 - 9y^2 + 32x + 36y - 164 = 0$

$\Rightarrow 16(x+1)^2 - 9(y^2 - 4y + 4) = 144$

$\Rightarrow \dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{16} = 1$

Centre $\rightarrow$ (– 1 , 2)

Foci $\rightarrow$ (4 , 2) & (– 6 , 2)

Directrix $\rightarrow$ $x = \dfrac{4}{5}$ & $x = \dfrac{-14}{5}$

Lotus rectum $\rightarrow$ $\dfrac{32}{3}$

Transverse axis $\rightarrow$ 6 ; equation y – 2 = 0

Conjugate axis $\rightarrow$ 8 ; equation x + 1 = 0

Asymptotes $\rightarrow$ 4x – 3y + 10 = 0 & 4x + 3y – 2 = 0

Q.27

Slope of tangent = (– 1)

Equation : $y = -x \pm \sqrt{36 \times 1 - 9}$

$y + x = \pm 3\sqrt{3}$

Q.28

Chord with ‘ θ_1 ’ & ‘ θ_2 ’ as and of extremities.

$$\frac{x}{a}\cos\frac{\theta_1 - \theta_2}{2} - \frac{y}{b}\sin\frac{\theta_1 + \theta_2}{2} = \cos\frac{\theta_1 + \theta_2}{2}$$

It passes through (ae , 0).

$$\text{So, } e\frac{\cos\theta_1 - \theta_2}{2} = \frac{\cos\ \theta_1 + \theta_2}{2}$$

$$e = \frac{\frac{\cos(\theta_1 + \theta_2)}{2}}{\frac{\cos(\theta_1 - \theta_2)}{2}}$$

$$\frac{e-1}{e+1} = \frac{\cos\left(\frac{\theta_1 + \theta_2}{2}\right) - \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$$

$$\frac{e-1}{e+1} = \frac{-2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}{2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}$$

$$\Rightarrow \boxed{\frac{e-1}{e+1} + \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = 0}$$

Q.29

A line through $\left(0\ ,\ \frac{5}{2}\right)$ is $y - \frac{5}{2} = mx$. This is tangent to $3x^2 - 2y^2 = 25$.

So on solving $3x^2 - 2\left(mx + \frac{5}{2}\right)^2 = 25$

$\Rightarrow\ 6x^2 - (3mx + 5)^2 = 50$

$\Rightarrow (6 - 4m^2)x^2 - (20m)x - 75 = 0$

$D = 0 \ \Rightarrow\ 400m^2 + 300(6 - 4m^2) = 0$

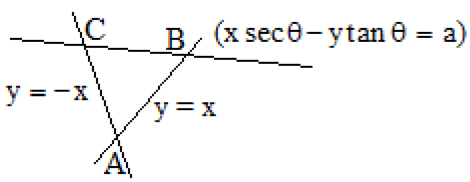
$\Rightarrow\ 800m^2 = 6 \times\ 300$

$\Rightarrow m = \pm\ \frac{3}{2}$

So equation(s) are $\Rightarrow 2y = \pm\ 3x + 5$

Q.30

A tangent to $x^2 - y^2 = a^2$ is $x\sec\theta - y\tan\theta = a$



A : (0 , 0)

$B : \left(\frac{a}{\sec\theta - \tan\theta}\ ,\ \frac{a}{\sec\theta - \tan\theta}\right)$

$C : \left(\frac{a}{\sec\theta + \tan\theta}\ ,\ \frac{-a}{\sec\theta + \tan\theta}\right)$

A (0 , 0) : $B(a(\sec\theta + \tan\theta),\ a(\sec\theta + \tan\theta))$

$C(a()\sec\theta - \tan\theta),\ -\ a(\sec\theta - \tan\theta))$

Area of Δ ABC

$= \frac{1}{2}\left|[a^2(\sec^2\theta - \tan^2\theta) - a^2(\sec^2\theta - \tan^2\theta)]\right| = a^2$

HYPERBOLA

EXERCISE – 3

Q.1

Let middle point is P(h , k).

The equation of chord is T = S’

$xh - yk = h^2 - k^2$

Let this is normal at $P(a\sec\theta , a\tan\theta)$. So equation of normal is $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$.

On comparison : $h\sec\theta = -k\tan\theta = \frac{h^2 - k^2}{2a}$

Eliminating $'\theta'$; we get $\left(\frac{h^2 - k^2}{2a}\right)^2 \left[\frac{1}{h^2} - \frac{1}{k^2}\right] = 1$

$\Rightarrow (x^2 - y^2)^3 = 4a^2x^2y^2$

Q.2

Let vertices of triangle are $A(t_1) , B(t_2) , C(t_3)$

$AB : (t_1t_2)y + x = C(t_1 + t_2)$

$BC : (t_2 + t_3)y + x = C(t_1 + t_3)$

$AC : (t_1 + 3)y + x = C(t_1 + t_3)$

If $(t_1t_2)y + x - c(t_1 + t_2) = 0$ is tangent to $y^2 = 4ax$. Then $\frac{y^2}{4a} + (t_1t_2)y - C(t_1 + t_2) = 0$ has discriminate equal to zero.

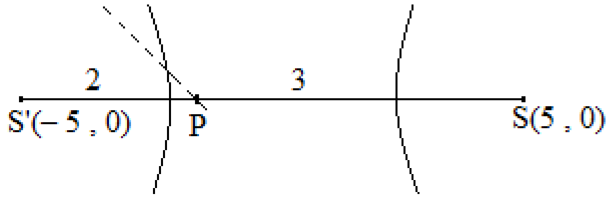
So D = 0 $\Rightarrow$ $a(t_1t_2)^2 + Ct_1 + Ct_2 = 0$

Similarly $a(t_2t_3)^2 + Ct_2 + Ct_3 = 0$

$a(t_1t_3)^2 + Ct_1 + Ct_3 = 0$

There are infinitely possible solutions for t_1 , t_2 & t_3

Q.3



$$\frac{x^2}{16} - \frac{y^2}{5} = 1$$

$$e = \sqrt{1 + \frac{5}{16}}$$

$$e = \frac{5}{4}$$

P is (– 1 , 0)

Equation of line is $(y - 0) = -1 (x + 1)$

$\Rightarrow y + x + 1 = 0$

Asymptotes of hyperbola are $9x^2 - 16y^2 = 0$

$\Rightarrow 9x^2 - 16(x+1)^2 = 0$

$\Rightarrow (3x - 4x - 4)\ (3x + 4x + 4) = 0$

$\Rightarrow x = -4 \quad \& \quad x = \frac{-4}{7}$

$\Rightarrow y = 3 \quad \& \quad y = \frac{-3}{7}$

So points are $(-4 , 3)$ & $\left(\frac{-4}{7} , \frac{-3}{7}\right)$

Q.4

Asymptote of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are $y = \pm \frac{3}{4}x$

Diameters of the ellipse perpendicular to this asymptotes are $y = \pm \frac{4}{3}x$

Passing through I[st] & III[rd] is $y = \frac{4}{3}x$.

Length of diameter of slope m $= 2ab\sqrt{\frac{1+m^2}{b^2+a^2m^2}}$

Hence required length is $= \frac{150}{\sqrt{481}}$.

Q.5

Let P is $(a\sec\theta\ ,\ b\tan\theta)$.

Tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Let it meet $y = \frac{b}{a}x$ at θ .

So $x = \frac{a}{\sec\theta - \tan\theta} = a(\sec\theta + \tan\theta)$

$y = b(\sec\theta + \tan\theta)$

Mid – point of PQ be (h , k)

$h = a\left(\sec\theta + \frac{\tan\theta}{2}\right)$; $k = g\left(\frac{\sec\theta}{2} + \tan\theta\right)$

Eliminating $\theta \Rightarrow 4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 3$

Q.6

$x\cos\alpha + y\sin\alpha = p$ is tangent to $x^2 + y^2 = p^2$

Q.7

Chord is : $\frac{x\cos\theta - \varphi}{2} - \frac{y\sin\theta + \varphi}{2} = \frac{\cos\theta + \varphi}{2}$

Normal at P : $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

So $\dfrac{\cos\dfrac{\theta-\varphi}{2}}{}\cdot\sec\theta = \dfrac{-\sin(\theta+\varphi)}{2}\tan\theta$

$$= \frac{\cos\frac{\theta+\varphi}{2}}{a^2+b^2}$$

On simplifying we get $\boxed{\tan\varphi = \tan\theta(4\sec^2\theta - 1)}$

Q.8

Let middle point is (h , k)

Chord is : $\dfrac{xh}{a^2} - \dfrac{yk}{b^2} = \dfrac{h^2}{a^2} - \dfrac{k^2}{b^2}$

Chord of contact from $(r\cos\theta\ , r\sin\theta)$ is $\dfrac{x\,r\cos\theta}{a^2} - \dfrac{y\,r\sin\theta}{b^2} = 1$

$$\frac{h}{r\cos\theta} = \frac{k}{r\sin\theta} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \frac{x^2+y^2}{r^2}$$

Q.9

Equation of pair of asymptotes of hyperbola differ from the equation of the hyperbola by a constant. Let the equation of pair of asymptotes be

$2x^2 - 3xy - 2y^2 + 3x - y + \lambda = 0$. It passes through the centre of the hyperbola.

$$\left.\begin{aligned} \frac{ds}{dx} &= 4x - 3y + 3 = 0 \\ \frac{ds}{dy} &= -3x - 4y - 1 = 0 \end{aligned}\right\} \quad \text{solving we get } \left(y = \frac{1}{5}\ , \ x = -\frac{12}{5}\right)$$

Asymptotes pass through $\left(-\dfrac{12}{5}, \dfrac{1}{5}\right)$

$\boxed{\lambda = -6}$

Q.10

Let asymptotes are $(2x + 3y + \lambda_1) = 0$ and $(3x+2y+\lambda_2) = 0$. Asymptote pass through (1, 2). So, $\lambda_1 = -8$, $\lambda_2 = -7$.

Hyperbola is $(2x+3y-8)(3x+2y-7) + \lambda = 0$. It passes through (5, 3).

So we get $(11)(14) + \lambda = 0 \Rightarrow \lambda = -154$.

So hyperbola is $6x^2 + 13xy + 6y^2 - 38x - 37y - 98 = 0$

Q.11

Let P is $(a\sec\theta, b\tan\theta)$. Let $\tan\alpha = m$ A point PQR at a distance 'r' from P is

$(a\sec\theta + r\cos\alpha , b\tan\theta + r\sin\alpha)$. It lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$. So,

$b^2(a\sec\theta + r\cos\alpha)^2 = a^2(b\tan\theta + r\sin\alpha)^2$

$\Rightarrow r^2\left[b^2\cos^2\alpha - a^2\sin^2\alpha\right] + 2r(b^2a\sec\theta\cos\alpha - a^2b\tan\theta\sin\alpha) + b^2a^2\sec^2\theta - a^2b^2\tan^2\theta$

$$\Rightarrow r_1r_2 = PQ \cdot PR = \frac{b^2a^2(\sec^2\theta - \tan^2\theta)}{(b^2\cos^2\alpha - a^2\sin^2\alpha)}$$

$$(QP)\cdot(PR) = \frac{b^2a^2}{b^2\cos^2\alpha - a^2\sin^2\alpha}$$

$$\tan\alpha = m \quad \Rightarrow \sin^2\alpha = \frac{m^2}{1+m^2}$$

$$\Rightarrow \cos^2\alpha = \frac{1}{1+m^2}$$

So, $(PQ)(PR) = \dfrac{b^2a^2(1+m^2)}{b^2 - a^2m^2}$

Q.12

Equation of tangents from (3 , 2) $y = mx + \sqrt{9m^2 - 1}$. It goes through (3 , 2).

So, $(2-3m)^2 = (9m^2 - 1)$

$\Rightarrow\ 4+9m^2-12m=9m^2-1$

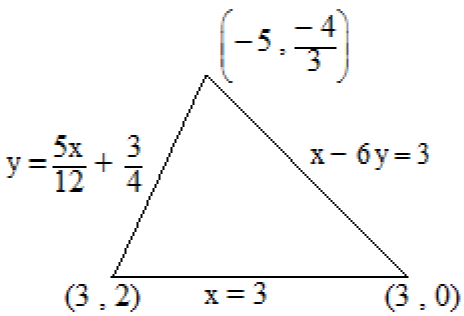
$\Rightarrow\ 12m=5$

$\Rightarrow\ m=\frac{5}{12}$ other root is ∞.

So tangent are $\boxed{y=\frac{5x}{12}+\frac{3}{4}}$ & $\boxed{x=3}$

Chord of contact is 3x – 9y(2) = 9 i.e. $\boxed{x-6y=3}$

Area of triangle can now be obtained



Which is 8 sq. units.

Q.13

Let the chords be $y=m(x-ae)$ & $y=-\frac{1}{m}(x+ae)$

Eliminating m gives $y^2=-(x+ae)(x-ae)$ or $x^2+y^2=a^2e^2$ as the required locus.

Q.14

$x=t^2+t+1, y=t^2-t+1\Rightarrow x-y=2t$

$\Rightarrow x=\left(\frac{x-y}{2}\right)^2+\frac{x-y}{2}+1$

$\Rightarrow \frac{x+y-2}{2}=\left(\frac{x-y}{2}\right)^2$

The required locus is in standard form of equation of parabola.

Q.15

Let the mid-point be (h, k), then equation of chord $(T = S_1)$ be $hx - ky = a^2$.

Also equation of any tangent to $y^2 = 4ax$ be $ty = x + at^2$.

Comparing the two equations gives, $\frac{k}{t} = h = -\frac{a}{t^2}$ or $k^2 = -ah$

Hence required locus is $y^2 = -ax$.

Q.16

Tangent to the hyperbola at $P(\theta)$ is $\frac{x}{2}\sec\theta - \frac{y}{3}\tan\theta = 1$.

Comparing this with $3x - y = c$ gives

$\frac{\sec\theta}{6} = \frac{\tan\theta}{3} \Rightarrow \sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$.

Q.17

Let the common tangent be $y = mx + c$, then

for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ c^2 = a^2m^2 - b^2$ & for $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1,\ c^2 = a^2 - b^2m^2$.

Hence $a^2m^2 - b^2 = a^2 - b^2m^2$ or $m = \pm 1$.

Hence the common tangents are $y = \pm x \pm \sqrt{a^2 - b^2}$.

Q.18

Let any point on S_1 be $(a\sec\theta, b\tan\theta)$.

Chord of contact of S_2 w.r.to this point will be $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 2$ ...(i)

Also asymptotes of S_1 are $\frac{x}{a} \pm \frac{y}{b} = 0$.

Solving these with (i) gives points of intersections as
$\left(2a(\sec\theta + \tan\theta), 2b(\sec\theta + \tan\theta)\right)$ & $\left(2a(\sec\theta - \tan\theta), -2b(\sec\theta - \tan\theta)\right)$

Now area of triangle formed by these and the origin

$$=\frac{1}{2}\left\|\begin{matrix}1 & 0 & 0\\ 1 & 2a(\sec\theta+\tan\theta) & 2b(\sec\theta+\tan\theta)\\ 1 & 2a(\sec\theta-\tan\theta) & -2b(\sec\theta-\tan\theta)\end{matrix}\right\| = 4ab.$$

Q.19

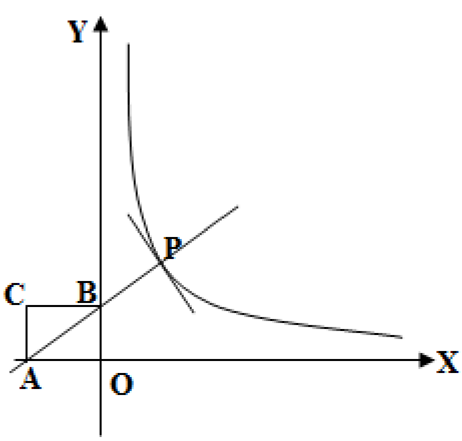
$$(10x-5)^2+(10y-2)^2=9(3x+4y-7)^2 \Rightarrow \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{5}\right)^2=\frac{9}{4}\left(\frac{3x+4y-7}{5}\right)^2$$

Hence one focus is $\left(\frac{1}{2},\frac{1}{5}\right)$, corresponding directrics is 3x + 4y = 7 and eccentricity is $\frac{3}{2}$.

So the latus rectum will be parallel to directrics passing through the focus

i.e. 3x + 4y = $\frac{23}{10}$.

Q.20



Let P be $\left(t,\frac{1}{t}\right)$, then normal at P will be $t^3x-ty=t^4-1$.

Hence coordinates of A & B will be $\left(\frac{t^4-1}{t^3},0\right)\&\left(0,-\frac{t^4-1}{t}\right)$ and

coordinates of P will be $x=\frac{t^4-1}{t^3}, y=-\frac{t^4-1}{t}$.

Eliminating 't' gives the required locus as $\left(x^2-y^2\right)^2+x^3y^3=0$.

Q.21

Equation of chord joining $P(\theta_1) \& Q(\theta_2)$ will be

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2}$$

If it passes through ($\pm$ ae, 0), then $\dfrac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}=\dfrac{1}{\pm e}$.

$$\Rightarrow \frac{\cos\frac{\theta_1-\theta_2}{2}-\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}+\cos\frac{\theta_1+\theta_2}{2}}=\frac{1\mp e}{1\pm e} \text{ or } \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{1\mp e}{1\pm e}.$$

Q.22

Locus of P, such that $|PA - PB| = k$, is a hyperbola if $0 < k < AB$.

Hence k must be less than the distance between (0, -1) & (0, 1)

i.e. $0 < k < 2$.

Q.23

Let P be $\left(t, \frac{1}{t}\right)$, then tangent and normal at P will be $x+t^2y=2ct$ & $t^3x-ty=t^4-1$.

Now $a_1=2ct, b_1=\dfrac{2c}{t}, a_2=\dfrac{c\left(t^4-1\right)}{t^3}$ & $b_2=-\dfrac{c\left(t^4-1\right)}{t}$

$$\Rightarrow a_1a_2+b_1b_2=2ct\times\frac{c\left(t^4-1\right)}{t^3}-\frac{2c}{t}\times\frac{c\left(t^4-1\right)}{t}=0.$$

Q.24

Let P be $\left(ct_1, \frac{c}{t_1}\right)$, then normal at P will be $t_1{}^3x-t_1y=c\left(t_1{}^4-1\right)$.

If this normal meets the curve again at $\left(ct_2, \frac{c}{t_2}\right)$, then

$$t_1^3 t_2 - \frac{t_1}{t_2} = t_1^4 - 1$$

$$\Rightarrow t_1^3 t_2 \left(t_2 - t_1\right) = t_1 - t_2$$

$$\Rightarrow t_1^3 t_2 = -1.$$